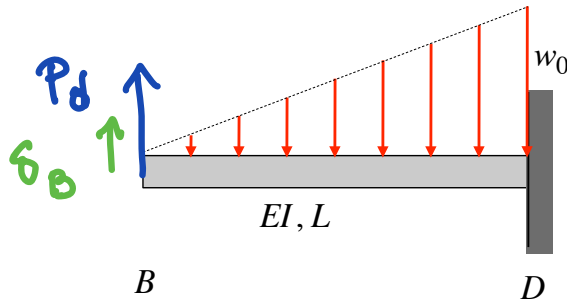


**Example 16.6**

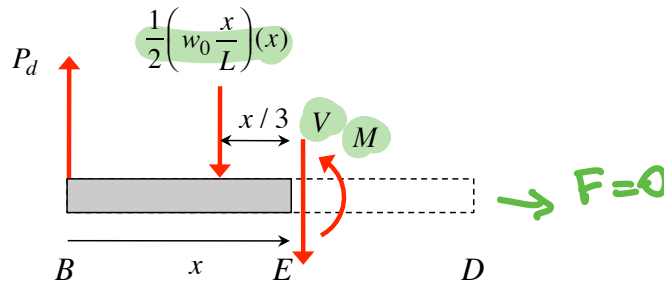
Determine the vertical deflection and beam rotation at end B of the cantilevered beam shown.



$$U(P_d, w_0)$$

$$\delta_B = \frac{\partial U}{\partial P_d} \Big|_{P_d=0}$$

Vertical deflection at B: apply “dummy” load  $P_d$  at end B



Determining internal bending moment

$$\sum M_E = M - P_d x + \left( \frac{w_0 x^2}{2L} \right) \left( \frac{x}{3} \right) = 0 \Rightarrow M(x) = -\frac{w_0 x^3}{6L} + P_d x$$

Strain energy in beam (ignoring contributions from shear stress/strain)

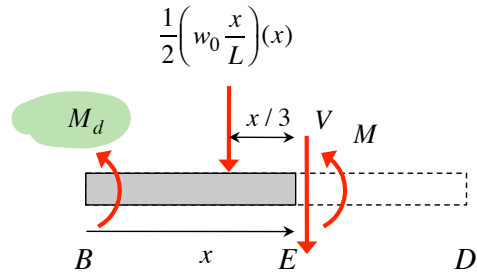
$$U = \frac{1}{2} \int_0^L \frac{M^2}{EI} dx = \frac{1}{2EI} \int_0^L \left( -\frac{w_0 x^3}{6L} + P_d x \right)^2 dx$$

Castigliano's theorem

$$v_B = \frac{\partial U}{\partial P_d} \Big|_{P_d=0} = \left[ \frac{1}{2EI} \int_0^L 2 \left( -\frac{w_0 x^3}{6L} + P_d x \right) (x) dx \right]_{P_d=0} = \left[ \frac{1}{EI} \int_0^L \left( -\frac{w_0 x^4}{6L} + P_d x^2 \right) dx \right]_{P_d=0}$$

$$= \left\{ \frac{1}{EI} \left[ -\frac{w_0 x^5}{30L} + \frac{P_d}{3} x^3 \right]_{x=0}^{x=L} \right\}_{P_d=0} = \frac{1}{EI} \left[ -\frac{w_0 L^4}{30} + \frac{P_d L^3}{3} \right]_{P_d=0} = -\frac{w_0 L^4}{30EI} \text{ (DOWN)}$$

**Beam rotation at B:** need to apply “dummy” couple  $M_d$  at end B



Determining internal bending moment

$$\sum M_E = M + M_d + \left( \frac{w_0 x^2}{2L} \right) \left( \frac{x}{3} \right) = 0 \Rightarrow M(x) = -\frac{w_0 x^3}{6} - M_d$$

Strain energy in beam (ignoring contributions from shear stress/strain)

$$U = \frac{1}{2} \int_0^L \frac{M^2}{EI} dx = \frac{1}{2EI} \int_0^L \left( -\frac{w_0 x^3}{6L} - M_d \right)^2 dx$$

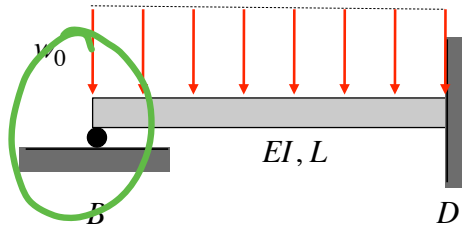
Castigliano's theorem

$$\theta_B = \frac{\partial U}{\partial M_d} \Big|_{M_d=0} = \left[ \frac{1}{2EI} \int_0^L 2 \left( -\frac{w_0 x^3}{6L} - M_d \right) (-1) dx \right]_{M_d=0} = \left[ \frac{1}{EI} \int_0^L \left( \frac{w_0 x^3}{6L} + M_d \right) dx \right]_{M_d=0}$$

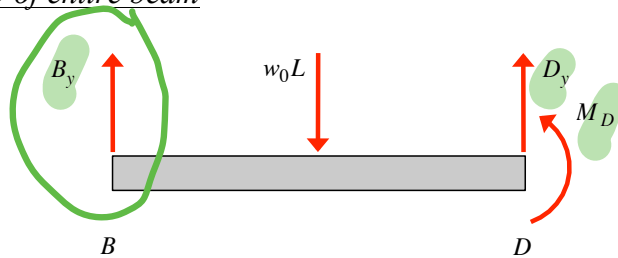
$$= \left\{ \frac{1}{EI} \left[ \frac{w_0 x^4}{24L} + M_d x \right]_{x=0}^{x=L} \right\}_{M_d=0} = \frac{1}{EI} \left[ \frac{w_0 L^3}{24} + M_d L \right]_{M_d=0} = \frac{w_0 L^3}{24EI} \text{ (CCW)}$$

### Example 16.7

Determine the reaction at end B of the beam shown.



Equilibrium – FBD of entire beam



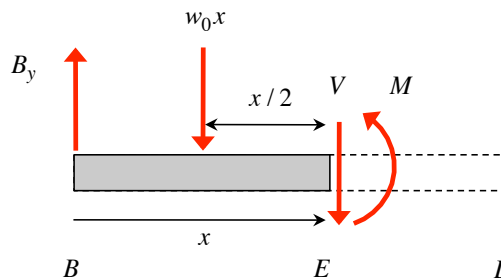
From here, we see that the problem is statically indeterminate: 3 unknowns ( $B_y$ ,  $D_y$  and  $M_D$ ) and only two equations. Here, we will choose  $B_y$  to be our redundant reaction:

#1  $\sum F_y = B_y - w_0 L + D_y = 0 \Rightarrow D_y = w_0 L - B_y$

#2  $\sum M_B = -(w_0 L) \left( \frac{L}{2} \right) + D_y L + M_D = 0 \Rightarrow M_D = -(w_0 L - B_y) L + \frac{1}{4} w_0 L^2$

3 unknowns  
2 equations  
1 redundant force

Determining internal bending moment



$\sum M_E = M - B_y x + (w_0 x) \left( \frac{x}{2} \right) = 0 \Rightarrow M(x) = -\frac{w_0 x^2}{2} + B_y x$

Strain energy in beam (ignoring contributions from shear stress/strain)

$U = \frac{1}{2} \int_0^L \frac{M^2}{EI} dx = \frac{1}{2EI} \int_0^L \left( -\frac{w_0 x^2}{2} + B_y x \right)^2 dx$

Castigliano's theorem

With  $B_y$  being our choice for the redundant reaction:

equation #3

$$0 = \frac{\partial U}{\partial B_y} = \frac{1}{2EI} \int_0^L 2 \left( -\frac{w_0 x^2}{2} + B_y x \right) (x) dx = \frac{1}{EI} \int_0^L \left( -\frac{w_0 x^3}{2} + B_y x^2 \right) dx$$
$$= \frac{1}{EI} \left[ -\frac{w_0 x^4}{8} + \frac{B_y x^3}{3} \right]_{x=0}^{x=L} = \frac{1}{EI} \left[ -\frac{w_0 L^4}{8} + \frac{B_y L^3}{3} \right] \Rightarrow B_y = \frac{3}{8} w_0 L$$

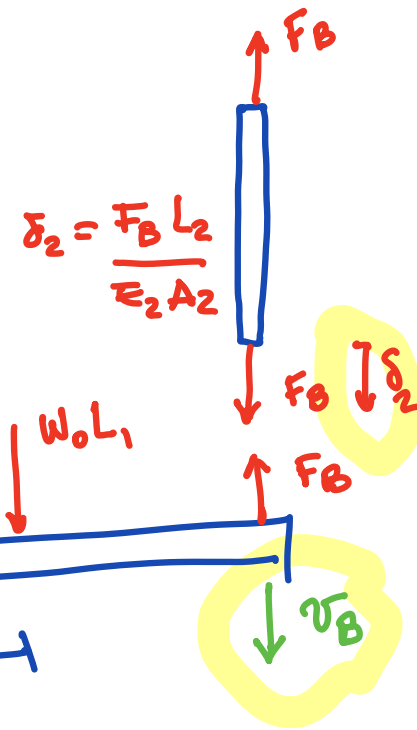
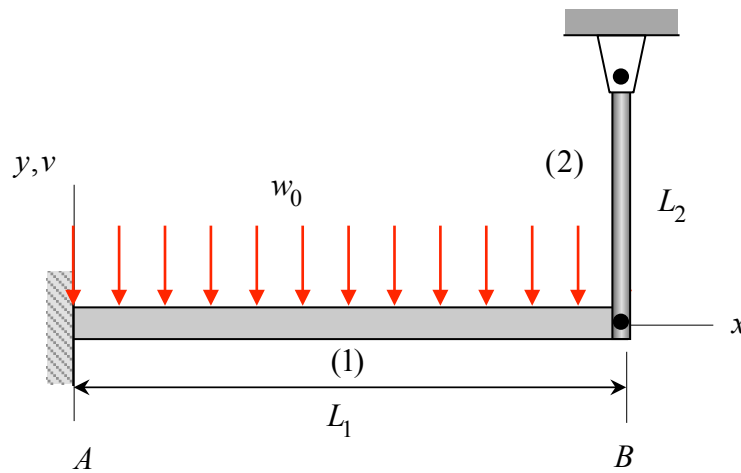
**Example 16.8**

For the following examples, set up the problem for determining the requested deflections using Castigliano's method:

- draw appropriate FBDs;
- determine internal results for each section;
- set up the integrals for calculating the required deflections;
- explain how Castigliano's method is used to solve. Discuss the application of dummy forces (when needed) and how to handle redundant forces for indeterminate structures.

**Problem A**

Find the load carried by member (2) of the structure below. Let  $E$  and  $A$  be the Young's modulus and cross-sectional area, respectively, of member (2), whereas  $E$  and  $I$  are the Young's modulus and second area moment of the cross section of (1), respectively.

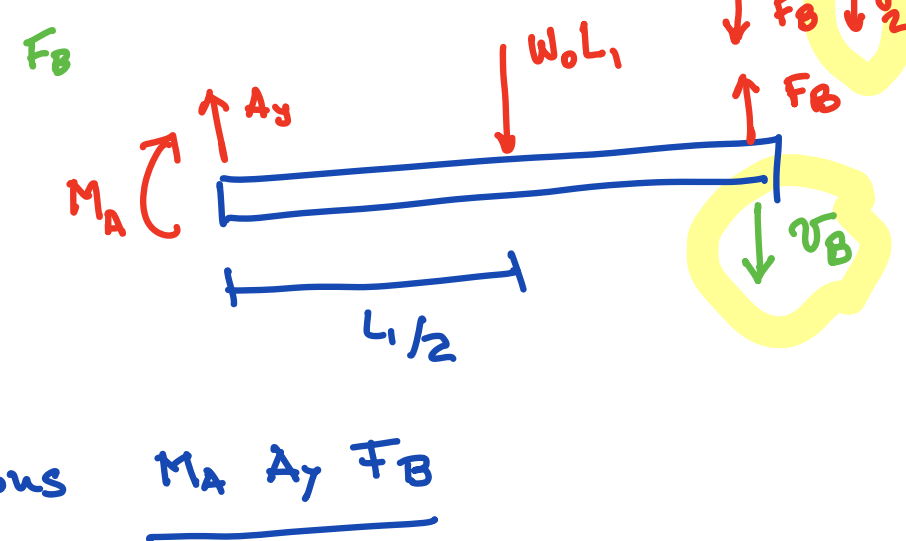


$$\sum M|_A = 0$$

$$M_A + \frac{w_0 L_1^2}{2} - F_B L_1 = 0 \quad (1)$$

$$\sum F_y = 0$$

$$A_y - w_0 L_1 + F_B = 0 \quad (2)$$



2 equations      3 unknowns       $M_A \quad A_y \quad F_B$

Find  $U \rightarrow$

$$\frac{\partial U}{\partial F_B} = -\delta_2$$

direction of  $F_B$

$$\delta_2 = \delta_2$$

$\downarrow$  more equation we solve

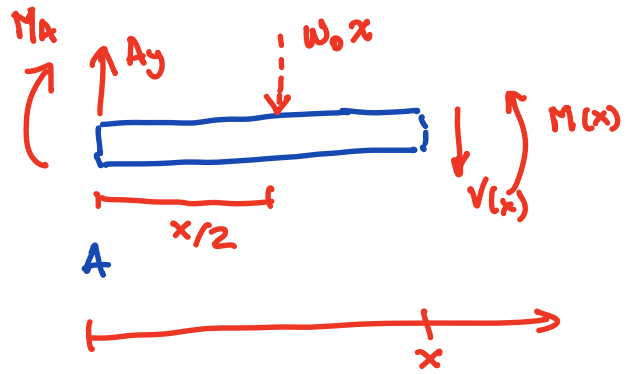
$$\delta_2 = \frac{F_B L_2}{E_2 A_2}$$

MA  $A_7$   $F_B$

$$U_2 = \frac{F_B^2 L_2}{2 E_2 A_2}$$

member 2

$U = U_1 + U_2$



$$M(x) = M_A + A_7 x - w_0 x \cdot \frac{x}{2}$$

$$V(x) = A_7 - w_0 x$$

$$\sum M_{F_0} = 0 \quad M(x) + \frac{w_0 (L_1 - x)^2}{2} - F_B (L_1 - x) = 0$$

$$M(x) = F_B (L_1 - x) - \frac{w_0 (L_1 - x)^2}{2}$$

Member (2)

$$U = \frac{F_B^2 L_2}{2 E_2 A_2} + \int_0^{L_1} \frac{1}{2 E_1 I} M^2(x) dx + (\text{neglect shear})$$

member (1)

$$-V_B = \frac{\partial U}{\partial F_B} = \frac{F_B L_2}{E_2 A_2} + \int_0^{L_1} \frac{1}{E_1 I_1} M(x) \cdot \frac{\partial M(x)}{\partial F_B} dx$$

$$-V_B = \frac{F_B L_2}{E_2 A_2} + \int_0^{L_1} \frac{1}{E_1 I_1} \left( F_B (L_1 - x) - \frac{w_0 (L_1 - x)^2}{2} \right) (L_1 - x) dx$$

Solve this integral

$$V_B = \delta_{(2)} = \frac{F_B L_2}{E_2 A_2} =$$

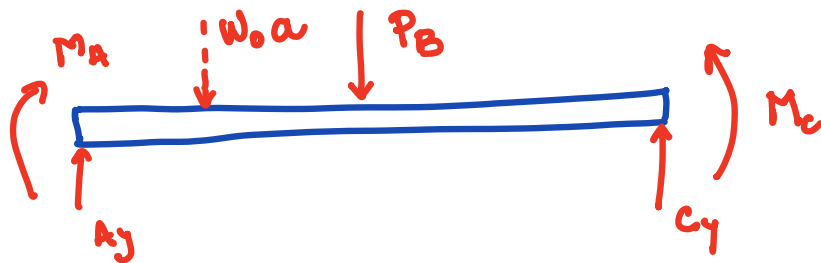
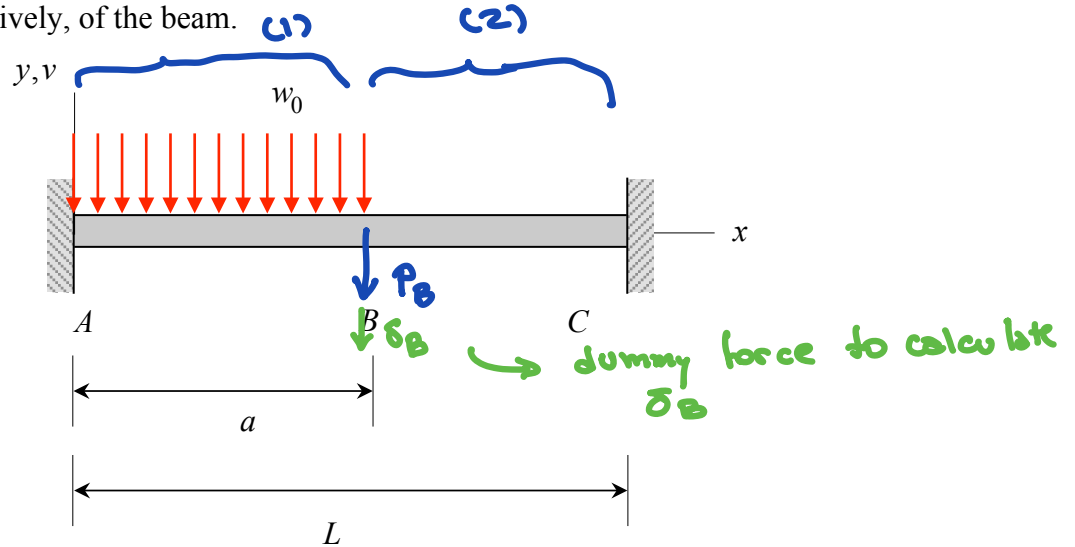
Solve for  $F_B$

Remember  
the "-"  
sign



**Problem B**

Find the vertical deflection of the beam at point B and the angle of rotation of the beam at B. Let  $E$  and  $I$  be the Young's modulus and second area moment of the beam cross section, respectively, of the beam.

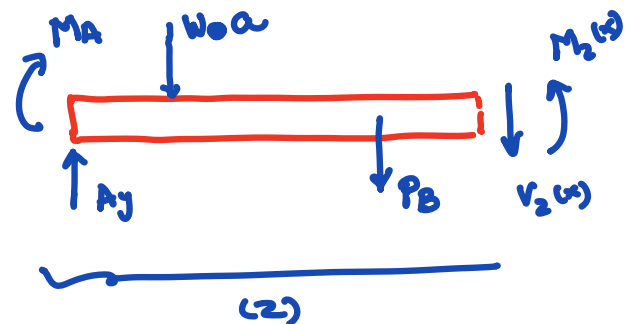
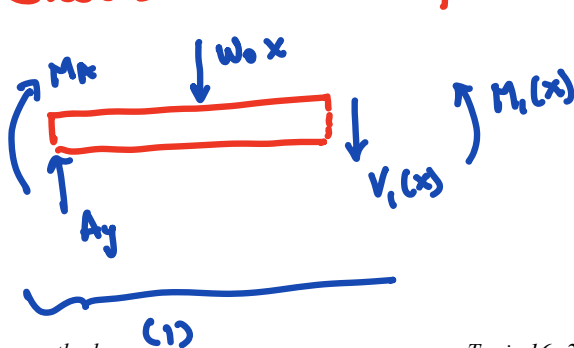


$$\sum M|_A = 0 \quad - M_A - w_0 a \frac{a}{2} - P_B \cdot a + C_y \cdot L + M_C \quad (1)$$

$$\sum F_y = 0 \quad A_y - w_0 a - P_B + C_y = 0 \quad (2)$$

2 eq.  $M_A$   $A_y$   $M_C$   $C_y$  4 unknowns

Choose  $M_A$   $A_y$



$$M_1(x) = M_A + A_7 x - \frac{1}{2} w_0 x^2$$

$$M_2(x) = M_A + A_7 x - w_0 a \left(x - \frac{a}{2}\right) - P_B (x - a)$$

$$U = \int_0^a \frac{M_1(x)^2}{2EI} dx + \int_a^L \frac{M_2(x)^2}{2EI} dx$$

$$U = U(M_A, A_7, P_B)$$

$$\delta_B = \left. \frac{\partial U}{\partial P_B} \right|_{P_B=0}$$

$$\left. \frac{\partial U}{\partial A_7} \right|_{P_B=0} = 0$$

$$\left. \frac{\partial U}{\partial M_A} \right|_{P_B=0} = 0$$

Find  $M_A, A_7$   
 $\downarrow$   
 To calculate  $M_C, C_7$   
 use eq. (1) & (2)  
 with  $P_B = 0$

$$\frac{\partial U}{\partial P_B} = \delta_B = \frac{a^4}{12L^3} (L-a)^2 (L-2a) w_0$$

Find  $M_A, A_7$

$$0 = \frac{\partial U}{\partial A_7} = \frac{1}{EI} \int_0^a \frac{\partial M_1}{\partial A_7} \cdot M_1 dx + \frac{1}{EI} \int_a^L \frac{\partial M_2}{\partial A_7} M_2 dx$$

$$\frac{\partial M_1}{\partial A_7} = x$$

$$\frac{\partial M_2}{\partial A_7} = x$$

$$(3) \quad 0 = 8A_7 L^3 - 2a^4 \omega_0 - 4aL^3 \omega_0 + 3L^2 (4M_A + a^2 \omega_0)$$

↑  
( $P_B = 0$ )

$$\frac{\partial U}{\partial M_A} = 0 \quad \frac{\partial M_1}{\partial M_A} = 1 \quad \frac{\partial M_2}{\partial M_A} = 1$$

$$\hookrightarrow \int_0^a M_1(x) dx + \int_a^L M_2(x) dx = 0$$

$$(4) \quad 0 = 6A_7 L^2 - 2a^3 \omega_0 - 3aL^2 \omega_0 + 3L (4M_A + a^2 \omega_0)$$

→ ( $P_B = 0$ )

Solve (3) & (4)

$$M_A = \frac{-6a^4 \omega_0 + 8a^3 L \omega_0 - 3a^2 L^2 \omega_0}{12L^2}$$

$$A_7 = \frac{2a^4 \omega_0 - 2a^3 L \omega_0 + aL^3 \omega_0}{2L^3}$$

Now calculate  $\delta_B$

$$\delta_B = \frac{\partial U}{\partial P_B} \quad \frac{\partial M_1}{\partial P_B} = 0$$

$$\frac{\partial M_2}{\partial P_B} = -(x-a)$$

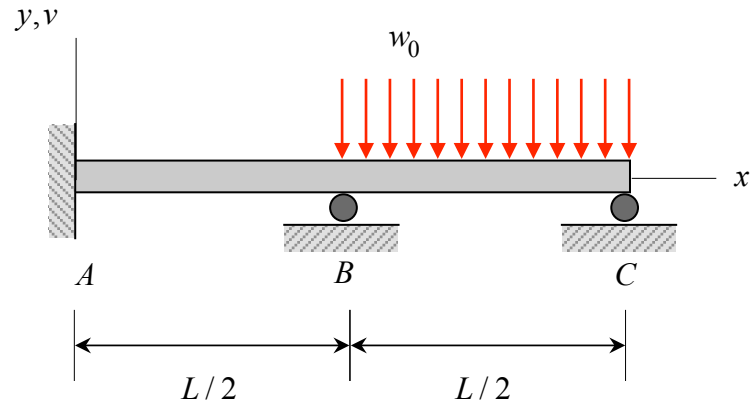
$$\delta_B = \frac{1}{EI} \int_a^L M_2(x) (a-x) dx$$

$$= \frac{(L-a)^2}{24 EI} (8 A_7 L + 12 M_A + a^2 w_0 + 4a (A_7 - L w_0))$$

↳ Replace  $A_7$   $M_A$

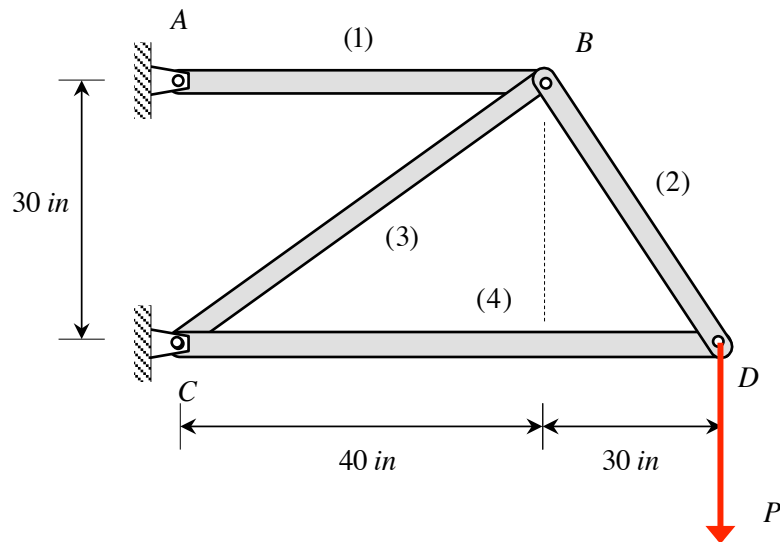
**Problem C**

Determine the reactions at rollers B and C on the beam below. Let  $E$  and  $I$  be the Young's modulus and second area moment of the beam cross section, respectively, of the beam.



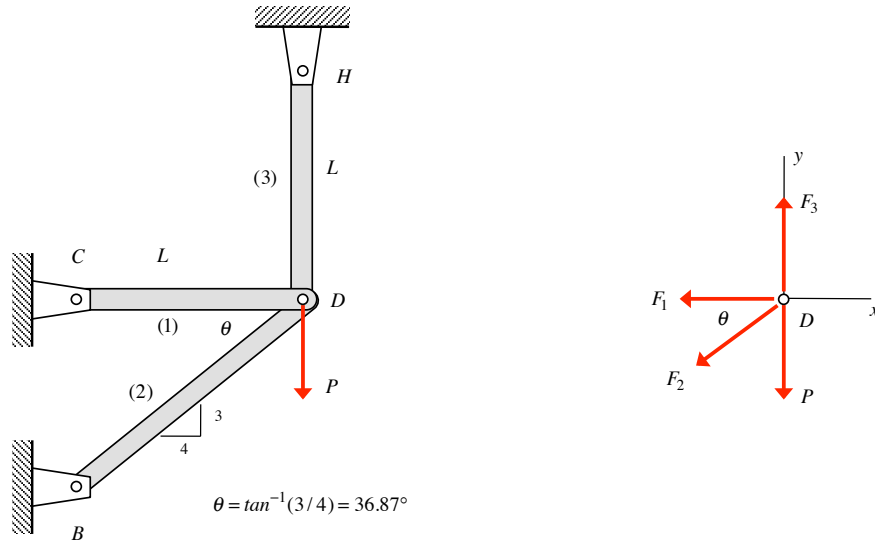
***Problem D***

Determine the vertical and horizontal deflection of the truss at joint D. All members of the truss have a cross-sectional area of  $A$  and are made of a material with a Young's modulus of  $E$ .



### Example 16.9

Determine the vertical component of deflection of node D on the truss. All members have the same cross-sectional area  $A$  and are made of the same material having a Young's modulus of  $E$ .



#### Equilibrium analysis of joint D

$$\sum F_y = -P - F_2 \sin\theta + F_3 = 0 \Rightarrow F_3 = P + F_2 \sin\theta \quad (1)$$

$$\sum F_x = -F_1 - F_2 \cos\theta = 0 \Rightarrow F_1 = -F_2 \cos\theta \quad (2)$$

From this we see that the problem is INDETERMINATE (two equilibrium equations and three unknowns). We will consider the force  $F_2$  to be the “redundant” force (this was an arbitrary choice).

#### Strain energy in truss

$$\begin{aligned} U &= U_1 + U_2 + U_2 = \frac{1}{2} \frac{F_1^2 L_1}{EA} + \frac{1}{2} \frac{F_2^2 L_2}{EA} + \frac{1}{2} \frac{F_3^2 L_3}{EA} \quad ; \quad L_2 = \frac{L}{\cos\theta} = \frac{5L}{4} \\ &= \frac{1}{2EA} \left[ F_1^2 L + F_2^2 \left( \frac{5L}{4} \right) + F_3^2 L \right] \\ &= \frac{L}{2EA} \left[ (-F_2 \cos\theta)^2 + \frac{5}{4} F_2^2 + (P + F_2 \sin\theta)^2 \right] \quad ; \quad \text{using (1) and (2)} \end{aligned}$$

#### Castigliano's theorem as applied to indeterminate structures

Since we chose  $F_2$  as our redundant force:

$$0 = \frac{\partial U}{\partial F_2} = \frac{L}{EA} \left[ F_2 \cos^2 \theta + \frac{5}{4} F_2 + (P + F_2 \sin \theta) \sin \theta \right] \Rightarrow$$

$$0 = F_2 \cos^2 \theta + \frac{5}{4} F_2 + (P + F_2 \sin \theta) \sin \theta \Rightarrow$$

$$\left( \cos^2 \theta + \frac{5}{4} + \sin^2 \theta \right) F_2 = -P \sin \theta \Rightarrow F_2 = -\frac{4}{9} P \sin \theta$$

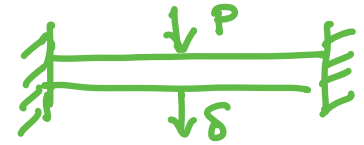
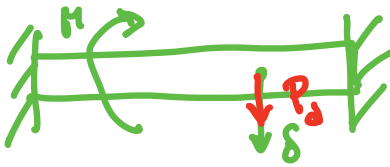
Therefore, the strain energy function becomes:

$$U = \frac{L}{2EA} \left\{ \left[ -\left( -\frac{4}{9} P \sin \theta \right) \cos \theta \right]^2 + \frac{5}{4} \left( -\frac{4}{9} P \sin \theta \right)^2 + \left( \frac{5}{9} P \sin \theta \right)^2 \right\}$$

$$= \frac{P^2 L \sin^2 \theta}{162EA} \left( \frac{5}{9} + \frac{16}{81} \cos^2 \theta \right)$$

Using Castigliano's theorem gives:

$$v_D = \frac{\partial U}{\partial P} = \frac{PL \sin^2 \theta}{81EA} \left( \frac{5}{9} + \frac{16}{81} \cos^2 \theta \right) \quad (\text{since "+" , in same direction as P - DOWN})$$



## Deflection analysis – Castigliano’s method

The procedure for deflection analysis using Castigliano’s method:

- i) First determine if you need to include any “dummy” loads (recall that the Castigliano’s method can produce deflections/rotations at points on the structures at which applied forces/moments act and in directions in which these forces/moments act). Add in ALL of the needed dummy loads from the start; this can save you a lot of time down the road.
- ii) Draw a free body diagram (FBD) of the entire structure, and from this FBD write down the equilibrium equations; these equilibrium equations will be written in terms of the external reactions.
  - If **DETERMINATE**, solve these equations for the external reactions.
  - If **INDETERMINATE**, establish the “order”  $N_R$  of the indeterminacy (i.e., equal to the number of additional equations needed to solve for external reactions). From your external reactions, choose a set of  $n$  redundant reactions ( $R_i ; i = 1, 2, \dots, N_R$ ). Write the remaining reactions in terms of these  $N_R$  redundant reactions.
- iii) Divide beam into sections:  $x_i < x < x_{i+1}$ . This section division is dictated by: support reactions, beam geometry changes, and/or load changes (concentrated forces/moments, line load definition changes, etc.).
- iv) For each section, draw an FBD of either the left or right side of the body from a cut through that section of the beam. From this FBD, determine the distribution of bending moment  $M_i(x)$ , shear force  $V_i(x)$  and axial force  $F_{Ni}(x)$  through that section of the structure. Using these, write down the strain energy in that section of the structure using:

$$U_i = \frac{1}{2EI} \int_{x_i}^{x_{i+1}} M_i^2 dx + \frac{f_s}{2GA} \int_{x_i}^{x_{i+1}} V_i^2 dx + \frac{1}{2EA} \int_{x_i}^{x_{i+1}} F_{Ni}^2 dx$$

From these strain energy terms, write down the total strain energy for the structure:  $U = U_1 + U_2 + U_3 + \dots$ . It is recommended that you do NOT expand out the “squared” terms in these integrals at this point.

- v) If the problem is **INDETERMINATE**, first set up the additional algebraic equations for the reactions of the problems using Castigliano:

$$0 = \frac{\partial U}{\partial R_i} ; \quad i = 1, 2, \dots, N_R$$

Be sure to set any dummy loads to zero in the end. Solve these equations with the equilibrium equations from i) above.

- vi) Determine the desired deflections/rotations using Castigliano’s method:  $\delta_i = \partial U / \partial P_i$ . Be sure to set any dummy loads to zero in the end.

***Additional notes:***