Discussion for indeterminate beams

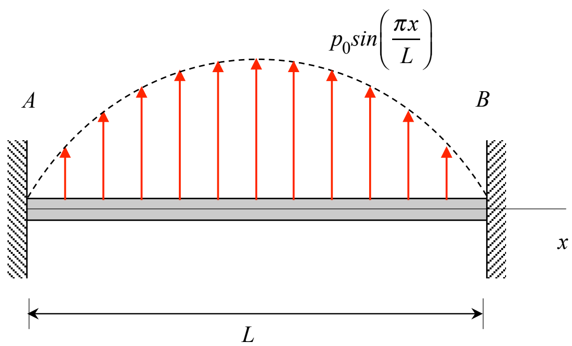
Statically indeterminate beams are those for which we are unable to solve for reactions simply from the rigid body equilibrium analysis. This is the case since the number of unknowns exceeds the number of equilibrium equations available. As we found for axially-loaded rods and torque-loaded shafts, we need to solve deformation analysis equations simultaneously with the equilibrium equations in order to determine the reactions on the beams.

Reread the earlier discussion on the analysis of determinate beams. All of the points made there also apply to indeterminate beams, with the exception of point a). In point a), we need to revise the wording for indeterminate beams to state that we are not able to solve for reactions at the first step. Here, for indeterminate beams, we will leave the reactions as unknowns, and later enforce compatibility relations to produce the additional equations needed for a solution.

Note that reactions for determinate beams are independent of material and cross section properties of the beam since these reactions depend only on the distribution of applied loads. For indeterminate beams, material and cross section properties (EI) influence the values found for the reactions.

Example 11.8

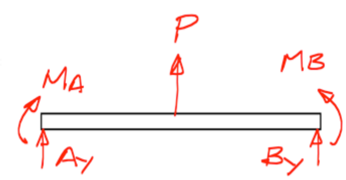
Determine the deflection curve for the beam shown below. The beam is made up of a material with an elastic modulus E and has a cross-sectional second area moment I, both of which are constant along the length of the beam.



SOLUTION

Here we will pursue the "fourth-order" integration approach due to a somewhat complicated loading on the beam.

1. Equilibrium



$$P = single\ force\ equivalent\ for\ p(x)$$

$$= \int_0^L p(x)\,dx = p_0 \int_0^L sin\pi\frac{x}{L}\,dx = -\frac{p_0 L}{\pi}\left[cos\pi\frac{x}{L}\right]_{x=0}^{x=L} = 2\frac{p_0 L}{\pi}$$

$$\sum M_A = P\left(\frac{L}{2}\right) + B_y(L) - M_A + M_B = 0 \;\Rightarrow\; B_y L - M_A + M_B = -\frac{p_0 L^2}{\pi} \qquad (1)$$

$$\sum F_y = A_y + B_y - P = 0 \;\Rightarrow\; A_y + B_y = P \qquad (2)$$

Have two equations and four unknowns (A_y, B_y, M_A, M_B). Therefore, the problem is indeterminate.

2. Load/deflection (using fourth-order integration):

$$V(x) = V(0) + \int_0^x p(x)\,dx = V(0) + \int_0^x p(x)\,dx = A_y + p_0\int_0^L sin\pi\frac{x}{L}\,dx$$

$$= A_y - \frac{p_0 L}{\pi}\left[cos\pi\frac{x}{L}\right]_0^x = A_y + \frac{p_0 L}{\pi}\left[1 - cos\pi\frac{x}{L}\right]$$

$$= \left(A_y + \frac{p_0 L}{\pi}\right) - \frac{p_0 L}{\pi}cos\pi\frac{x}{L}$$

$$M(x) = M(0) + \int_0^x V(x)\,dx = M(0) + \int_0^x \left[\left(A_y + \frac{p_0 L}{\pi} \right) - \frac{p_0 L}{\pi} cos\pi \frac{x}{L} \right] dx$$

$$= M_A + \left(A_y + \frac{p_0 L}{\pi} \right) x - \frac{p_0 L^2}{\pi^2} sin\pi \frac{x}{L}$$

$$\theta(x) = \theta(0) + \frac{1}{EI} \int_0^x M(x)\,dx = 0 + \frac{1}{EI} \int_0^x \left[M_A + \left(A_y + \frac{p_0 L}{\pi} \right) x - \frac{p_0 L^2}{\pi^2} sin\pi \frac{x}{L} \right] dx$$

$$= \frac{1}{EI} \left[M_A x + \frac{1}{2} \left(A_y + \frac{p_0 L}{\pi} \right) x^2 + \frac{p_0 L^3}{\pi^3} \left(cos\pi \frac{x}{L} - 1 \right) \right]$$

$$v(x) = v(0) + \int_0^x \theta(x)\,dx = 0 + \frac{1}{EI} \int_0^x \left[M_A x + \frac{1}{2} \left(A_y + \frac{p_0 L}{\pi} \right) x^2 + \frac{p_0 L^3}{\pi^3} \left(cos\pi \frac{x}{L} - 1 \right) \right] dx$$

$$= \frac{1}{EI} \left[\frac{1}{2} M_A x^2 + \frac{1}{6} \left(A_y + \frac{p_0 L}{\pi} \right) x^3 + \frac{p_0 L^4}{\pi^4} sin\pi \frac{x}{L} - \left(\frac{p_0 L^3}{\pi^3} \right) x \right]$$

3. Compatibility

Enforce the no-displacement/no-slope BCs at end B:

$$\theta(L) = 0 = \frac{1}{EI} \left[M_A L + \frac{1}{2} \left(A_y + \frac{p_0 L}{\pi} \right) L^2 - 2 \frac{p_0 L^3}{\pi^3} \right]$$

$$v(L) = 0 = \frac{1}{EI} \left[\frac{1}{2} M_A L^2 + \frac{1}{6} \left(A_y + \frac{p_0 L}{\pi} \right) L^3 - \frac{p_0 L^4}{\pi^3} \right]$$

or:

$$2 M_A + A_y L = -\left(1 - \frac{4}{\pi^2} \right) \frac{p_0 L^2}{\pi} \qquad (3)$$

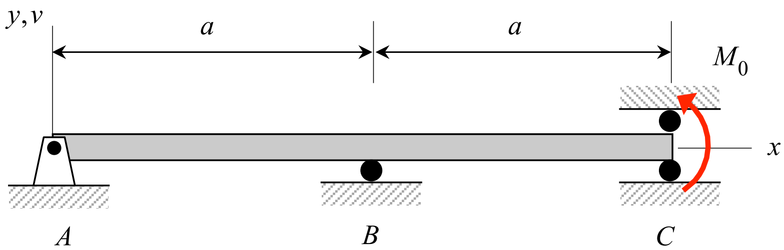
$$3 M_A + A_y L = -\left(1 - \frac{6}{\pi^2} \right) \frac{p_0 L^2}{\pi} \qquad (4)$$

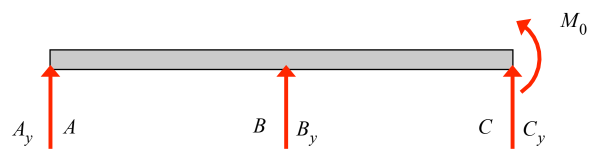
4. Solve

Solve equations (3) and (4) for A_y and M_A. With those results, the expression for $\theta(x)$ and $v(x)$ are then complete.

Example 11.9

The beam is made up of a material with an elastic modulus E that is constant throughout the beam. The beam has a square cross section of dimensions $h \times h$. Using the second-order integration method, determine the reactions at A, B and C.





1. Equilibrium

$$\sum M_A = B_y\left(a\right) + C_y\left(2a\right) + M_0 = 0 \;\;\Rightarrow\;\; B_y + 2C_y = -\frac{M_0}{a} \qquad (1)$$

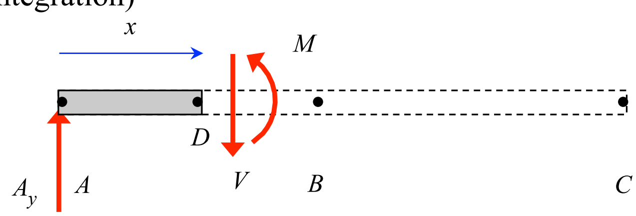
$$\sum M_A = A_y + B_y + C_y = 0 \qquad (2)$$

Have two equations and three unknowns (A_y, B_y, C_y). Therefore, the problem is indeterminate.

2. Load/deflection equations (using second-order integration)

<u>Section AB</u>:

$$\sum M_D = M - A_y x = 0 \;\;\Rightarrow\;\; M(x) = A_y x$$



Therefore:

$$\theta(x) = \theta(0) + \frac{1}{EI}\int_0^x \left(A_y x\right) dx = \theta_A + \frac{1}{2}\frac{A_y}{EI}x^2$$

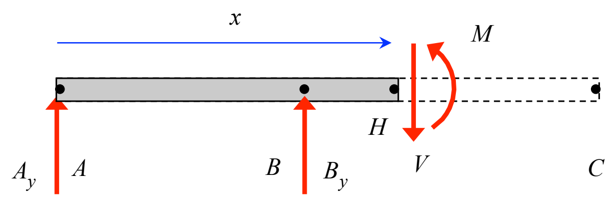
$$v(x) = v(0) + \int_0^x \left(\theta_A + \frac{1}{2}\frac{A_y}{EI}x^2\right) dx = 0 + \theta_A x + \frac{1}{6}\frac{A_y}{EI}x^3$$

From this, we have:

$$v_B = 0 = v(a) = \theta_A a + \frac{1}{6}\frac{A_y}{EI}a^3 \;\;\Rightarrow\;\; \theta_A = -\frac{1}{6}\frac{A_y}{EI}a^2$$

$$\theta_B = \theta(a) = \theta_A + \frac{1}{2}\frac{A_y}{EI}a^2 = -\frac{1}{6}\frac{A_y}{EI}a^2 + \frac{1}{2}\frac{A_y}{EI}a^2 = \frac{1}{3}\frac{A_y}{EI}a^2$$

Section BC:



$$\sum M_H = M - A_y x - B_y\left(x-a\right) = 0 \quad \Rightarrow$$
$$M(x) = \left(A_y + B_y\right)x - B_y a$$

Therefore:

$$\theta(x) = \theta(a) + \frac{1}{EI}\int_a^x \left[\left(A_y + B_y\right)x - B_y a\right]dx = \theta_B + \frac{1}{EI}\left[\frac{1}{2}\left(A_y + B_y\right)x^2 - B_y a x\right]_a^x$$
$$= \theta_B + \frac{1}{EI}\left[\frac{1}{2}\left(A_y + B_y\right)\left(x^2 - a^2\right) - B_y a\left(x-a\right)\right]$$
$$= \theta_B + \frac{1}{EI}\left[\frac{1}{2}\left(A_y + B_y\right)x^2 + \frac{1}{2}\left(-A_y + B_y\right)a^2 - B_y a x\right]$$
$$= \left[\theta_B + \frac{a^2}{2EI}\left(-A_y + B_y\right)\right] - \frac{a}{EI}B_y x + \frac{1}{2EI}\left(A_y + B_y\right)x^2$$

$$v(x) = v(a) + \int_a^x \left\{\left[\theta_B + \frac{a^2}{2EI}\left(-A_y + B_y\right)\right] - \frac{a}{EI}B_y x + \frac{1}{2EI}\left(A_y + B_y\right)x^2\right\}dx$$
$$= 0 + \left[\theta_B + \frac{a^2}{2EI}\left(-A_y + B_y\right)\right]\left(x-a\right) - \frac{a}{2EI}B_y\left(x^2 - a^2\right) + \frac{1}{6EI}\left(A_y + B_y\right)\left(x^3 - a^3\right)$$

3. Compatibility

Enforce the zero-displacement at end C:

$$v(2a) = 0 = \left[\theta_B + \frac{a^2}{2EI}\left(-A_y + B_y\right)\right]\left(2a-a\right) - \frac{a}{2EI}B_y\left(4a^2 - a^2\right)$$
$$+ \frac{1}{6EI}\left(A_y + B_y\right)\left(8a^3 - a^3\right)$$
$$= a\theta_B + \frac{a^3}{2EI}\left(-A_y + B_y\right) - \frac{3a^3}{2EI}B_y + \frac{7a^3}{6EI}\left(A_y + B_y\right)$$
$$= a\theta_B + \left(-\frac{a^3}{2EI} + \frac{7a^3}{6EI}\right)A_y + \left(\frac{a^3}{2EI} - \frac{3a^3}{2EI} + \frac{7a^3}{6EI}\right)B_y$$

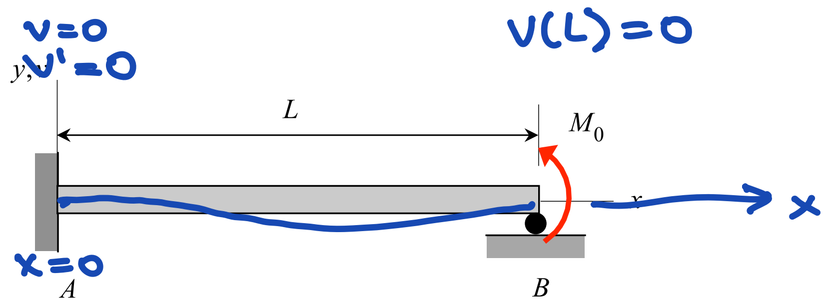
$$0 = \frac{1}{3}\frac{A_y}{EI}a^3 + \frac{2a^3}{3EI}A_y + \frac{a^3}{6EI}B_y = \frac{a^3}{EI}A_y + \frac{a^3}{6EI}B_y \quad \Rightarrow \quad B_y = -6A_y \qquad (3)$$

4. Solve

Solve equations (1), (2) and (3) for the reactions A_y, B_y and C_y.

Example 11.10

The beam is made up of a material with an elastic modulus E and has a cross-sectional second area moment I, both of which are constant along the length of the beam. Determine the reactions at A and B.



$\sum M_A = 0$ $\quad M_A - B_y \cdot L - M_0 = 0$ ①

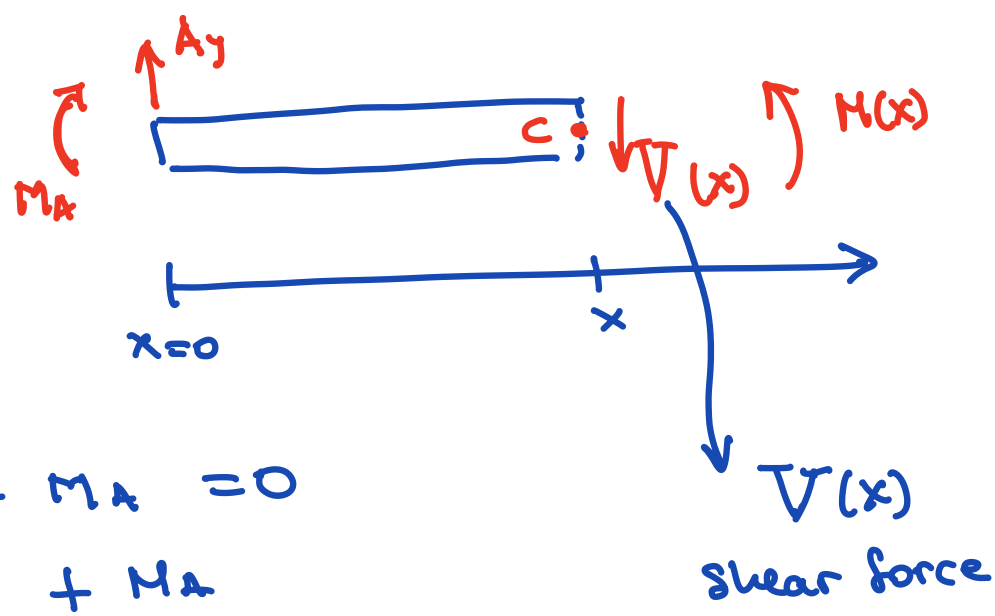
$\sum F_y = 0$ $\quad A_y = -B_y$ ②

M_A, A_y, B_y UNK (3)

① ② 2 eq

We will find the deflection by integrating M(x)

Find M(x)



$\sum M_c = 0$

$M(x) - A_y \cdot x - M_A = 0$

$M(x) = A_y x + M_A$

Find $v(x)$ (deflection)

$$EI\, v''(x) = M(x)$$

$$\int_0^x v''(\xi)\, d\xi = \int_0^x \frac{M(\xi)}{EI}\, d\xi$$

$$v'(x) - v'(0) = \left(\frac{A_y}{EI}\frac{\xi^2}{2} + \frac{M_A \xi}{EI}\right)\Bigg|_0^x$$

wall

$$v'(x) = \frac{A_y}{EI}\frac{x^2}{2} + \frac{M_A x}{EI}$$

$$\int_0^x v'(\xi)\, d\xi = \int_0^x \frac{A_y}{EI}\frac{\xi^2}{2}\, d\xi + \int_0^x \frac{M_A}{EI}\xi\, d\xi$$

$$v(x) - v(0) = \frac{A_y}{EI}\frac{x^3}{6} + \frac{M_A}{EI}\frac{x^2}{2}$$

=0 wall

Indefinite integral

$$EI\, v''(x) = M(x)$$

$$v''(x) = \frac{A_y}{EI}x + \frac{M_A}{EI}$$

$$v(x) = \frac{A_y}{EI}\frac{x^3}{6} + \frac{M_A}{EI}\frac{x^2}{2} + Ax + B$$

$v(0) = 0$ ✓ → $A = 0$ ✓

$v'(0) = 0$ ✓ → $B = 0$ ✓

$v(L) = 0$ ③

$$v(L) = \frac{1}{EI}\left(\frac{A_y L^3}{6} + \frac{M_A L^2}{2}\right) = 0$$

$$\frac{M_A}{2} + \frac{A_y L}{6} = 0$$

Eq (1) $$M_A = M_o + B_y \cdot L$$

Eq (2) $$A_y = -B_y$$

3 eq
3 UNK

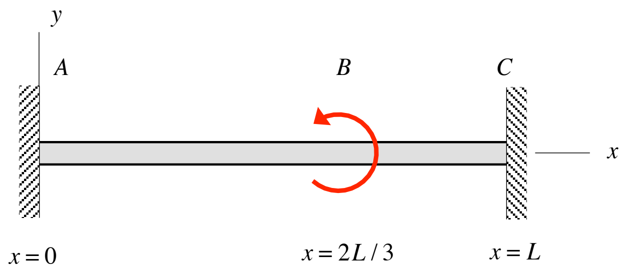
$$A_y = \frac{3}{2}\frac{M_o}{L}$$

$$B_y = -\frac{3}{2}\frac{M_o}{L}$$

$$M_A = -\frac{M_o}{2}$$

Example 11.11

Determine the deflection curve for the beam shown below. The beam is made up of a material with an elastic modulus E and has a cross-sectional second area moment I, both of which are constant along the length of the beam.



Summary: beam deflection – second-order integration method

The procedure to determine the deflection of a bending beam using the second-order integration method:

i) Before starting, write down the boundary conditions (BCs) for the problem.

ii) Draw a free body diagram (FBD) of the entire structure, and from this FBD write down the equilibrium equations; these equilibrium equations will be written in terms of the external reactions. If *DETERMINATE*, solve these equations for the external reactions.

iii) Divide beam into sections: $x_i < x < x_{i+1}$, where this section division is dictated by: support reactions, beam geometry changes, and/or load changes (concentrated forces/moments, line load definition changes, etc.).

iv) For each section, draw free an FBD of either the left of right side of the body from a cut through that section of the beam. From this FBD, determine the distribution of bending moment $M(x)$ through that section of the beam.

v) Use the following integrals to determine slope and deflection of the beam over $x_i < x < x_{i+1}$:

$$\theta(x) = \theta(x_i) + \frac{1}{EI}\int_{x_i}^{x} M(x)\,dx$$

$$v(x) = v(x_i) + \int_{x_i}^{x} \theta(x)\,dx$$

$\theta(x) = v'(x)$

vi) The final values of slope and displacement for one section $\theta(x_{i+1})$ and $v(x_{i+1})$ become the initial values of slope and displacement for the next section (continuity conditions).

vii) Enforce any remaining boundary conditions to determine any remaining integrations constants. For the case of *INDETERMINATE* beams, additional equations needed for determining external reactions are also produced through the enforcement of boundary conditions. These equations are solved with the equilibrium equations in ii) above.

Calculation of beam deflections using superposition

In many situations, the loadings on beams can be recognized as the combination of simple loadings. You might discover that the beam deflection due to each of the simple loadings individually have already been computed by others before you, with some of these results appearing in the Appendix of the textbook (with these results repeated here in these notes). Since the Euler-Bernoulli theory used for the deflection of beams here is *linear*, the deflection of the beam due the combination of simpler loads is a "superposition" of the deflections due of the beam as a result of each simpler load acting individually.

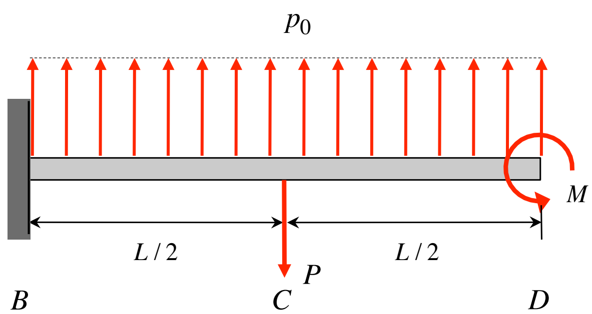
For determinate beams, we need only to find the deflection due to each loading individually, and then add together these deflections.

For indeterminate beams, we will first need to recognize which loadings (both applied and reactive) combine together to give the indeterminate problem. We next find the deflections due to the simple loadings individually, and add together through superposition. Prescribed displacements/rotations are then imposed on the combined deflection relations to determine the unknown reactions.

Please note that linear superposition is a generic and widely applicable method in linear elasticity. This method originates from the assumed linear relationship between applied loads and deformations. Although we will use linear superposition here in combining known deflection relationships of simple problems in determining the deflection of more complicated loadings, the concept of "superposition" extends well beyond this limited application.

Example 11.17

Determine the deflection of the beam loaded as below using superposition.



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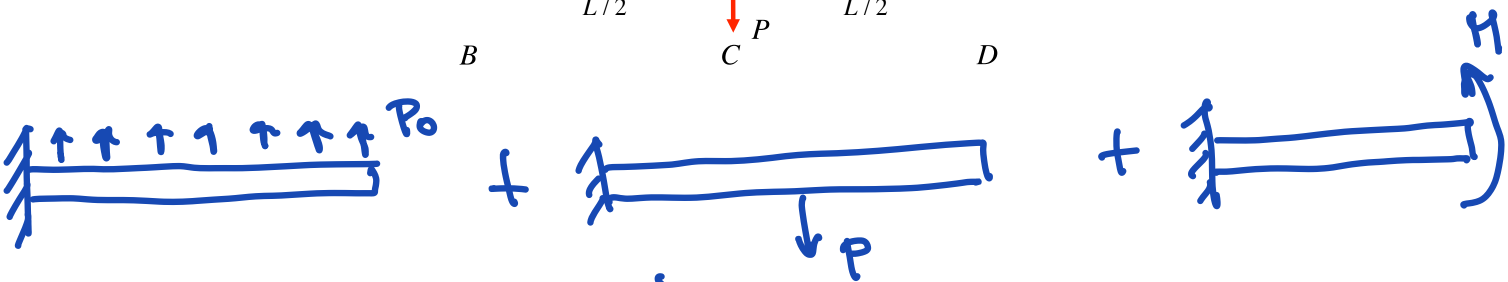

$$v_a(x) = \frac{p_0}{24EI} x^2 (6L^2 - 4Lx + x^2)$$

$$v_p(x) = \begin{cases} -\dfrac{P}{6EI} x^2 \left(\dfrac{3L}{2} - x\right) & x < \dfrac{L}{2} \\ -\dfrac{P}{6EI} x^2 \left(3x - \dfrac{L}{2}\right) & x \geq \dfrac{L}{2} \end{cases}$$

$$v_M(x) = \frac{M x^2}{2EI}$$

Example 11.19

Determine the load carried by rod BD using superposition. The beam has a flexural rigidity of *EI*. Rod DB is made of a material with a Young's modulus of *E*, cross-sectional area of *A* and length *L*.

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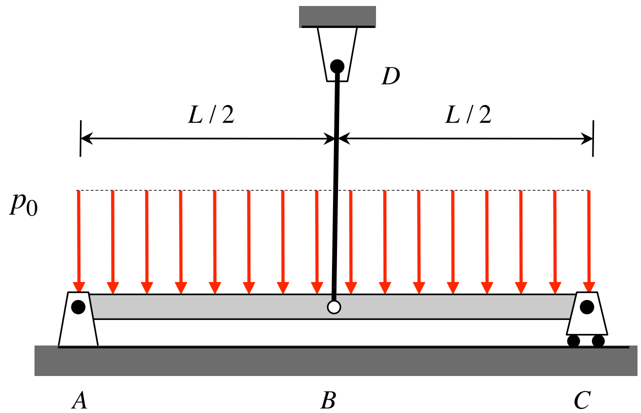
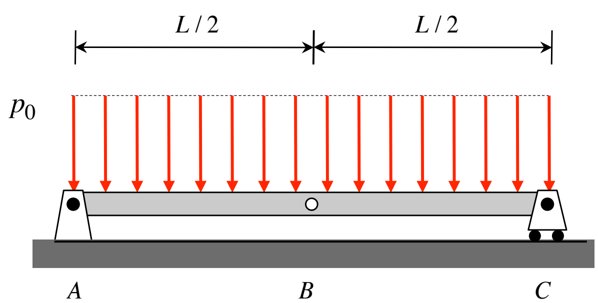


SOLUTION

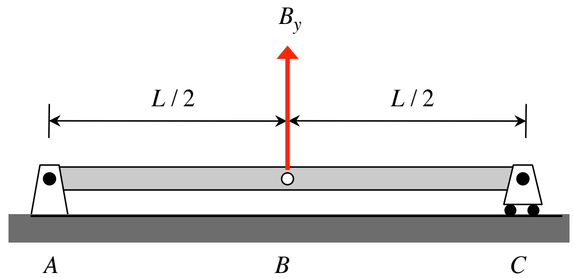
Will break the problem into two loadings, as shown in the following:

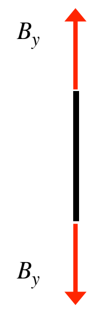
<u>*Loading 1*</u>



$$v_1(x) = -\frac{p_0 x}{24EI}\left(L^3 - 2Lx^2 + x^3\right)$$

Loading 2





$$v_2(x) = \frac{B_y x}{48EI}\left(3L^2 - x^2\right) \quad ; \quad 0 \le x \le L/2$$

where: $e = \dfrac{B_y L}{EA} \Rightarrow \quad B_y = \dfrac{eEA}{L}$

Therefore for $0 \le x \le L/2$ we have:

$$v(x) = v_1(x) + v_2(x)$$

$$= -\frac{p_0 x}{24EI}\left(L^3 - 2Lx^2 + x^3\right) + \frac{eAx}{48LI}\left(3L^2 - x^2\right)$$

Enforcing the boundary condition at B:

$$v\left(\frac{L}{2}\right) = -e = -\frac{p_0(L/2)}{24EI}\left[L^3 - 2L\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^3\right] + \frac{eA(L/2)}{48LI}\left[3L^2 - \left(\frac{L}{2}\right)^2\right]$$

$$= -\frac{5p_0L^4}{384EI} + \frac{11eAL^2}{384I} \quad \Rightarrow$$

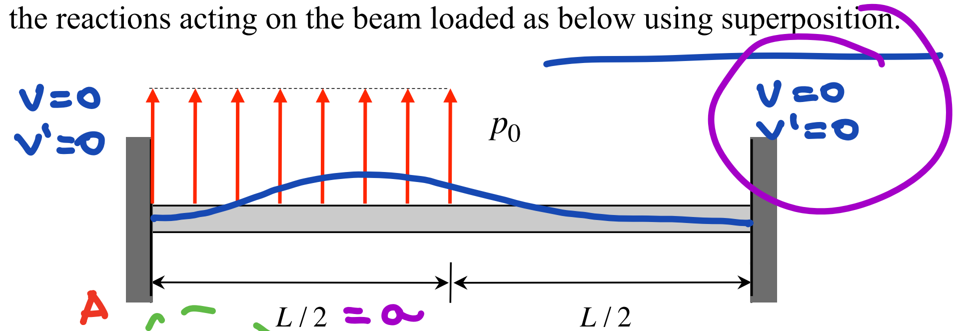
$$\left(1 + \frac{11AL^2}{384I}\right)e = \frac{5p_0L^4}{384EI} \quad \Rightarrow \quad e = \frac{5p_0L^4/384EI}{1 + 11AL^2/384I} = \frac{5p_0L^4/E}{384I + 11AL^2}$$

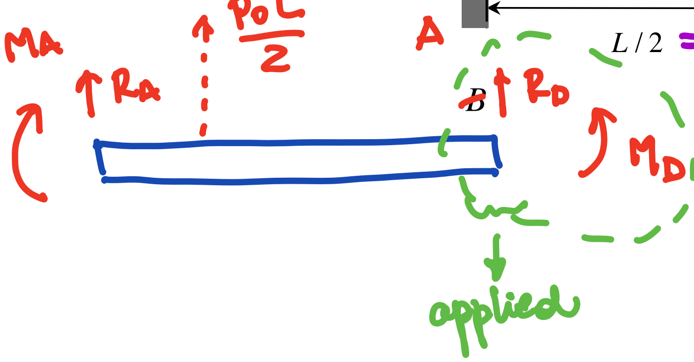
Therefore,

$$B_y = \frac{eEA}{L} = \frac{5p_0AL^3}{384I + 11AL^2}$$

Example 11.20

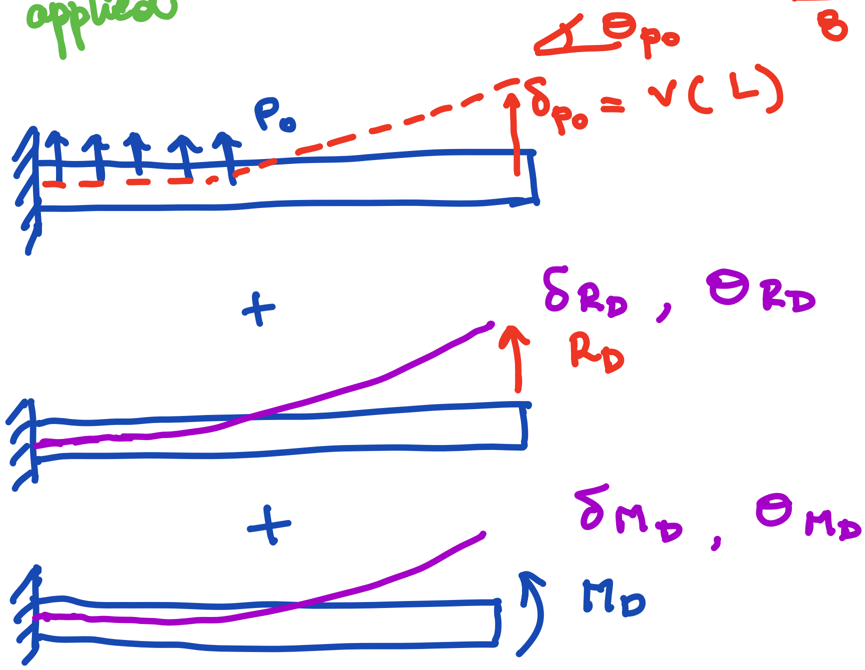
Determine the reactions acting on the beam loaded as below using superposition.





① $R_A = -R_D - \frac{p_0 L}{2}$

② $M_D = M_A - \frac{p_0 L^2}{8} - R_D L$



$\delta_{R_D} + \delta_{M_D} + \delta_{p_0} = 0 \rightarrow v(L) = 0$

$\theta_{R_D} + \theta_{M_D} + \theta_{p_0} = 0 \rightarrow \theta(L) = 0$

$$\delta_{po} = \frac{p_o a^3}{24EI}(4L - a)$$

$$= \frac{p_o L^3}{192EI}\left(4L - \frac{L}{2}\right)$$

$$\delta_{R_D} = \frac{R_D L^3}{3EI}$$

$$\delta_{M_D} = \frac{M_D L^2}{2EI}$$

$$\theta_{po} = \frac{p_o L^3}{48EI}$$

$$\theta_{R_D} = \frac{R_D L^2}{2EI}$$

$$\theta_{M_D} = \frac{M_o L}{EI}$$

$$\delta_{po} + \delta_{R_D} + \delta_{M_D} = 0 \quad (3)$$

$$\theta_{po} + \theta_{R_D} + \theta_{M_D} = 0 \quad (4)$$

Solve M_A, R_A, M_D R_D

$$R_D = -\frac{3}{32} p_o L$$

$$R_A = -\frac{13}{32} p_o L$$

$$M_D = \frac{5}{192} p_o L^2$$

$$M_A = \frac{11}{192} p_o L^2$$

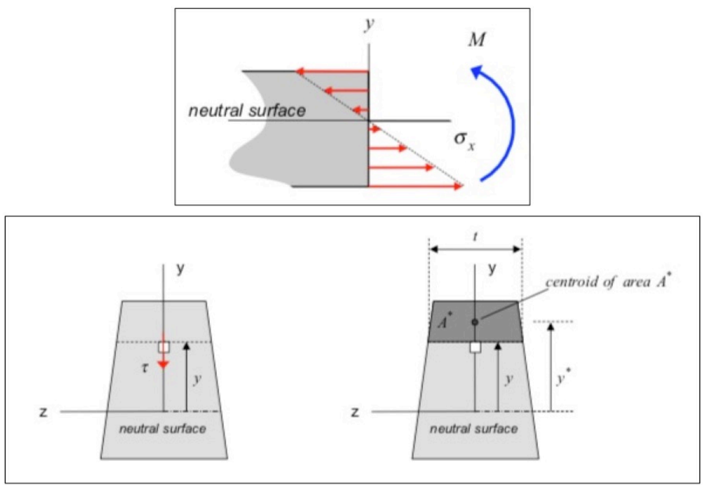
Stress analysis for indeterminate beam structures

As we have seen in the last chapter, the determination of the normal and shear components of stress at a point on the cross-section of a beam relies on our ability to calculate the shear force *V* and bending moment *M* at that point. With knowledge of these two internal resultants, we find:

$$\sigma = -\frac{My}{I}$$

$$\tau = \frac{VA^*\bar{y}^*}{It}$$

where *y* is the *y*-component of the location on the beam cross-section at which the stress is desired, *I* is the centroidal second area moment of the cross-section, A^* is the area of the cross-section above *y*, $\bar{y}^*$ is the centroid of the area above *y* and *t* is the thickness of the beam in the *z*-direction.



The process for determining these components of stress in *determinate* beams is the following:

i) Determine the external reactions acting on the beam.

ii) Determine the internal resultants along the length of the beam:

   - If the stress is to be determined at a single point, the shear force and bending moment resultants can found directly from equilibrium analysis on a section of the beam created by a mathematical cut through the beam at the location of interest.

   - If the maximum components of stress along the length of the beam are desired, then the shear force/bending moment diagrams are first drawn using methods outlined in an earlier chapter.

iii) Using these internal resultants, the normal and shear stress components are calculated using the stress equations shown above.

For stress analysis of *indeterminate* beams, we encounter an obstacle on the first step above – for an indeterminate beam we are NOT able to calculate the external reactions from equilibrium considerations alone. How do we proceed?

What we have seen thus far in this chapter is that the inclusion of deformation analysis along with compatibility conditions, we are able to determine the external reactions. Once these reactions are known, we can follow through with the remaining steps for finding the normal and shear components of stress.

For the deformation analysis in this chapter, we have explored a number of different approaches:

i) Deflections found through *integration* of the deflection curve differential equations. We used both the second-order and fourth-order integration methods. Both methods are fundamentally identical, although differing somewhat in procedure.

   - *Fourth-order integration methods*: With these methods, we start out by considering the known distributed loadings, $p(x)$, on the beam. These loadings become part of the sequence of four integrations leading up to the beam deflection over a given section of the beam, $x_i < x < x_{i+1}$:

$$V(x) = V(x_i) + \int_{x_i}^{x} p(x)\,dx$$

$$M(x) = M(x_i) + \int_{x_i}^{x} V(x)\,dx$$

$$\theta(x) = \theta(x_i) + \frac{1}{EI}\int_{x_i}^{x} M(x)\,dx$$

$$v(x) = v(x_i) + \int_{x_i}^{x} \theta(x)\,dx$$

   - *Second-order integration methods*: With these methods, we use standard FBD/equilibrium methods to determine the bending moment distribution over the section of the beam, $x_i < x < x_{i+1}$, and then using the last two equations above to develop the deflection equation:

$$\theta(x) = \theta(x_i) + \frac{1}{EI}\int_{x_i}^{x} M(x)\,dx$$

$$v(x) = v(x_i) + \int_{x_i}^{x} \theta(x)\,dx$$

For indeterminate problems, the forms of the $\theta(x)$ and $v(x)$ equations above will still be dependent at this point on a number of to-be-determined reaction loads on the beam. These loads are found by imposing the geometric boundary (compatibility) conditions on these expressions for $\theta(x)$ and $v(x)$.

ii) Direct usage of the *superposition principle* with tabulated deflection curves of relevant sub-problems. For determinate problems, the beam deflection equations come directly from this superposition. For indeterminate problems, a number of the reaction loads are initially represented as applied loads, with the deflection functions then being expressed in terms of these unknown reactions. The enforcement of displacement and rotation boundary conditions produce the equations needed to solve for the external reaction loads.

Once the external reactions are known, the shear force and bending moments can be found from either:

- the equilibrium equations of:

$$M(x) = EI\frac{d^2v}{dx^2}$$

$$V(x) = \frac{dM}{dx}$$

- or, through the construction of the shear force/bending moment diagrams, as detailed in an earlier chapter.

Using the results for *M* and *V*, the normal and shear stress components are found using the stress equations above:

$$\sigma = -\frac{My}{I}$$

$$\tau = \frac{VA^*\bar{y}^*}{It}$$

Summary of stresses in indeterminate beam structures

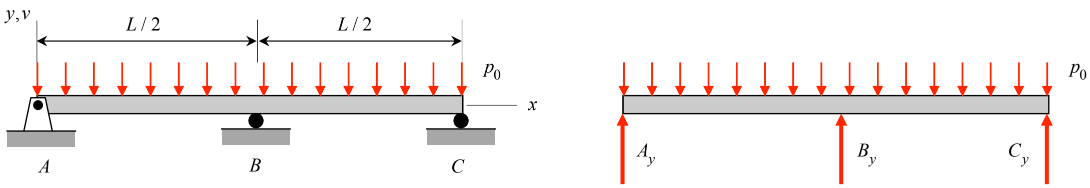
Indeterminate structures are characterized by having excessive or redundant reaction supports. That is, more supports are provided to the structure than are needed for equilibrium. The consequences of this indeterminacy are:

a) Deformation analysis is need along with equilibrium analysis in order to determine stresses.

b) An important, but somewhat less obvious, result is that material properties play a role in the stresses found in an indeterminate beam. Note that for a determinate beam, external reactions and internal resultants can be found directly from equilibrium. Material properties of the beam do not come into play, only the loadings, supports and geometry influence these resultant. For indeterminate beams, deformation information is needed to solve for reactions and resultants. Here the material properties can influence both the deformations and the external reactions/internal resultants leading up to the stresses.

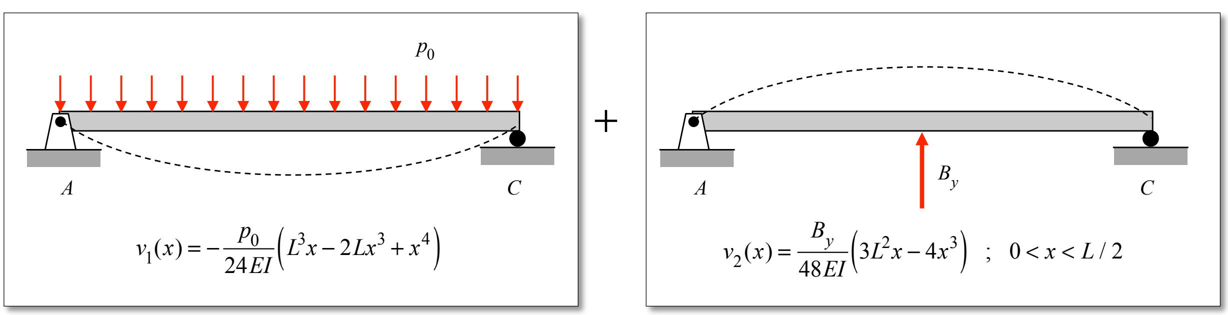
Example 11.21

Consider the beam shown below supported at locations A, B and C with a constant force per length of p_0. As seen in the FBD to the right, the system is INDETERMINATE since we have three unknown reaction forces (A_y , B_y and C_y) for two equations.



Based on our integration analysis methods for indeterminate systems from before, we can solve for the reactions at A, B and C. An alternative view is to see this problem as being the superposition of two simply-supported beam problems: one with a constant force per length and one with a concentrated (unknown) reaction force B_y , as below. Using the formulas from Appendix E of the textbook, we can write down the deflection as the sum of the deflections of the two problems, $v_1(x)$ and $v_2(x)$:

$$v(x) = v_1(x) + v_2(x) = -\frac{p_0}{24EI}\left(L^3x - 2Lx^3 + x^4\right) + \frac{B_y}{48EI}\left(3L^2x - 4x^3\right) \; : \; 0 < x < L/2 \qquad (1)$$



The internal bending moment in the beam is found from, for $0 < x < L/2$:

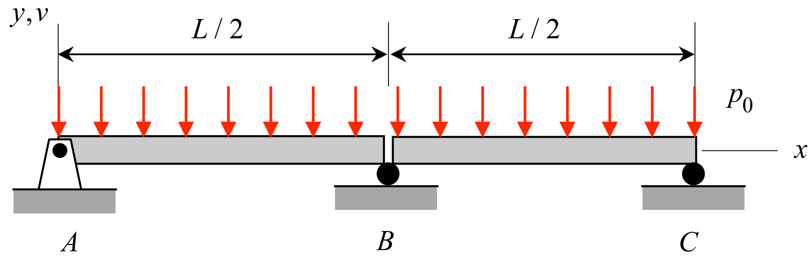
$$M(x) = EI\frac{d^2v}{dx^2} = -\frac{p_0}{24}\left(-12Lx + 12x^2\right) + \frac{B_y}{48}\left(-24x\right) = \frac{p_0}{2}\left(Lx - x^2\right) - \frac{B_y}{2}x \qquad (2)$$

Questions:

a) How do we determine the reaction B_y ?

b) What is the maximum internal bending moment? How do you determine the maximum flexural stress in the beam?

c) How does the maximum internal bending moment for this problem compare with the maximum internal bending moment for the determinate beam system shown below?



y, v
L / 2
L / 2
p0
x
A
B
C

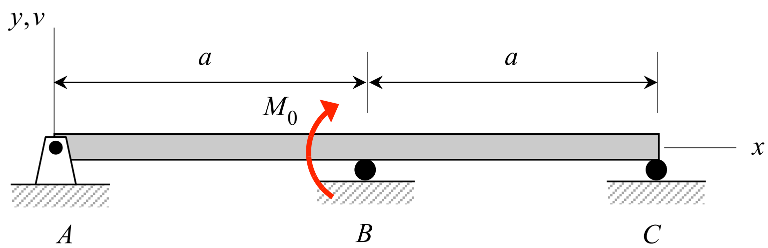
Example 11.22

Consider the beam below with a pin support at A and roller supports at B and C. The beam has a circular cross-section with a diameter of *d*. The material making up the beam has a Young's modulus of *E*.

a) Write down the equilibrium equations for the beams. Is this an indeterminate beam?

b) Use a second-order integration method to determine the reaction loads at A, B and C.

c) Using your results from Part b), determine the maximum magnitude normal and shear components of stress on the cross-section of the beam at $x = a/2$.



Example 11.23

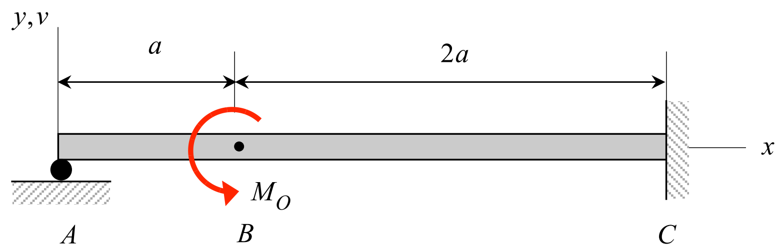
Consider the beam below with propped cantilevered supports and a couple applied at point B. The beam has a square cross-section with dimensions of $h \times h$.

a) Write down the equilibrium equations for the beams. Is this an indeterminate beam?

b) Use the superposition approach to determine the reaction loads at A and C.

c) Construct the shear force/bending moment diagram for the beam.

d) At which location(s) along the beam do the maximum magnitude normal and shear components of stress exist? What are the magnitude values of these maximum stress components?



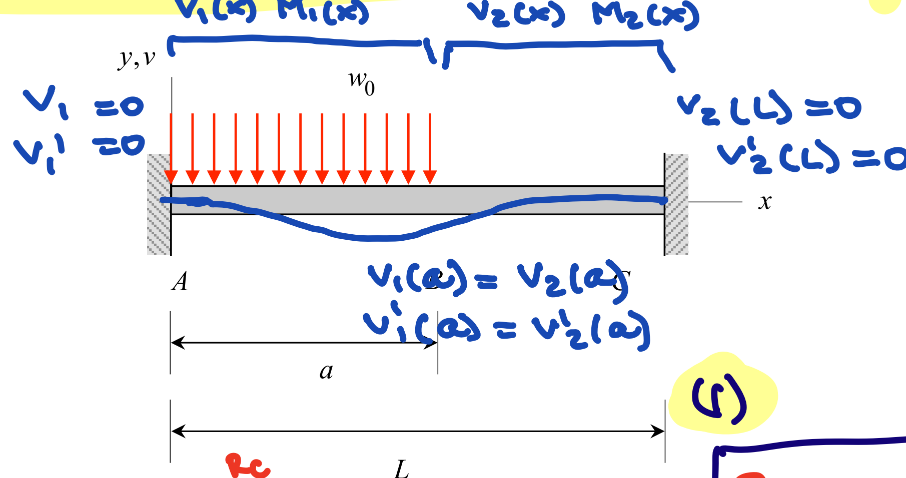
Example 11.24

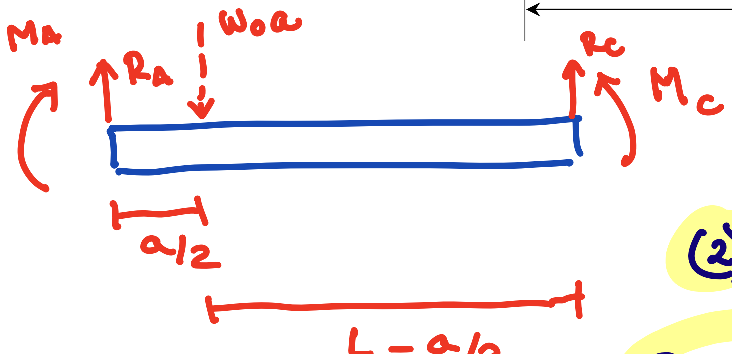
Consider the beam below with fixed-fixed supports and a constant line load along section AB. The beam has a circular cross-section with a diameter of *d*. The material making up the beam has a Young's modulus of *E*. For your analysis here, use $L = 2a$.

d) Write down the equilibrium equations for the beams. Is this an indeterminate beam?

e) Use a second-order integration method to determine the reaction loads at A and C.

f) Using your results from Part b), determine the maximum magnitude normal and shear components of stress on the cross-section of the beam at $x = a/2$.





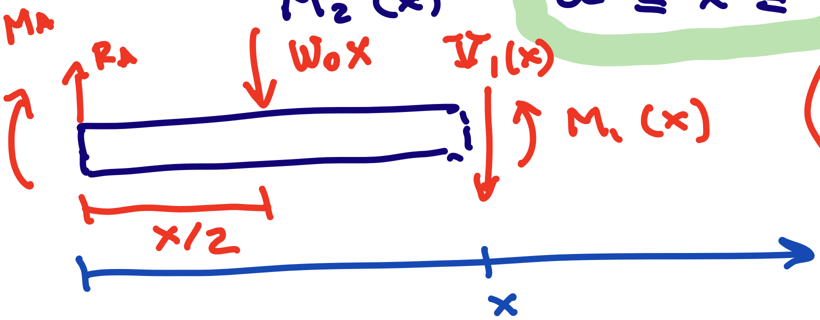
$\Sigma F_y = 0$ (1) $R_C = w_0 a - R_A$

$\Sigma M_C = 0$ (2) $M_A + R_A \cdot L - w_0 a\left(L - \frac{a}{2}\right) - M_C = 0$

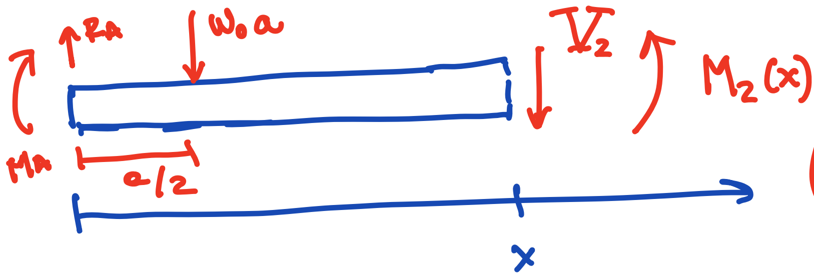
R_C, M_C, M_A, R_A — 4 unk, 2 eq.

Find $M_1(x)$ $0 \le x \le a$

$M_2(x)$ $a \le x \le L$



$$M_1(x) = M_A + R_A \cdot x - w_0 \frac{x^2}{2}$$



$$M_2(x) = M_A + R_A \cdot x - w_0 a\left(x - \frac{a}{2}\right)$$

Integrate to find $v_1(x)$ $v_2(x)$

$$EI\, v_1(x) = M_A \frac{x^2}{2} + R_A \frac{x^3}{6} - \frac{w_0 x^4}{24} + C_1 x + C_2$$

$0 \le x \le a$

$$EI\, v_2(x) = M_A \frac{x^2}{2} + R_A \frac{x^3}{6} - w_0 a\left(\frac{x^3}{6} - \frac{a x^2}{4}\right) + D_1 x + D_2$$

$a \le x \le L$

$v_1(0) = 0$ ③ $v_1'(0) = 0$ ④ $v_2(L) = 0$ ⑤ $v_2'(L) = 0$ ⑥

$v_1'(a) = v_2'(a)$ ⑦ $v_1(a) = v_2(a)$ ⑧

Now we have 8 unknowns

R_C M_C R_A M_A C_1 C_2 D_1 D_2

and equations ① - ⑧

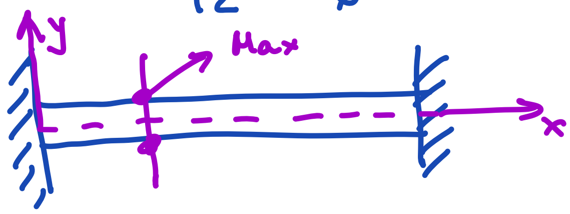
We can find the unknowns

$a = \frac{L}{2}$

$R_A = \frac{13 w_0 L}{32}$ $M_A = -\frac{w_0 L^2}{12} \frac{11}{16}$

$R_C = \frac{3 w_0 L}{32}$ $M_C = -\frac{w_0 L^2}{12} \frac{5}{6}$

c) $\sigma_x(x, y) = -\frac{M(x)\, y}{I}$



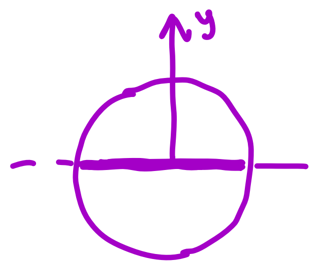
$\sigma^{max}(a/2, y) = \frac{M_1(a/2)\, d/2}{I}$ $I = \frac{\pi r^4}{4}$

$\tau(x\, y) = \frac{V(x)\, A^*(y)\, \bar{y}^*}{I t}$

$\tau^{max}(a/2, y=0) = \frac{16\, V_1(a/2)}{3\pi\, d^3}$

↓

Example 10.11



Example 11.25

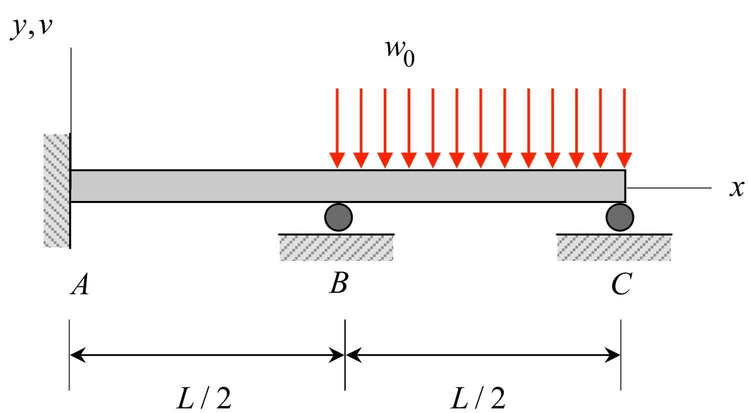
Consider the beam below with a doubly-propped cantilevered supports and a constant line load along section BC. The beam has a square cross-section with dimensions of $h \times h$.

e) Write down the equilibrium equations for the beams. Is this an indeterminate beam?

f) Use the superposition approach to determine the reaction loads at A, B and C.

g) Construct the shear force/bending moment diagram for the beam.

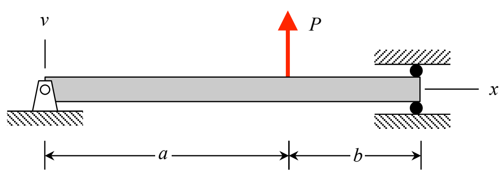
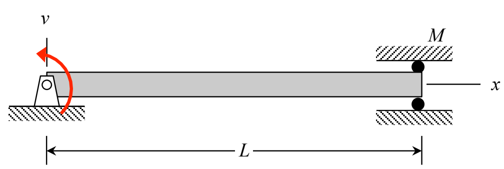
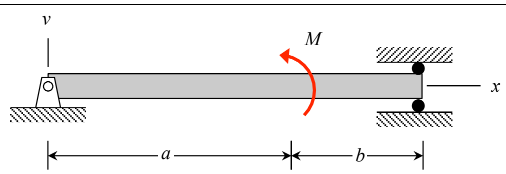
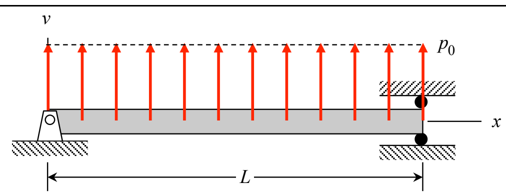
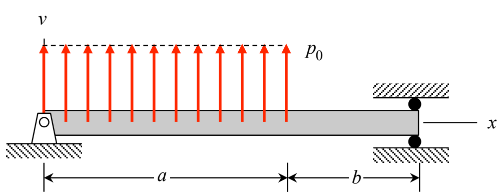
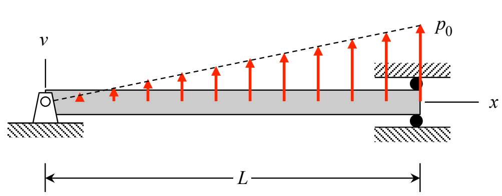
h) At which location(s) along the beam do the maximum magnitude normal and shear components of stress exist? What are the magnitude values of these maximum stress components?



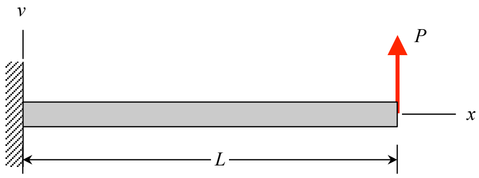
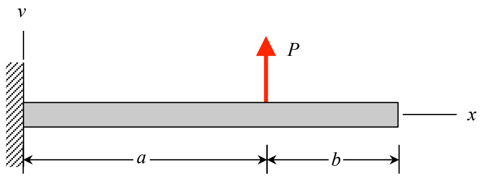
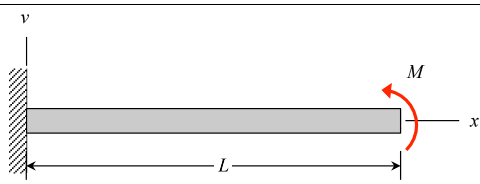
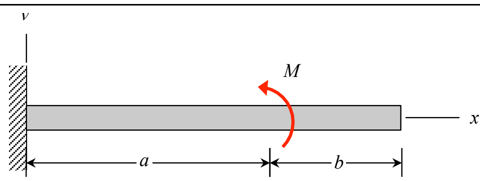
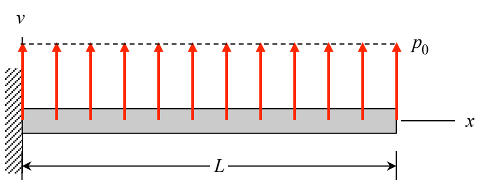
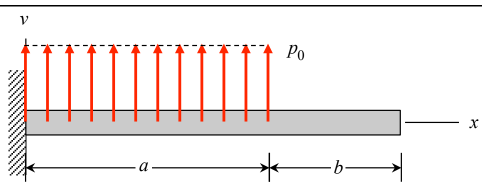
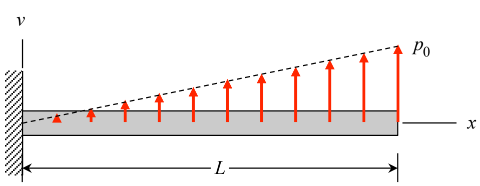
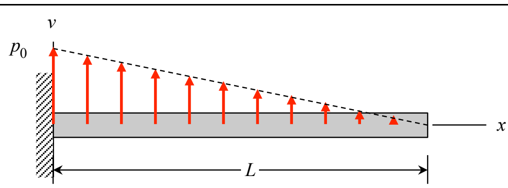
A.2 Beam deflection equations

Formulas are provided below for selected beams and beam loadings, where *EI* is the flexural rigidity for the beam material/cross section and *L* is the beam length.

SIMPLY-SUPPORTED BEAMS

Loading on beam	***Deflection equation***
	$v(x)=\frac{1}{6}\left[bx\left(L^2-b^2-x^2\right)\right]\frac{P}{LEI} \quad ; \quad 0<x<a$
	$v(x)=\frac{1}{6}\left[x\left(2L^2-3Lx+x^2\right)\right]\frac{M}{LEI}$
	$v(x)=\frac{1}{6}\left[x\left(-6aL+3a^2+2L^2-x^2\right)\right]\frac{M}{LEI} \quad ; \quad 0<x<a$
	$v(x)=\frac{1}{24}\left[x\left(L^3-2Lx^2+x^3\right)\right]\frac{p_0}{EI}$
	$v(x)=\frac{1}{24}\left[x\left(a^4-4a^3L+4a^2L^2 + 2a^2x^2-4aLx^2+Lx^3\right)\right]\frac{p_0}{LEI} \quad ; \quad 0<x<a$ $=\frac{1}{24}\left[a^2\left(-a^2L+4L^2x+a^2x -6Lx^2+2x^3\right)\right]\frac{p_0}{LEI} \quad ; \quad a<x<L$
	$v(x)=\frac{1}{360}\left[x\left(7L^4-10L^2x^2+3x^4\right)\right]\frac{p_0}{LEI}$

CANTILEVERED BEAMS

Loading on beam	***Deflection equation***
	$v(x)=\frac{1}{6}\left[x^2(3L-x)\right]\frac{P}{EI}$
	$v(x)=\frac{1}{6}\left[x^2(3a-x)\right]\frac{P}{EI}\quad ;\ 0<x<a$ $=\frac{1}{6}\left[a^2(3x-a)\right]\frac{P}{EI}\quad ;\ a<x<L$
	$v(x)=\frac{1}{2}\left[x^2\right]\frac{M}{EI}$
	$v(x)=\frac{1}{2}\left[x^2\right]\frac{M}{EI}\quad ;\ 0<x<a$ $=\frac{1}{2}\left[a(2x-a)\right]\frac{M}{EI}\quad ;\ a<x<L$
	$v(x)=\frac{1}{24}\left[x^2\left(6L^2-4Lx+x^2\right)\right]\frac{p_0}{EI}$
	$v(x)=\frac{x^2}{24}\left[6a^2-4ax+x^2\right]\frac{p_0}{EI}\quad ;\ 0<x<a$ $=\frac{a^3}{24}\left[4x-a\right]\frac{p_0}{EI}\quad ;\ a<x<L$
	$v(x)=\frac{1}{120}\left[x^3\left(20L^3-10L^2x+x^3\right)\right]\frac{p_0}{LEI}$
	$v(x)=\frac{1}{120}\left[x^2\left(10L^3-10L^2x+5Lx^2-x^3\right)\right]\frac{p_0}{LEI}$