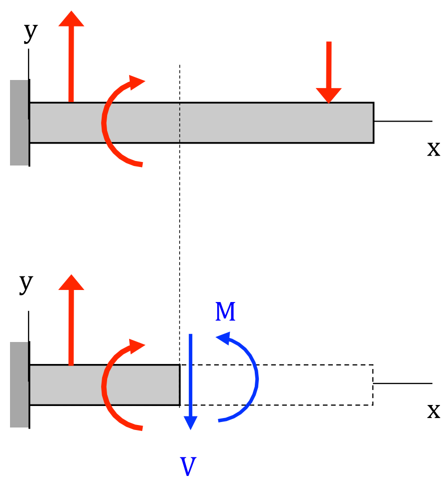
10. Beams: Flexural and shear stresses

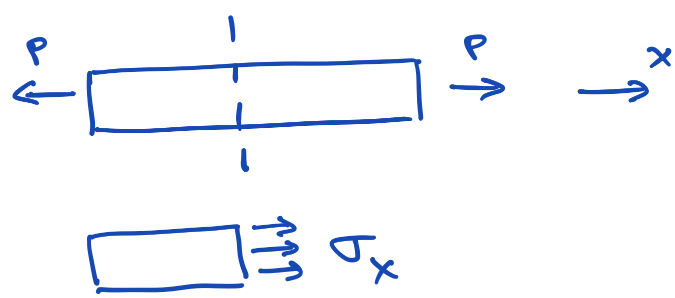
Objectives:
To develop relationships for the normal stresses and shear stresses corresponding to the internal bending moment and shear force resultants in beams.

Background:

- The bending moment M and shear force V at a cut through the cross section of a beam are couple and force *resultants* of the *normal* and *shear* stresses, respectively, at the cross section.



Uniaxial Loading



- Shear force/bending moment equation:

$$V = \frac{dM}{dx}$$

- Axial stress/strain relation:

$$\sigma_x = E\varepsilon_x$$

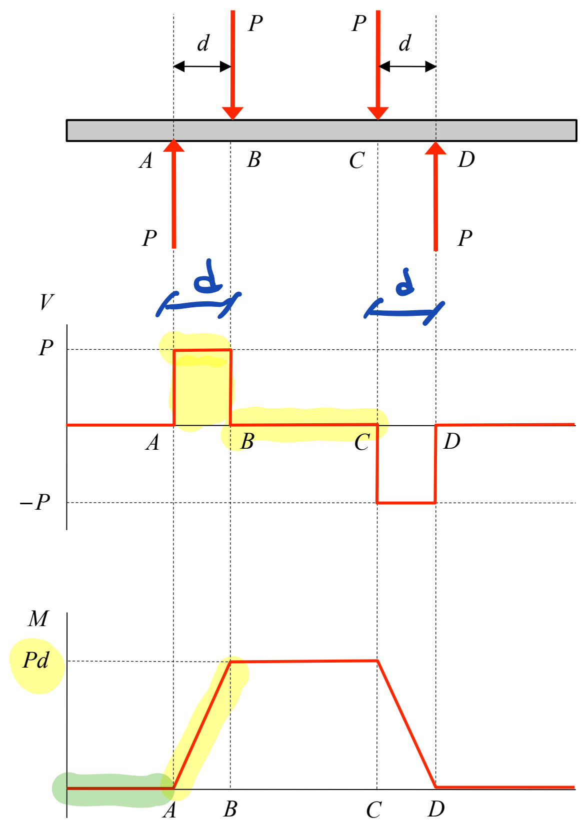
$$\sigma_x = \frac{P}{A}$$

Lecture topics:

a) Strains for pure bending in beams
b) Flexural stresses due to bending in beams
c) Stresses due general transverse force and bending-couple loading of beams

Lecture Notes

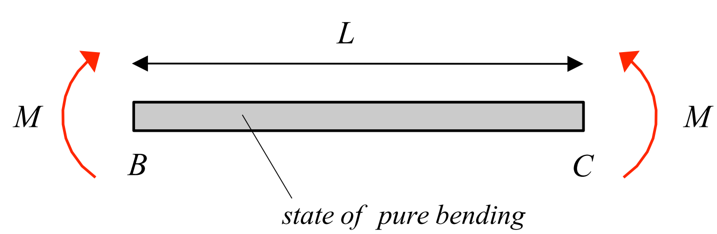
Suppose we consider an example of a beam acted upon by two force/couple pairs resulting from equal magnitude forces P at locations A, B, C and D.



As seen in the above shear-force/bending-moment diagrams:

- The shear force in the beam between B and C is zero.
- The bending moment between B and C is a constant value of $M = Pd$.

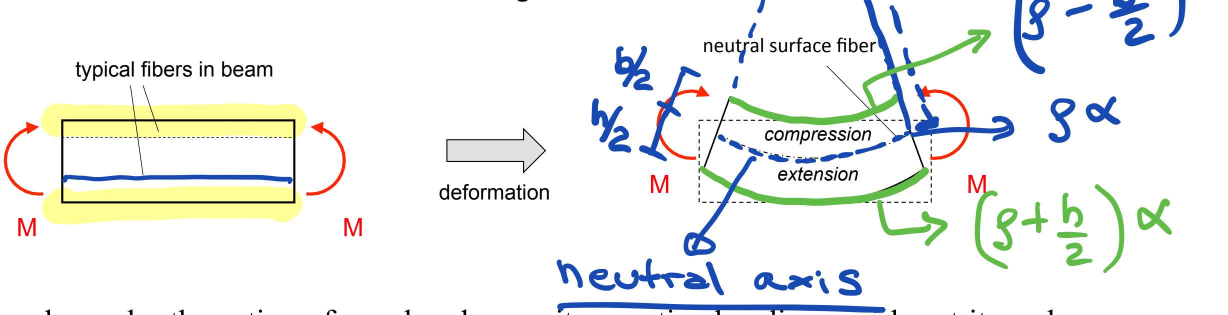
Therefore, a state of "pure bending" (zero shear force) exists between B and C in the beam. So long as we keep our focus on the section BC of the beam, we can represent the above loading as a beam with equal and opposite couples $M = Pd$ applied at its ends, as shown below.



a) Strains for pure bending in beams

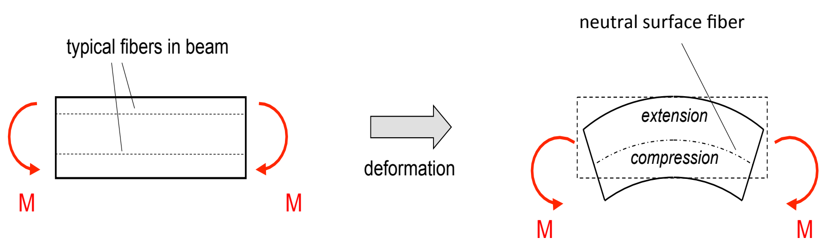
In order to view the beam deformations, it is convenient to imagine the beam to be made up of longitudinal fibers parallel to the longitudinal axis of the beam. Under the action of equal and opposite positive bending couples at its ends, the top fibers of the beam will shorten and the bottom fibers of the beam will stretch, as indicated below. The fiber that divides the region of compression from the region of stretch is said to lie on the "*neutral surface*" of the beam.

Positive **bending moment**



Conversely, under the action of equal and opposite negative bending couples at its ends, the top fibers of the beam stretch and the bottom fibers will shorten.

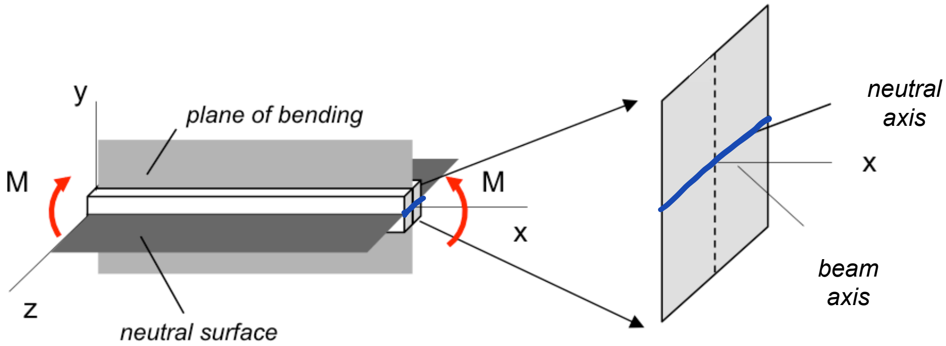
Negative **bending moment**



Euler- Bernoulli definitions and kinematic assumptions for thin beams

Consider the following assumptions related to the geometry and loading of a beam:

- The beam has a plane of longitudinal plane of symmetry (xy-plane as shown in following figure) called the "*plane of bending*". Loading and supports for the beam are assumed to be symmetrical about the plane of bending.
- The beam has a longitudinal plane (xz-plane as shown in following figure) perpendicular to the plane of bending on which there is zero longitudinal strain called the "*neutral surface*". The intersection of the neutral surface with the plane of the cross section is called the "*neutral axis*" for the cross section. In the following discussions, it will be assumed that the z-axis will be aligned with the neutral axis of the beam in its undeformed state. The intersection of the plane of bending and neutral surface is known as the beam axis. The deformation of the initially-straight beam axis is known as the "*deflection curve*" of the beam.

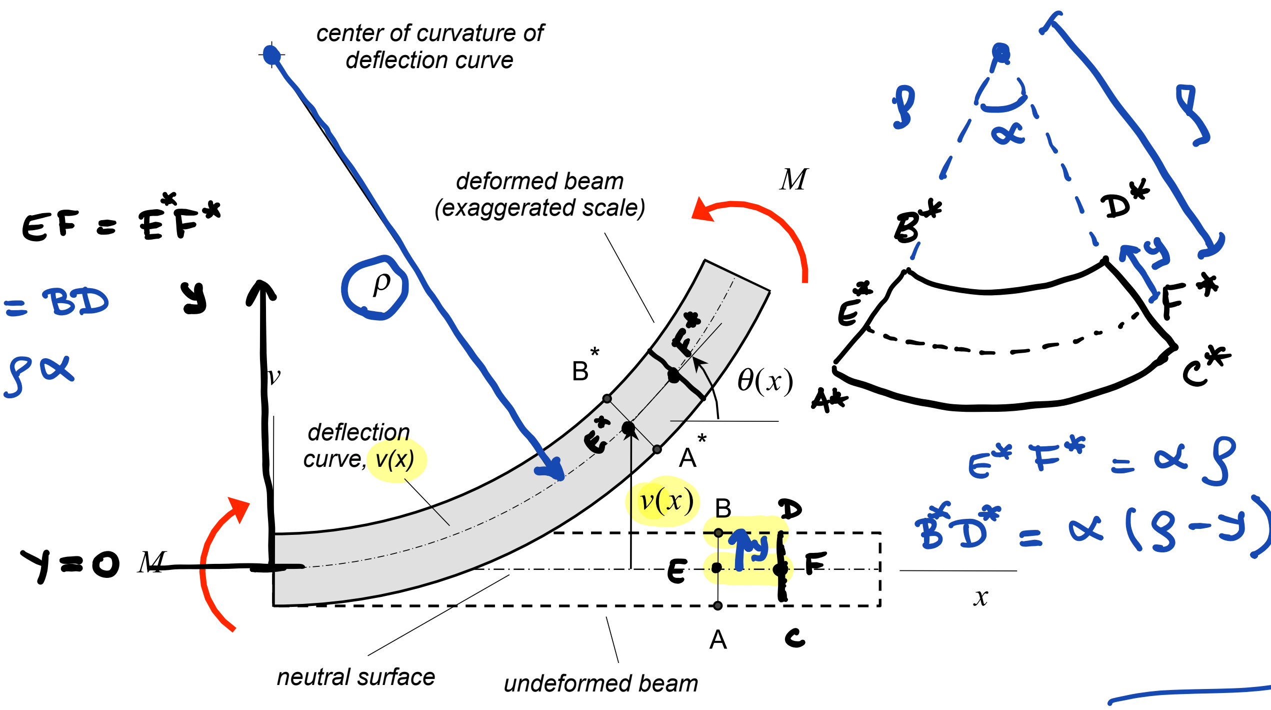


- Planar cross sections that are perpendicular to the beam axis before the beam deforms remain perpendicular to the beam axis after deformation. In the following figure are shown two points A and B on a cut made perpendicular to the neutral axis of the undeformed beam. As a result of the application of the bending moment M, cut *A-B* rotates in the counter-clockwise sense to produce $A^* - B^*$; however, as a result of this assumption, $A^* - B^*$ remains perpendicular to the deflection curve. Also, the radius of curvature of the deflection curve is denoted as ρ in the figure.

$EF = E^*F^*$

$EF = BD$

$BD = \rho\alpha$



$E^*F^* = \alpha\rho$

$B^*D^* = \alpha(\rho - y)$

$$\epsilon_x = \frac{B^*D^* - BD}{BD} = \frac{\alpha(\rho-y) - \alpha\rho}{\alpha\rho} = -\frac{y}{\rho}$$

b) Flexural stresses due to bending in beams

Consequences of the Euler-Bernoulli assumptions:

- As a result of the above Euler-Bernoulli assumptions, it can be shown that the axial strain ε_x across a perpendicular cut in the beam has the following distribution in y :

$$\varepsilon_x = -\frac{y}{\rho} \tag{1}$$

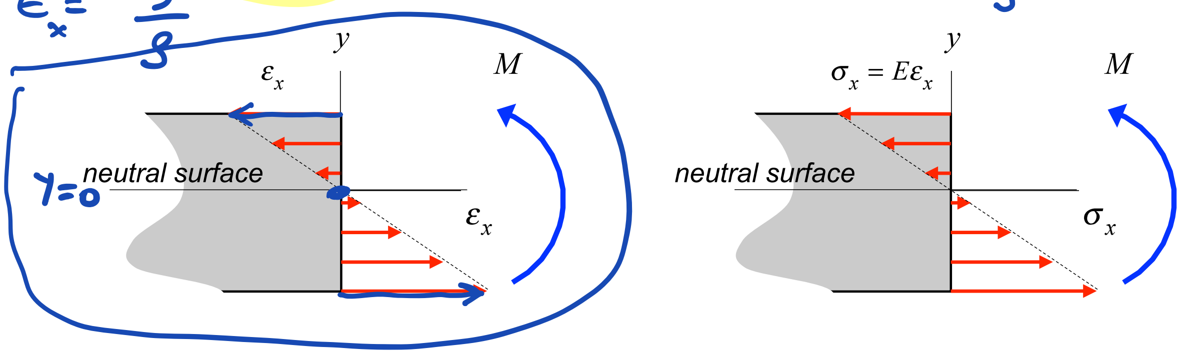
<u>*where y is measured from the neutral surface of the beam*</u> and ρ is the radius of curvature of the deflection curve for the loaded beam.

- For a linearly-elastic material for the beam, the normal stress distribution in y is therefore:

$$\sigma_x = E\varepsilon_x = -\frac{Ey}{\rho} \tag{2}$$

$\sigma_x = -\frac{Ey}{\rho}$

$\varepsilon_x = \frac{-y}{\rho}$



strain distribution across cut stress distribution across cut

- The resultant axial force on the face of the cut is found by:

$\int_A y\, dy\, dx = \bar{y} A$

$$F_N = \int_A \sigma_x \, dA = -\frac{E}{\rho}\int_A y\, dA = -\frac{E\bar{y}A}{\rho} = 0$$

where A is the area of the cross section at the cut and $\bar{y}$ is y-position of the centroid of the cut. Since the beam is known to be in pure bending, the resultant axial force on the face of the cut must be zero. Therefore, using the above, we see that:

$$\bar{y} = 0 \tag{3}$$

or, in words, the neutral axis must past through the centroid of the cross section of the cut.

RESULT: When studying the stress distribution in beams, determine first the location of the centroid of the cross section – the neutral axis passes through this point.

$\hookrightarrow \quad y = 0$

- The resultant moment about the neutral axis must be equal to the couple M. Therefore,

$$M_z = -\int_A \sigma_x y \, dA = \frac{E}{\rho}\int_A y^2 \, dA = \frac{EI}{\rho} \qquad (4)$$

$$\int_A y^2 \, dx\,dy = I$$

where:

$$I = \int_A y^2 \, dA = \textit{second area moment of cross section} \qquad (5)$$

- Combining equations (2) and (4) gives the desired relationship between the applied couple M and the distribution of normal stress across a cross section of the beam:

$$\sigma_x = -\frac{M\,y}{I} \qquad (6)$$

$$\left.\begin{aligned} \sigma_x &= -\frac{Ey}{\rho} \\ M &= \frac{EI}{\rho} \end{aligned}\right\} \quad \sigma_x = -\frac{M}{I}\,y$$

$$\sigma_x(x, y) = -\frac{M(x)\,y}{I}$$

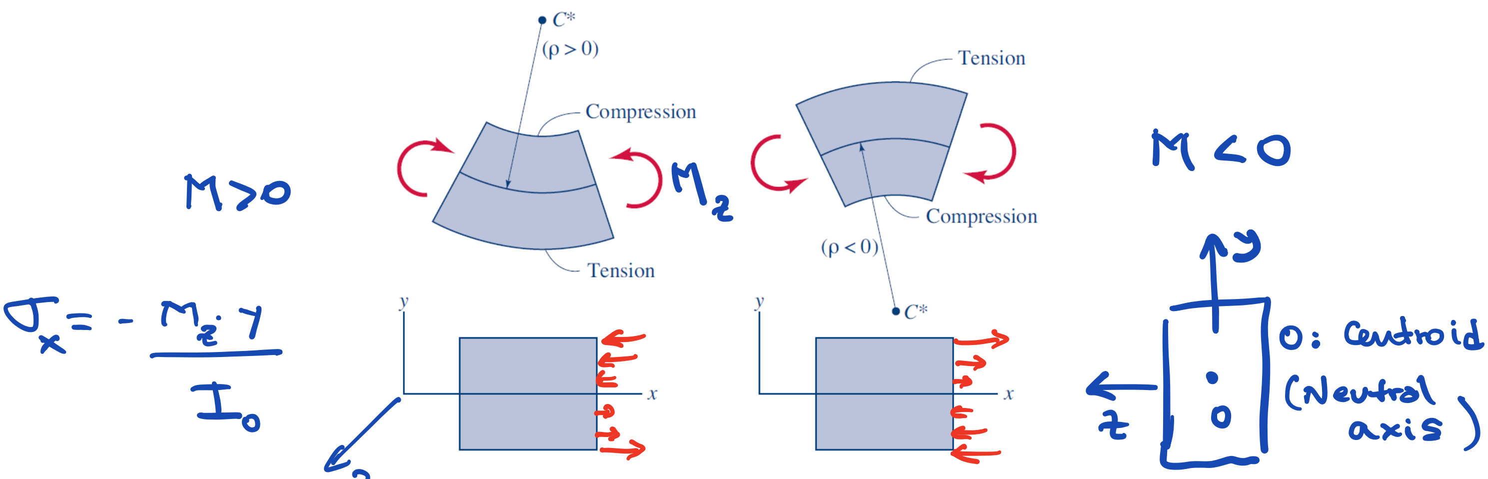
Summary: <u>pure bending</u> at a beam cross section

At a cut through a section of a beam experiencing pure bending (zero shear force, $V = 0$) and abiding by the Euler-Bernoulli assumptions, we can make the following observations (see following figure):

a) Even though loads are applied transverse to the beam, axial strains and stresses are produced. Only normal stresses σ_x exist at the cut.

b) The extensional strain $\varepsilon_x = -y/\rho$ is inversely proportional to the radius of curvature of the beam deflection curve at a cross section, *x*.

c) The signs of ρ and *y* govern the sign of ε_x. If ρ is positive, the center of curvature of the beam deflection lies above the beam, that is, on the +*y* side of the beam and the deformed beam is concave upward. Because of the negative sign in the equation of ε_x, the sections above the neutral surface are in compression, while the sections below the neutral surface are in tension.



(*a*) Positive curvature. (*b*) Negative curvature.

d) The axial strain is not uniform across the section but varies according the height of the point from the neutral axis. Flexural strain reaches maximum at the top and bottom of the beam and is zero at the neutral axis where there is no axial strain.

e) The neutral axis of the cross section (axis of zero strain) passes through the centroid of the cross section.

f) The normal stresses vary linearly in the y-direction: $\sigma_x(y) = -My/I$, where I is the second area moment of the cross section at the cut about the neutral axis. The negative sign in this equation results from sign conventions established earlier. For example, a positive bending moment results in negative (compressive) stress above the neutral axis and positive (tensile) stress below the neutral axis.

g) The normal stresses are constant in the z-direction (into the depth of the beam).

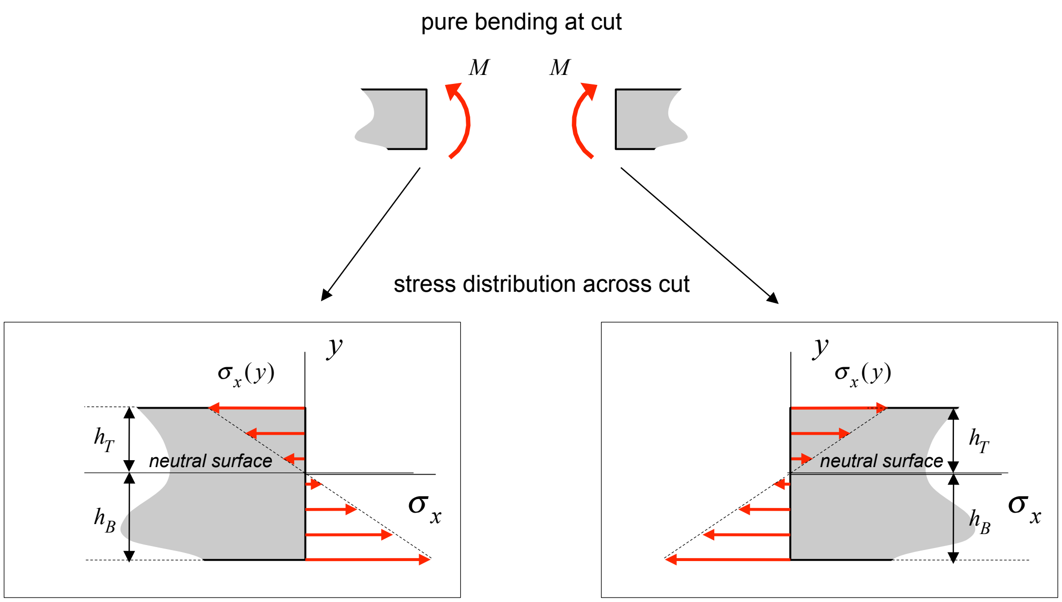
h) The normal stress is zero at the neutral axis.

i) The maximum (magnitude) normal stress exists at the most outer surface of the beam (as measured from the neutral axis). In particular,

$$\left|\sigma_x\right|_{max} = \frac{|M|\,|y|_{max}}{I}$$

where $|y|_{max} = max\left(h_T, h_B\right)$.

j) The bending moment M can be written in terms of the radius of curvature ρ of the beam deflection as: $M = EI/\rho$. Since M is a constant over the section of pure bending, the radius of curvature is also a constant. Hence, we conclude that *a section of pure bending of a beam takes on the shape of a circle* (circle = curve of constant radius of curvature).



Example 10.1

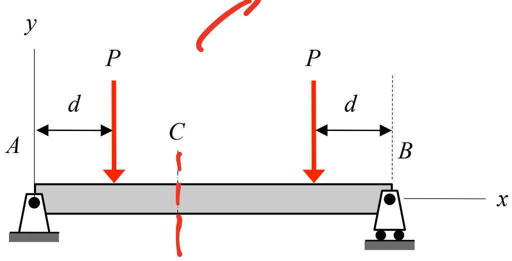
A simply-supported beam is loaded as shown. The cross section at location C of the beam is as shown below right, where C is somewhere between the two applied loads P. Point O on the cross section is on the neutral axis of the beam.

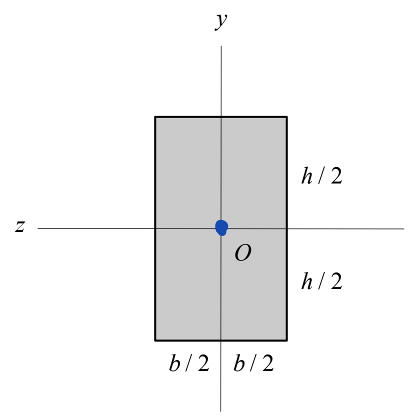
a) Determine the second area of moment of the beam cross section. Leave your answer in terms of b and h.

b) Determine the distribution of normal stress on the cross section of the beam as a function of y.

c) Determine the maximum (magnitude) of the normal stress on the cross section.

σ_x at C





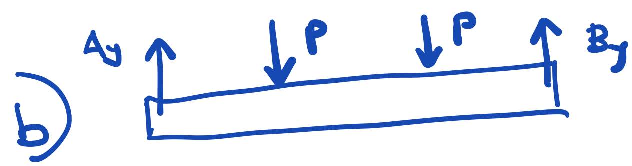
$y = 0$

a) $I = \int y^2 \, dA$

$= \int_{-h/2}^{h/2} y^2 \int_{-b/2}^{b/2} dz \, dy$

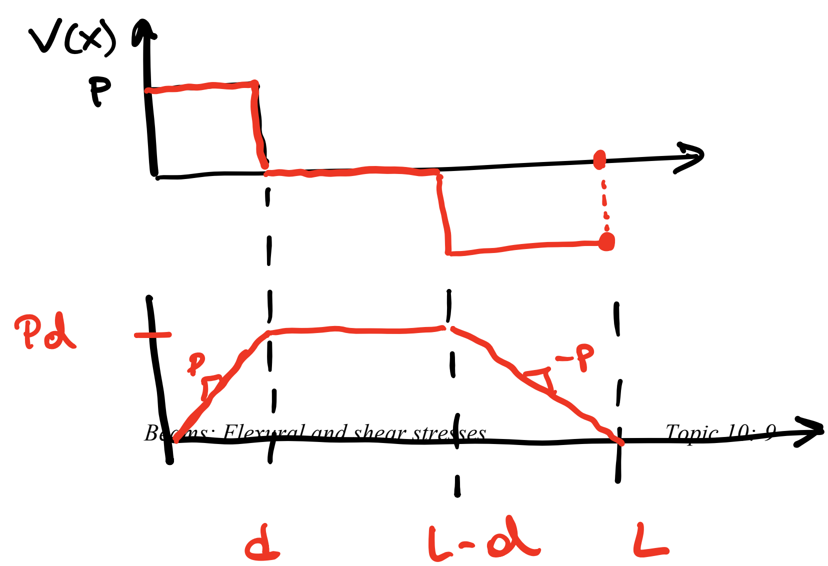
$$\boxed{I = \frac{bh^3}{12}}$$

b)



$A_y = B_y = P$

$$\sigma_x = -\frac{M \cdot y}{I}$$

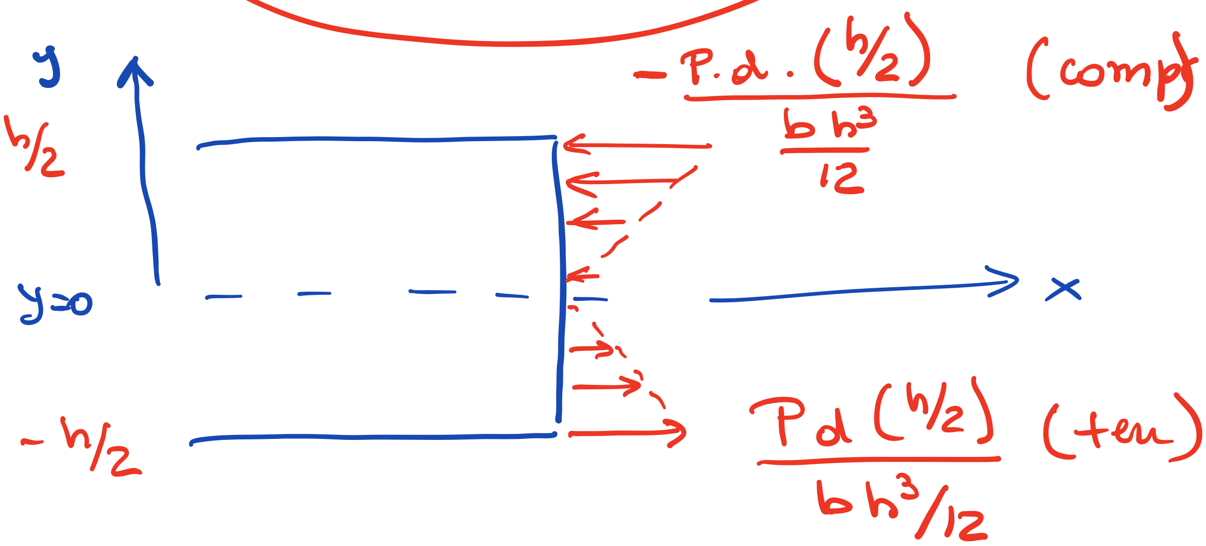


$M(x) = Pd$

$d \leq x \leq L-d$

$V(x) = 0$ pure bending

$$\sigma_x = -\frac{Pd \cdot y}{\frac{b h^3}{12}}$$



$$\sigma^{Max} = Pd \;\; \frac{6}{h^2 b}$$

$$\sigma_x = \frac{-My}{I_o}$$

$$I_p = \int_0^{2\pi}\int_0^{r} \rho^3 \, d\rho \, d\theta = \frac{\pi r^4}{2}$$

$\longrightarrow$ Torsion

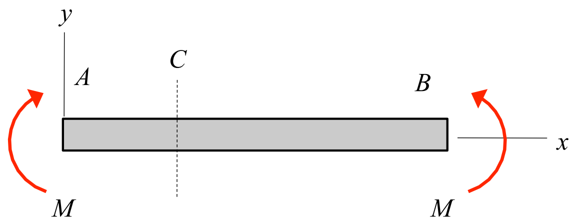
Example 10.2

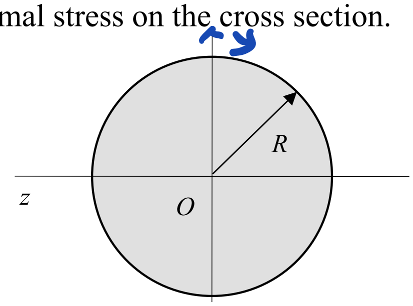
A beam is loaded in pure bending, as shown. The cross section at location C of the beam is as shown below right, where C is somewhere along the length of the beam. Point O on the cross section is on the neutral axis of the beam.

a) Determine the second area of moment of the beam cross section. Leave your answer in terms of R.

b) Determine the distribution of normal stress on the cross section of the beam as a function of y.

c) Determine the maximum (magnitude) of the normal stress on the cross section.

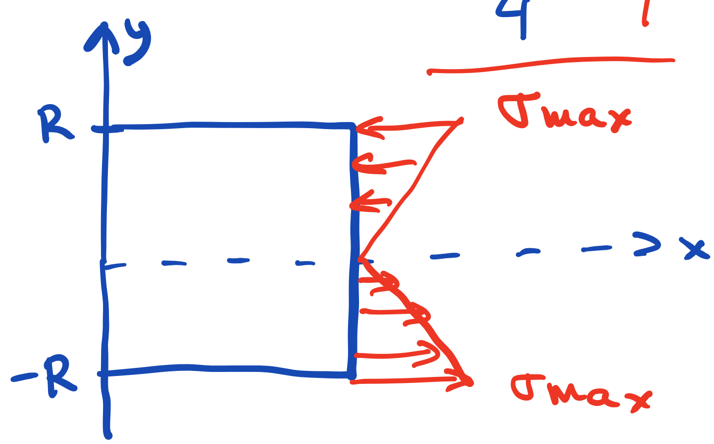




$$I_{oz} = \iint_{Area} y^2 \, dz \, dy \longrightarrow \sigma_x = -\frac{M_z y}{I_{oz}}$$

$$\boxed{I_o = \int_0^{R}\int_0^{2\pi} r^2 \sin^2\theta \; r \, dr \, d\theta = \frac{\pi R^4}{4}}$$

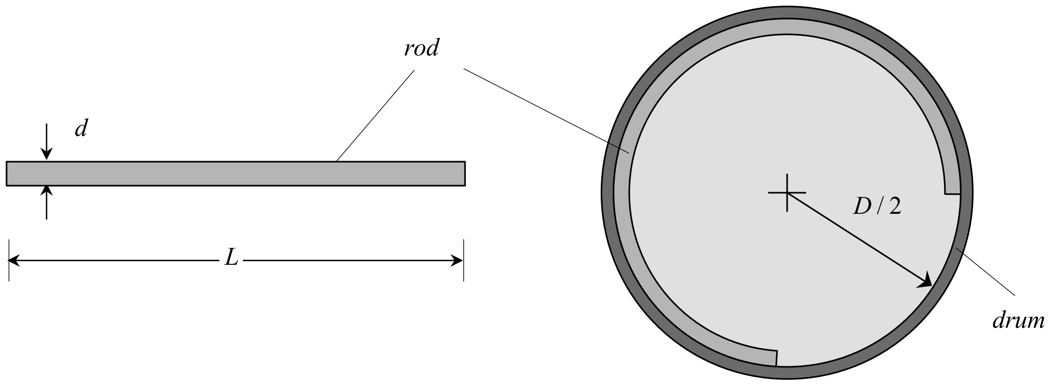
$$\sigma_x = -\frac{M \cdot y}{I_o}$$



$$\sigma_{max} = \frac{M \cdot R}{\pi R^4/4} = \frac{4M}{\pi R^3}$$

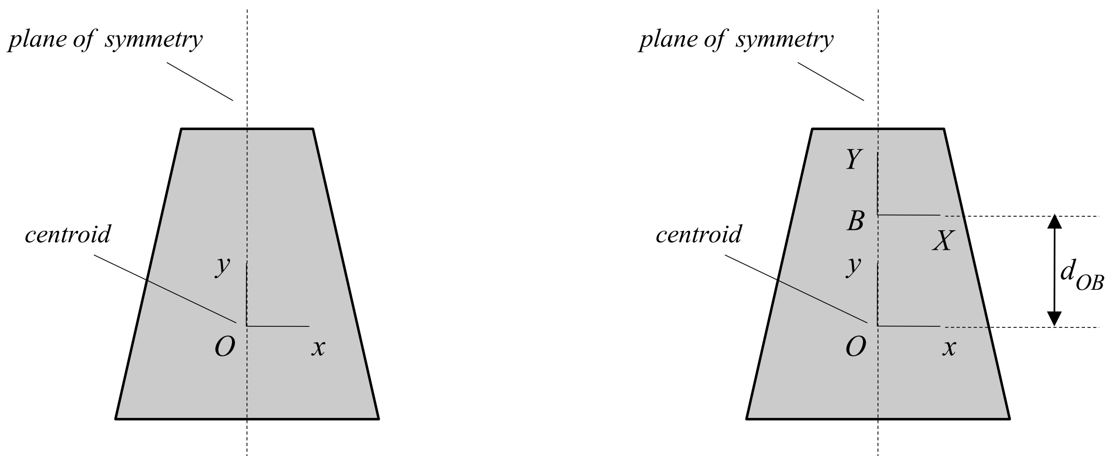
Example 10.4

A circular cross-sectioned, straight rod having a diameter of d, a length of L and of a material with a Young's modulus of E is stored by coiling the rod inside of drum with an inside diameter of D. Assuming that the yield strength of rod material is not exceeded, determine the maximum stress in the coiled rod, and the maximum bending moment in the rod.



Second area moment of a cross section

Consider the beam cross section shown below left that is symmetrical about the y-axis but with no symmetry assumptions about the x-axis, where the origin of the x-y axis, O, is placed at the centroid of the cross section.



In the preceding derivation of the stress distribution across a cross section:

$$\sigma_x = -\frac{M\,y}{I_O} \tag{6}$$

we saw that this relationship depends on the "second area moment" I_O for the cross section:

$$I_O = \int_A y^2\, dA$$

where y is measured from the centroid of the cross section. Note that this parameter depends solely on the shape of the cross section and does not depend on either the material properties of the beam or the strain in the beam.

Tabulated expressions for the centroidal second area moments for a number of common beam cross sections are provided on the following pages.

For reasons that we will discuss later on, we often times need to know the second area moment about points on the plane of symmetry but not at the centroid of the cross section. Consider point B shown in the figure above right that is located at a distance d_{OA} from the centroid O on the plane of symmetry. Suppose we place a set of *X*-*Y* coordinate axes with its origin at A such that $X = x$ and $Y = y - d_{OB}$. Therefore, the second area moment about point B is found from:

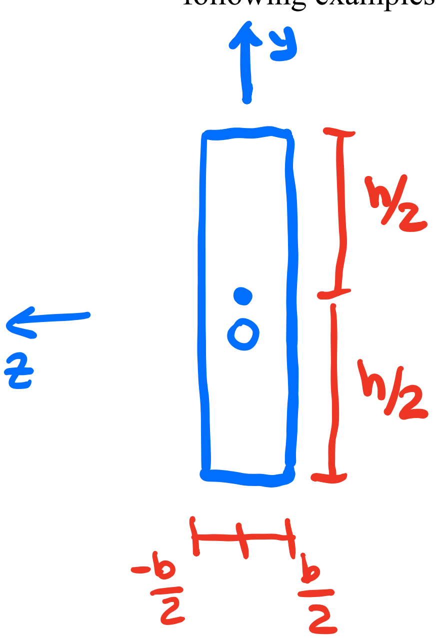
$$
\begin{aligned}
I_B &= \int_A Y^2\,dA = \int_A \left(y - d_{OB}\right)^2 dA \\
&= \int_A \left(y^2 - 2d_{OB}y + d_{OB}^2\right) dA \\
&= \int_A y^2\,dA - 2d_{OB}\int_A y\,dA + d_{OB}^2 \int_A dA \\
&= I_O - 2d_{OB}\bar{y}A + Ad_{OB}^2
\end{aligned}
$$

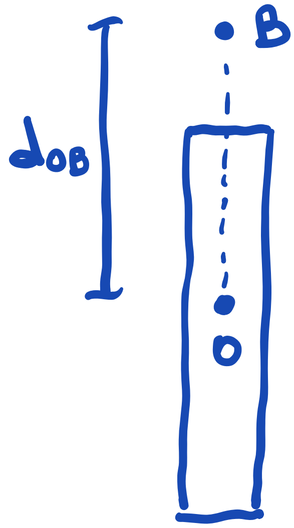
where A is the area of the cross section and $\bar{y}$ is the y-position of the centroid of the area. Since the origin O for the x-y axes is located at the centroid of the cross section, we have $\bar{y} = 0$. Therefore,

$$I_B = I_O + Ad_{OB}^2 \tag{7}$$

Equation (7) is the "*parallel axis theorem*" for second areas of moments. In words, in order to determine the second area moment about an arbitrary point B on the plane of symmetry, simply add Ad_{OB}^2 to the centroidal second area moment I_O, where d_{OB} is the distance between O and B.

In general, one needs to perform an integration over the cross section of the beam in order to evaluate this integral representation for I_O. We have seen this process in the earlier examples. However, for certain cross sections, we can use results from simple shapes to construct the overall second area moment for the cross section. To this end, we will need to use the above parallel axis theorem. This process is demonstrated in the following examples.





B is in y axis

$I_B = I_{Oz} + d_{OB}^2 A$

$A = b \cdot h$

$I_{Oz} = \dfrac{h^3 b}{12}$ $\sigma_x = -\dfrac{M_z y}{I_{Oz}}$

Example 10.5

The cantilevered beam shown below is loaded in pure bending. The beam has a cross section at location C on the beam as shown below right. The origin O is located on the neutral axis of the beam.

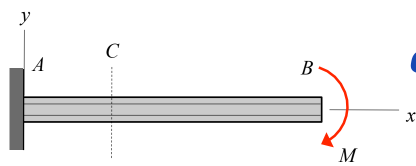
a) Determine the second area moment I_{Oz} corresponding to the neutral axis of the beam.

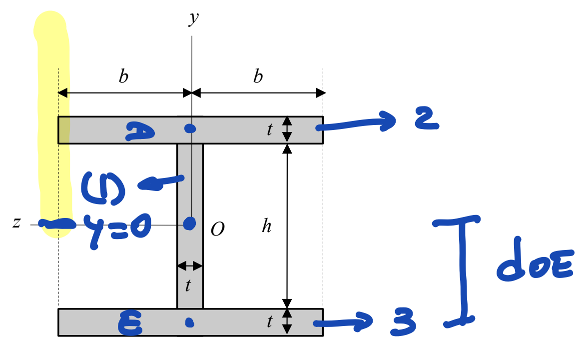
b) Determine the distribution of normal stress on the cross section of the beam as a function of y.

c) Determine the maximum (magnitude) normal stress occurring on the cross-sectional face at C.

$d_{OD} = \frac{h}{2} + \frac{t}{2}$ d

$d_{OE} = \frac{h}{2} + \frac{t}{2}$





cross section at C

$$I_o = I_1 + I_2 + d_{OD}^2 A_2 + I_3 + d_{OE}^2 A_3$$

$$= \frac{h^3 t}{12} + \frac{t^3 2b}{12} + \left(\frac{h}{2}+\frac{t}{2}\right)^2 \cdot t \cdot 2b$$

$$+ \frac{t^3 2b}{12} + \left(\frac{h}{2}+\frac{t}{2}\right)^2 t\, 2b$$

$$\sigma_x = -\frac{M y}{I_o}$$

M = 2000 N.m

t, h, b in mm

Use consistent units

Example 10.6

The beam shown below is loaded in pure bending. The beam has a cross section at location C on the beam as shown below right. The origin O is located on the neutral axis of the beam.

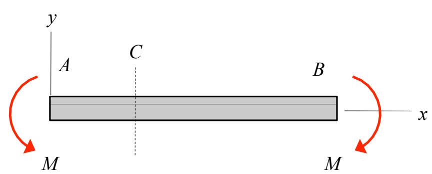
a) Determine the location of the centroid for the cross of this beam; i.e., what is the distance d?

b) Determine the second area moment I_{Oz} corresponding to the neutral axis of the beam.

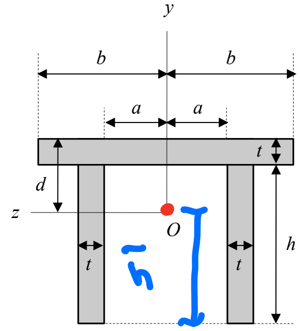
c) Determine the distribution of normal stress on the cross section of the beam as a function of y.

d) Determine the maximum (magnitude) normal stress occurring on the cross-sectional face at C.

$t = 20$ mm $b = 80$ mm $a = 40$ mm $h = 80$ mm

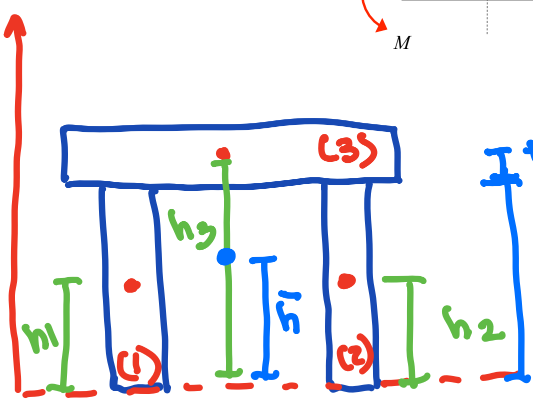


d



$\bar{h}$

cross section at C



$$\bar{h} = \frac{h_1 \cdot A_1 + h_2 A_2 + h_3 A_3}{A_1 + A_2 + A_3}$$

$A_1 = A_2 = 1600$ mm^2

$A_3 = 3200$ mm^2

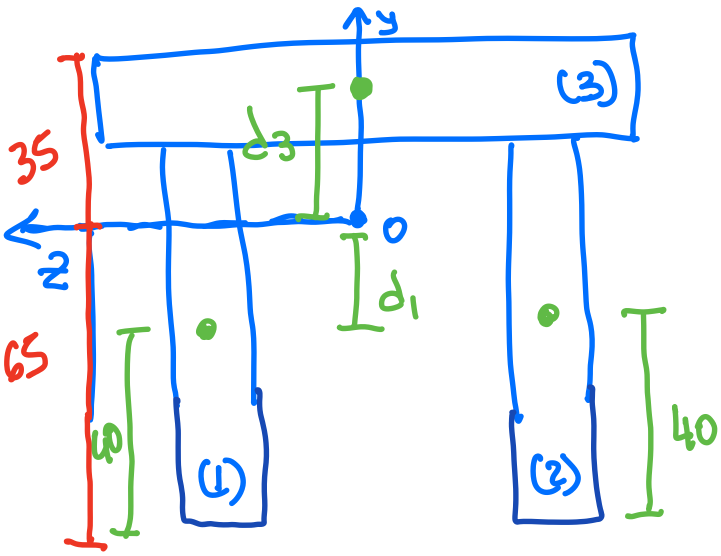
$h_1 = \frac{h}{2} = 40$ mm

$h_2 = \frac{h}{2} = 40$ mm

$h_3 = h + \frac{t}{2} = 90$ mm

$\bar{h} = 65$ mm

$d = 100 - 65 = 35$ mm

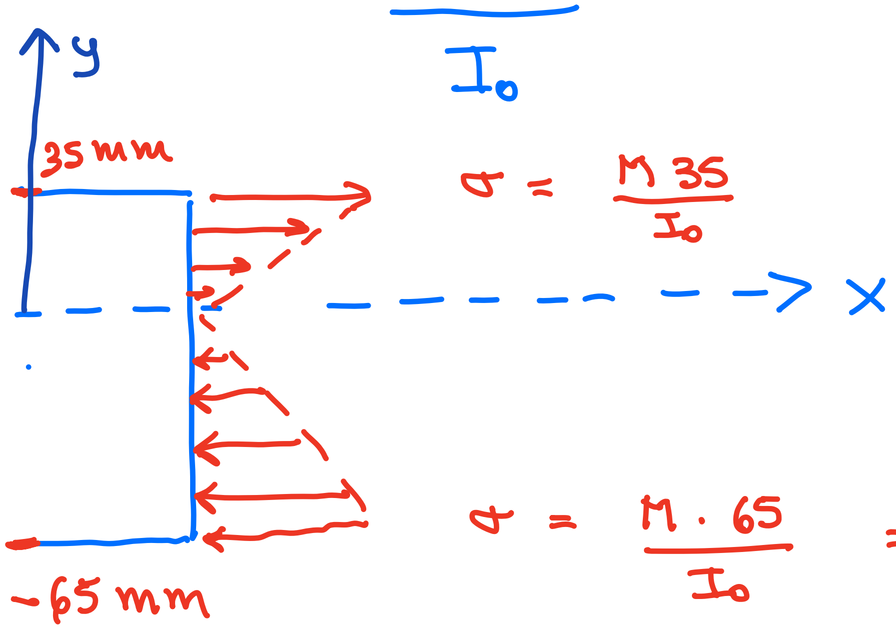


$d_1 = 25\ mm$

$d_3 = 25\ mm$

$$I_o = I_1 + A_1 d_1^2 + I_2 + A_2 d_1^2 + I_3 + A_3 d_3^2$$

$$\sigma_x = - \frac{M\ y}{I_o}$$

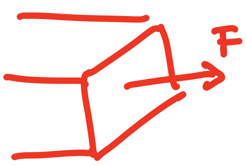


$$\sigma = \frac{M\ 35}{I_o}$$

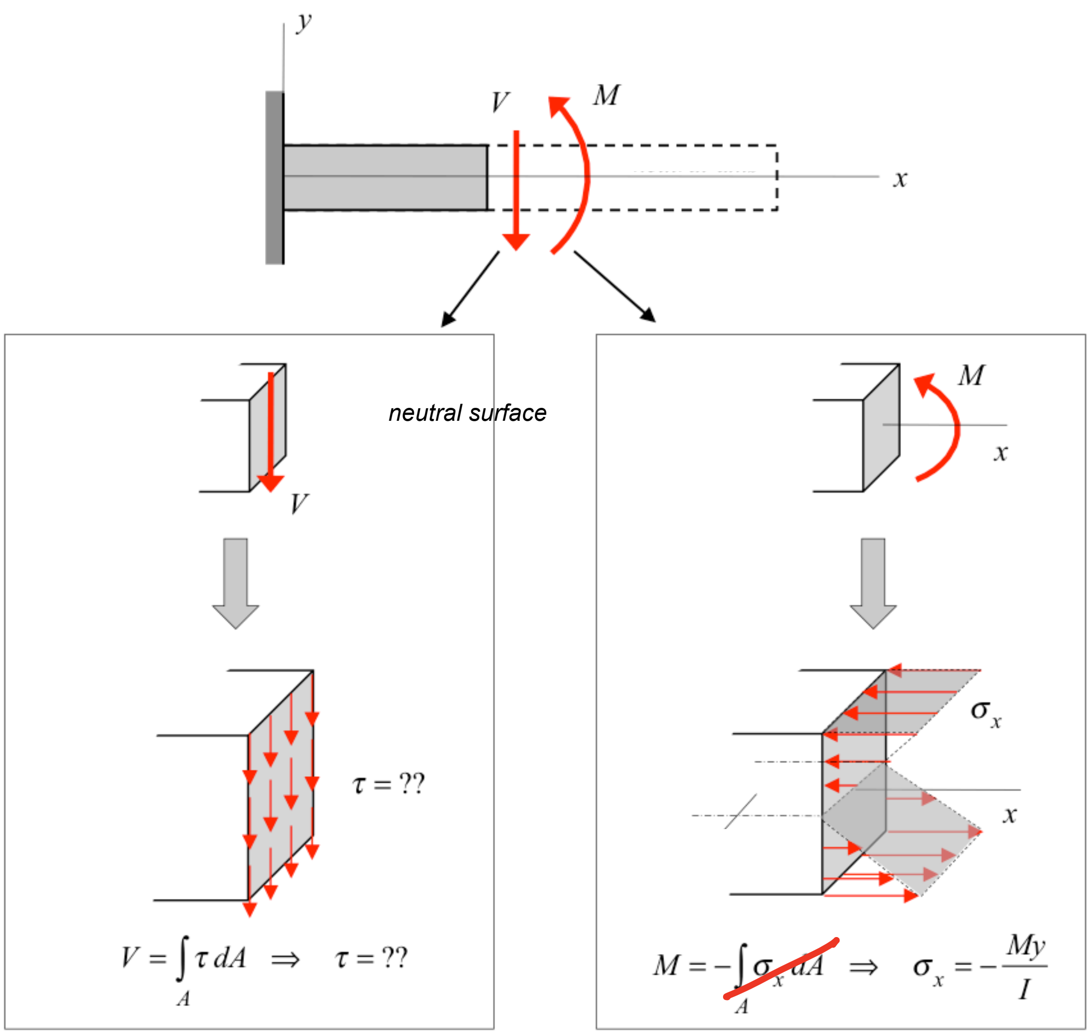
$$\sigma = \frac{M \cdot 65}{I_o} = \sigma^{max}$$

c) *Stresses due general transverse force and bending-couple loading of beams*

Earlier in the chapter, we considered the normal stress distribution within the cross section of a beam experiencing pure bending (i.e., in the absence of a shear force resultant on the cross-sectional cut). Here we will now consider the more general case of having both shear force and bending moment couples on the cross-sectional cut, as demonstrated by the figure below.



$\sigma_x = \frac{F}{A}$



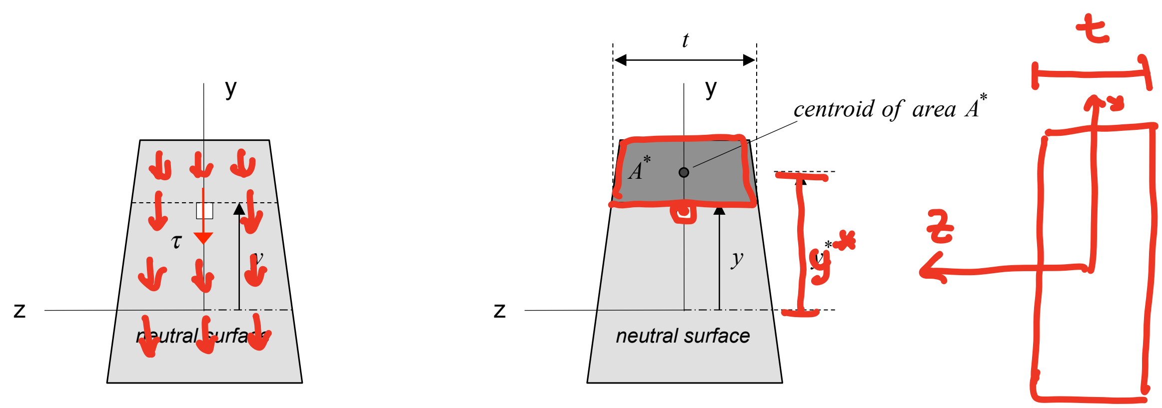
$\tau(y)$

$= -\int \sigma_x \cdot y \, dA$

$\sigma_x(y)$

We have seen that the normal stresses due to the bending moment M are linearly distributed over the cross section, with maximum magnitudes of normal stress occuring on the outer fibers of the beam and with zero normal stress at the neutral axis (the neutral axis passing through the centroid of the cross section).

With the shear force V now added to the cross-sectional cut, we now need to determine the shear stress distribution on the cross section. With our earlier assumptions of symmetry of the beam cross section about the xy-plane, we know that the distribution of the shear force will be constant through the depth of the beam (z-direction). For the case of direct shear (zero bending moment), the shear stress was also constant in the y-direction, making shear force constant throughout the cross section a constant. However, the presence of the bending moment induces a redistribution of shear stresses in the y-direction.



Consider the cross section shown above. We desire to know the shear stress τ acting on a stress element at a distance of y from the neutral surface. This shear stress along the axis of symmetry (the y-axis) can be expressed as:

$$\tau = \frac{VQ}{It} \qquad \tau = \frac{V(x)\,Q(y)}{I\,t} \tag{8}$$

where:

V = shear force at cross section

$Q = A^* \bar{y}^*$

A^* = cross-sectional area above the element

$\bar{y}^*$ = centroid of the area above the element

I = centroidal second area moment for the entire cross section

t = depth dimension of the beam at the location of the stress element of interest

The derivation of equation (8) will be presented on the following pages.

2/21 Review
2/23 No class.

Derivation of the shear stress distribution equation

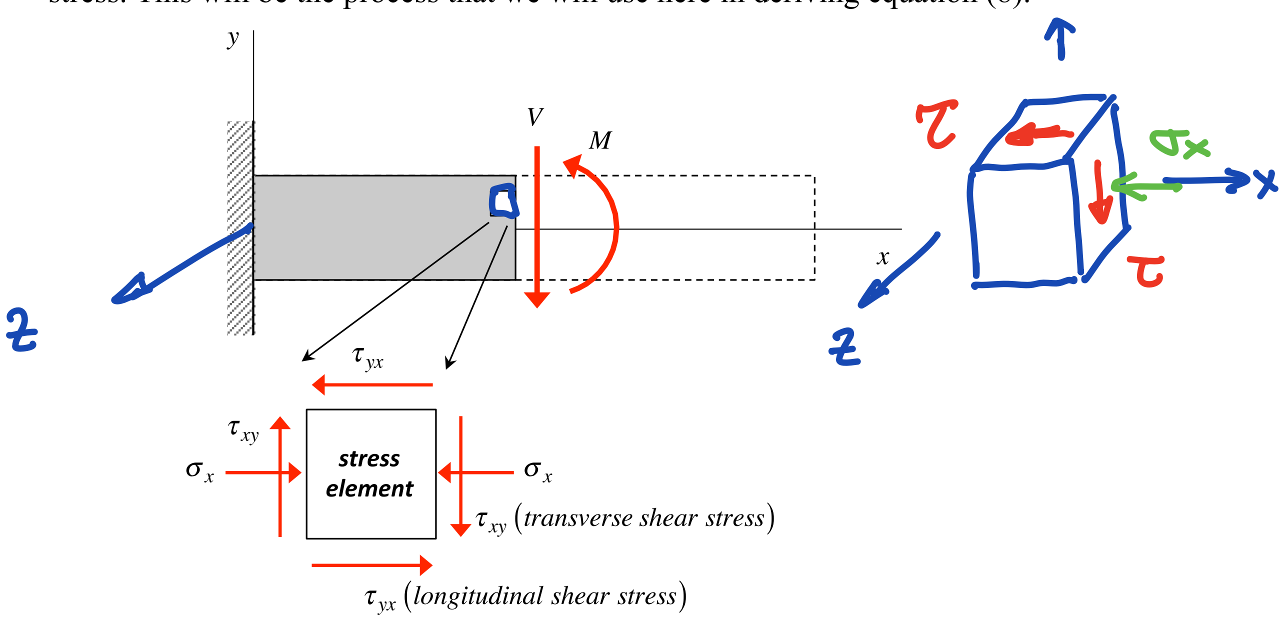
<u>Background</u>:

a) Recall that in the derivation of the equation for the normal stress distribution for pure bending:

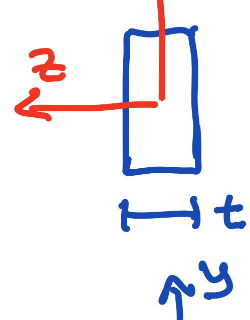
$$\sigma_x = -\frac{M\,y}{I} \qquad (6)$$

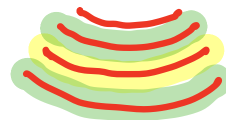
we assumed that plane sections of the cross section remain plane, and that they remain perpendicular to the deformed axis of the beam. For the more general situation in which a shear force V acts along with the bending moment M, a component of shear stress will exist. As we have seen earlier, the resulting shear strains correspond to a change in angle of the stress element. This angle change is somewhat in contradiction with the pure bending assumption of the cross section remaining perpendicular to the deformed beam axis. For our derivation, *we will assume that the shear strain effects will be slight and that, even in the presence of shear stress, the distribution of flexural stress on a given cross section is unaffected by the deformation due to shear* and that *equation (6) is still valid for computing the normal stresses on the cross section.*

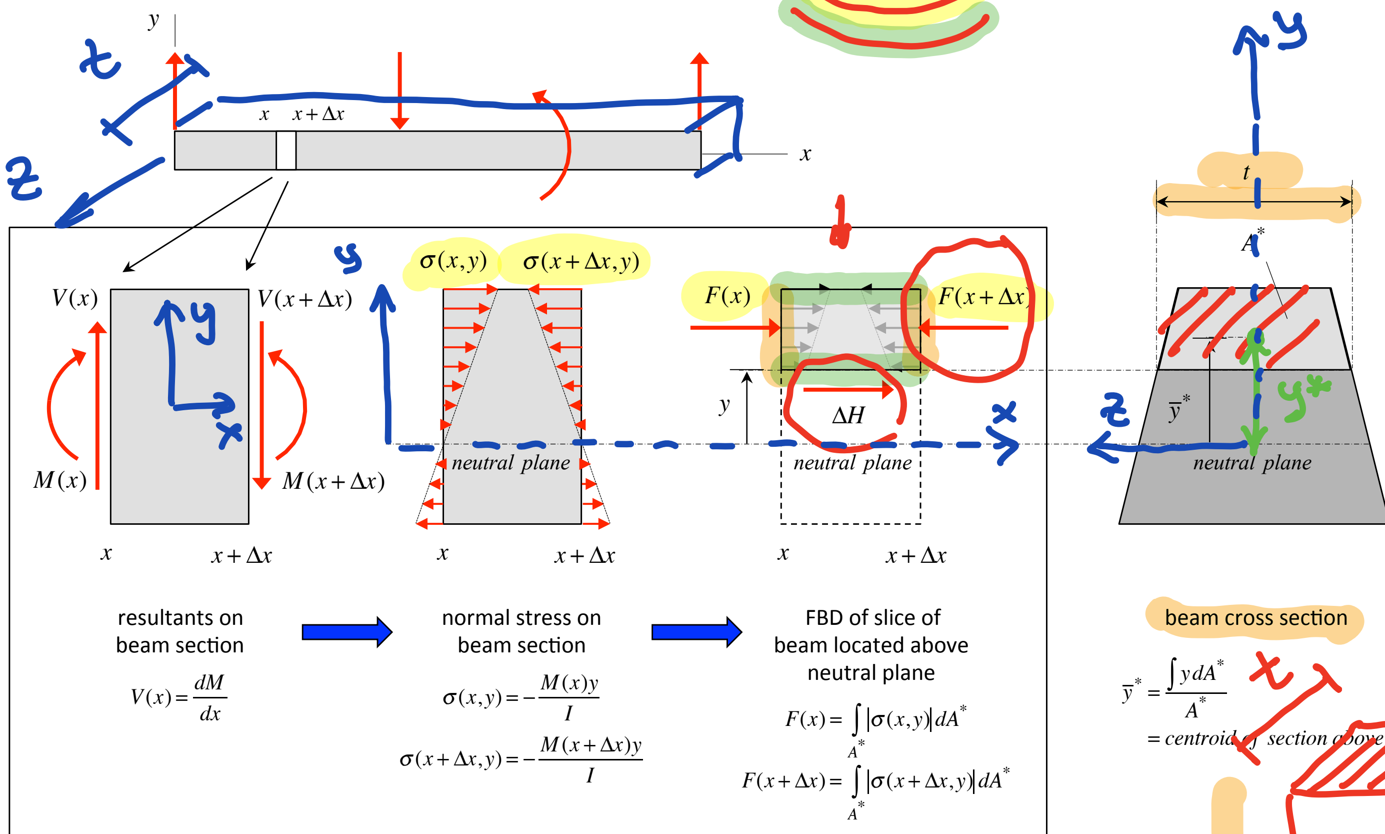
b) Suppose we consider a stress element on the side of a beam with a non-zero shear force resultant on the face of the cut. Our goal here is to determine the <u>*transverse*</u> shear stress component τ_{xy} that corresponds to the shear force resultant V. Note, however, that since $\tau_{yx} = \tau_{xy}$, the transverse shear stress component τ_{xy} is the same as the longitudinal shear stress component τ_{yx}. Stated in different words, we can determine the transverse shear stress by calculating the longitudinal shear stress. This will be the process that we will use here in deriving equation (8).

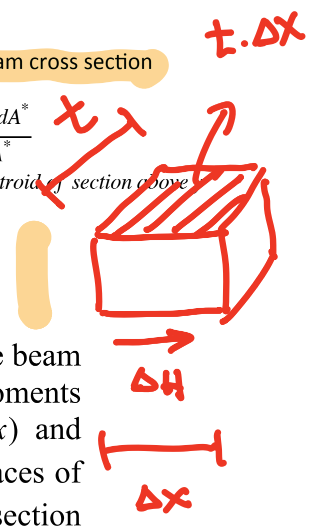


<u>*Derivation of shear stress equation*</u>:









Consider the aribitrarily-loaded beam shown above. Here we isolate a section of the beam between locations x and $x+\Delta x$, with the resultant shear forces and bending moments acting on this section, as shown above left. The resultant bending moments $M(x)$ and $M(x+\Delta x)$ produce normal stresses of $\sigma(x)$ and $\sigma(x+\Delta x)$ on the left and right faces of the beam section, respectively. Suppose we further isolate a slice of this beam section found <u>*above*</u> a given value of y on the beam cross section. As shown in the above figure, the resultants of the normal components of stress on the left and right faces are given by $F(x)$ and $F(x+\Delta x)$, respectively. A resultant longitudinal shear force ΔH also acts on the lower surface of the slice at y. From static equilibrium of the slice we have:

$$\sum F_x = F(x) - F(x+\Delta x) + \Delta H = 0 \;\Rightarrow\; \Delta H = F(x+\Delta x) - F(x)$$

The shear stress corresponding to this resultant shear force is found from the usual definition of stress in terms of the force resultant as:

$$\tau = \lim_{\Delta x \to 0}\left(\frac{\Delta H}{t\Delta x}\right) = \frac{1}{t}\lim_{\Delta x \to 0}\left(\frac{F(x+\Delta x) - F(x)}{\Delta x}\right) = \frac{1}{t}\frac{dF}{dx} \qquad (9)$$

From the above we have:

$$F(x) = \frac{M(x)}{I}\int_{A^*} y\, dA^* = \frac{\bar{y}^* A^*}{I} M(x) \qquad (10)$$

$F(x) = \int \sigma_x \, dA$

where A^* and $\overline{y}^*$ are the area and the centroid of the area of the cross section above y. Combining equations (9) and (10) gives:

$$\tau = \frac{A^*\overline{y}^*}{It}\frac{dM}{dx} \qquad (11)$$

Finally, recall that from equilibrium analysis that $V = dM / dx$. Therefore, (11) becomes:

$$\tau = \frac{VA^*\overline{y}^*}{It} \qquad (8)$$

Comments on the usage of the shear stress equation

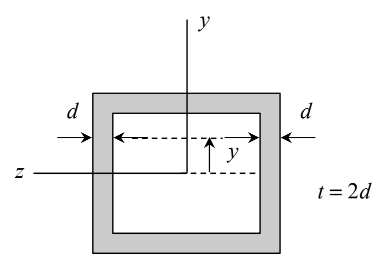
a) Note that this derivation was based on considering a slice of the beam section ABOVE the location y; hence, we ended up with $A^*\overline{y}^*$ representing the area *above* y. Alternately, we could have easily kept a slice of the section BELOW position y. In that case $A^*\overline{y}^*$ in the equation would then represent that area *below* y. We will get the same magnitude for the shear stress using the area below y as if we consider the area above y.

b) There are limitations on the usage of this shear stress equation, as listed below.

- Effect of load distribution: The assumptions of plane sections remaining plane and perpedicular to the neutral surface are valid for beams that are long compared to their depth. This assumption limits the influence of shear deformations in the beam and, hence, limits the error in the flexural stresses.
- Effect of cross section shape: The shear stress equation derived is particularly accurate for beams that are thin in the depth dimension ("t") and for which this dimension t does not vary rapidly with y. For thin-walled beams, the shear stress equation is valid for sections of the cross section that are aligned with the y-axis, and most accurately so near the neutral plane.

c) Other remarks on the shear stress equation:

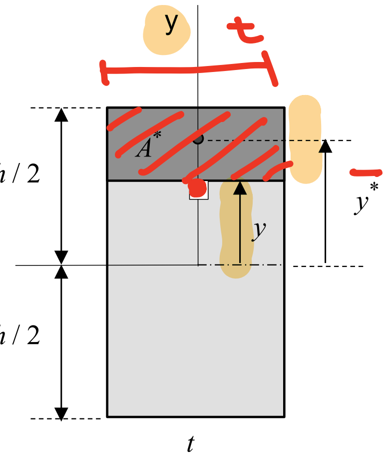
- The sign of τ is the same as the sign on V. Also, recall that V is the force resultant of the shear stress: $V = \int \tau \, dA$
- I is the second area moment of the cross-section (independent of the location y).
- t is the net thickness of the beam at the location y.
- Regardless of the cross section, $\tau = 0$ at the top and bottom fibers of the beam.
- If the beam cross section is symmetric about the neutral axis, the maximum shear stress occurs at the neutral axis.



$$\tau = \frac{V(x)\ Q(y)}{I_0 t} \qquad Q(y) = A^* \bar{y}^*$$

Example – shear stress distribution in a rectangular cross section

As an example, consider a rectangular cross section beam of dimensions of thickness h and depth t. From before, we know that the centroidal second area moment for a rectangular beam of these dimensions is $I = th^3/12$. For a stress element at y, we have:



$$A^* = \left(\frac{h}{2} - y\right)t \qquad \left(\frac{h}{2} - y\right) \quad I$$

$$\bar{y}^* = \frac{1}{2}\left(\frac{h}{2} + y\right) \qquad \bar{y}^* = y + \left(\frac{h}{2} - y\right)\frac{1}{2}$$

Combining the above gives:

$$\tau = \frac{V\left[(h/2 - y)t\right]\left[(h/2 + y)/2\right]}{\left(th^3/12\right)t}$$

$$= \frac{6}{h^3 t}\left(\frac{h^2}{4} - y^2\right)V = \frac{6}{Ah^2}\left(\frac{h^2}{4} - y^2\right)V$$

$$I_0 = \frac{h^3 \cdot t}{12}$$

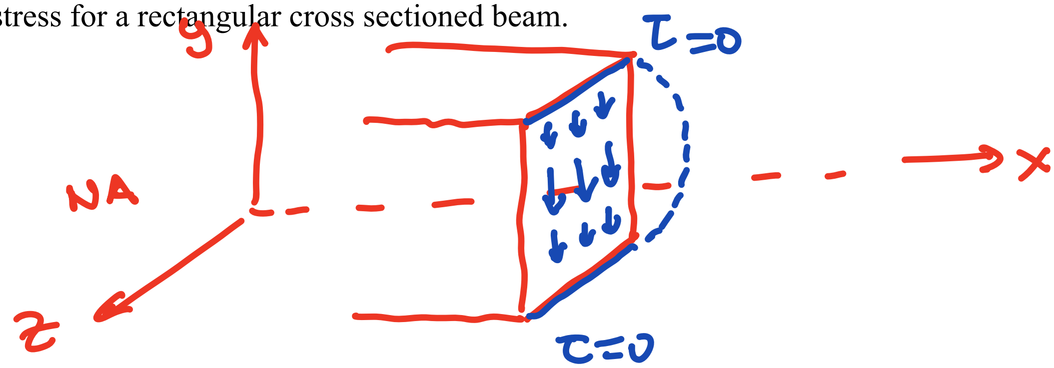
From this result, we observe the following for the shear stress distribution across a cut of a rectangular cross section beam experiences a shear force V:

- The stress distribution is quadratic with location y of the stress element.
- The shear stress is zero at the outer fibers of the beam ($y = \pm h/2$), as expected since these fibers experience no horizontal loads.
- The shear stress is a maximum at the neutral axis ($y = 0$). This maximum shear stress is given by:

$$\tau_{max} = \frac{3V}{2A}$$

$$\tau(x,y) = \frac{V(x)\,6}{(t \cdot h)\,h^2}\left(\frac{h^2}{4} - y^2\right)$$

- Recall that the average shear stress across the cut is given by $\tau_{ave} = V/A$, which would be the shear stress on the cut in the absence of a bending moment. From this we see that the bending moment produces a 50% increase in the maximum shear stress for a rectangular cross sectioned beam.



$$\tau^{max}(x, y=0) = \frac{6V\ h^2}{4\,t \cdot h\ h^2}$$

$\cancel{\tau^{ave} = \frac{V}{A}}$ → No $\underline{M}$ $= \frac{3}{2} \frac{V}{t \cdot h}$

Summary: stress distribution due to combined shear force and bending couple at cut
At a cut through a section of a beam experiencing both a shear force V and bending moment M, and abiding by the Euler-Bernoulli assumptions, we can make the following observations (see following figure):

a) Both normal stresses σ_x and shear stresses τ exist at the cut.

b) The normal stresses vary linearly in the *y*-direction as in the pure bending case. All previous observations about the normal stresses due to pure bending also apply in the case.

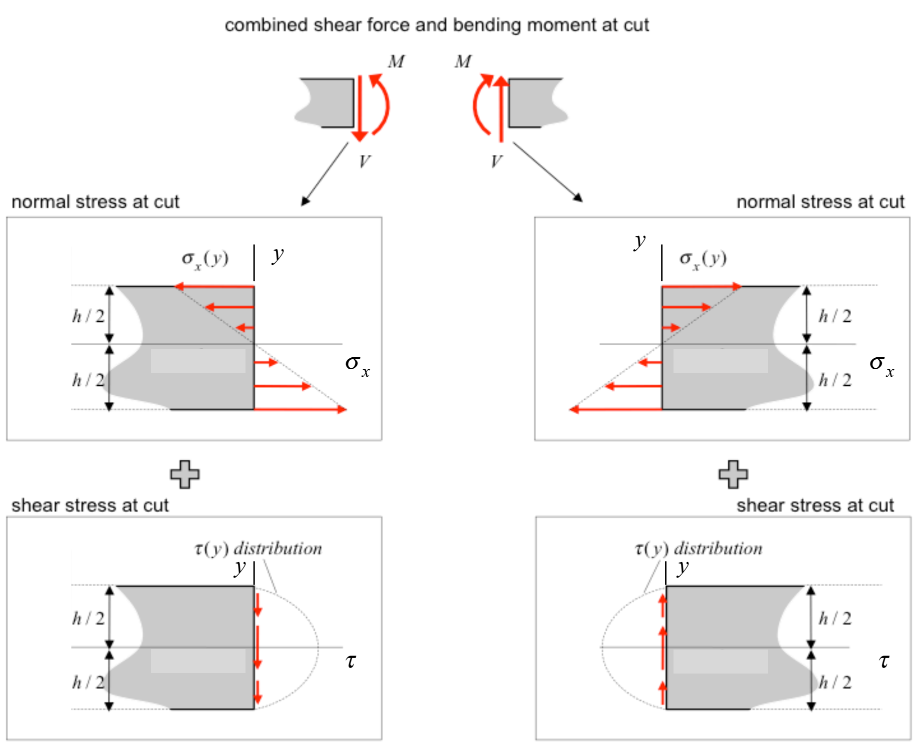
c) The shear stresses are approximately constant in the *z*-direction (into the depth of the beam) for "narrow beams", $t > 2h$.

d) The shear stress is zero at the outer surfaces of the beam.

e) For rectangular cross-section beams, the shear stress distribution at a cut is parabolic in the y-direction:

$$\tau = \frac{6}{Ah^2}\left(\frac{h^2}{4} - y^2\right)V$$

where A is the area of the cross section. The maximum shear stress, $\tau_{max} = 3V / 2A$, occurs at the neutral axis ($y = 0$).



Shown below is a rectangular cross section cantilevered beam with a single transverse applied load *P*. A cut is made at one location along the beam. What are the stress states at stress elements a, b, c, d and e at the cut?

The cut in the beam exposes both a bending moment M and a shear force V, where:

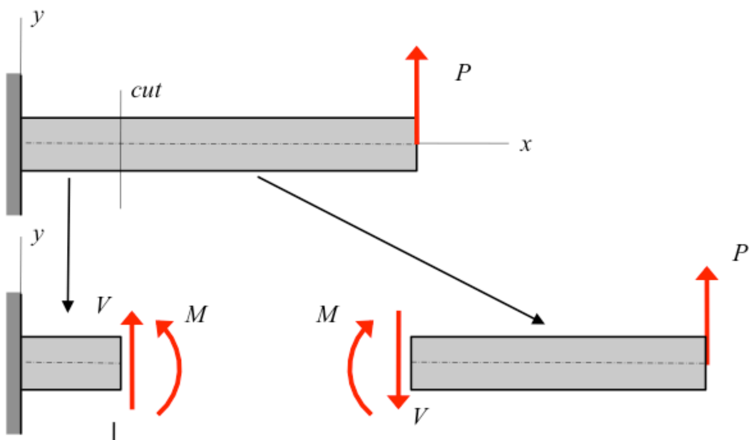
$$M(x) = P(L - x)$$

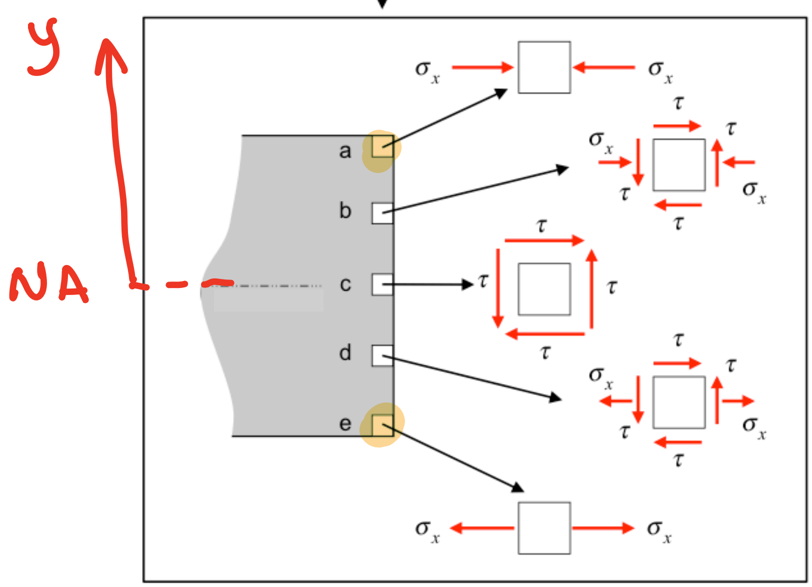
$$V = -P = constant\ along\ length\ of\ beam$$

where x is the location of the beam cut. A combination of normal stress σ and shear stress τ is expected, in general, at the stress elements. Based on our earlier analysis, we observe:

- Stress elements a and e experience only normal stress since shear stress is zero at the outer fibers. At a the normal stress is compressive, and at e the normal stress is tensile. The magnitudes of these normal stresses are equal and are their maximum values on the cross section.
- Stress element c experiences only shear stress since the normal stress is zero at the neutral axis. The shear stress τ at c is the maximum of all stress elements on the cut.
- Stress elements b and d experience a combination of normal and shear stress. The normal stress at b is compressive, and the normal stress at d is tensile.

Note that the maximum shear stress at a cut is constant along the length of the beam. The magnitude of the maximum normal stress at a cut decreases as the cut is moved away from the wall.





$$M(x) = P(L - x)$$

$$V = -P = \text{constant along length of beam}$$

$$\sigma_x = -\frac{M\,y}{I}$$

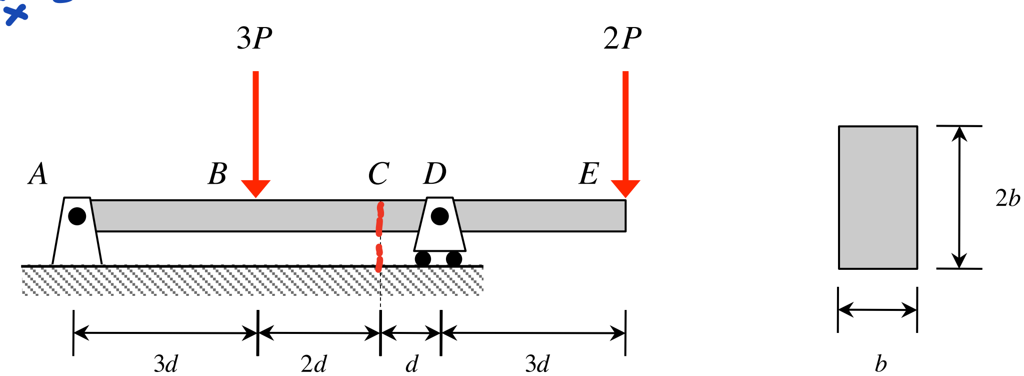
$$\tau = \frac{6}{Ah^2}\left(\frac{h^2}{4} - y^2\right)V$$

Example 10.7

A rectangular cross-section timber beam AE has dimensions and loading shown. Determine the normal and shear stress distributions at location C on the beam.

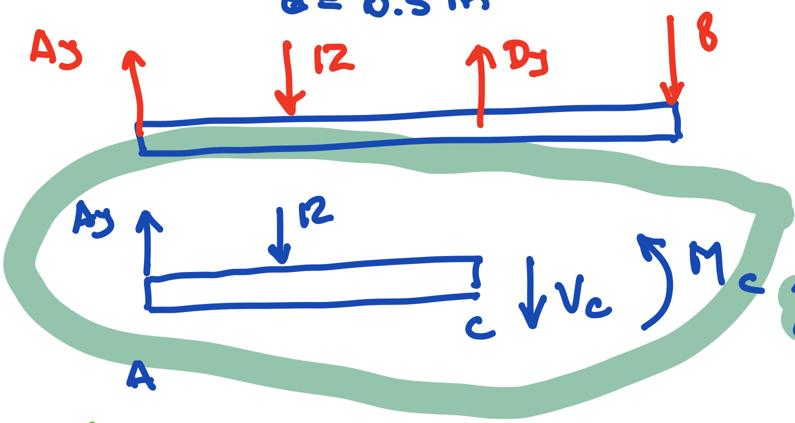
σ_x $\quad$ τ

$P = 4\ kN$
$b = 0.09$



beam cross section

$d = 0.5\ m$



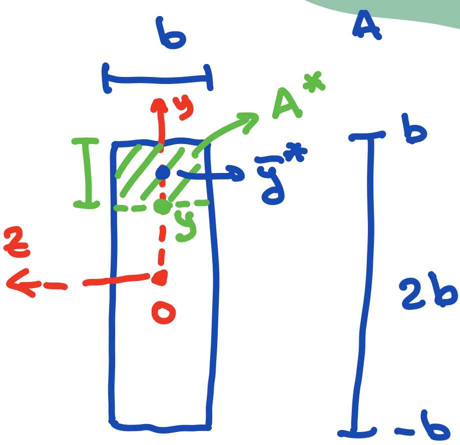
$D_y = 18\ kN$

$A_y = 2\ kN$

$V_c = -10\ kN$

$M_c = -7\ kN \cdot m$

?



$$\bar{y}^* = y + \left(\frac{b-y}{2}\right) = \frac{y+b}{2}$$

$$A^* = b \cdot (b-y)$$

$$A^*\bar{y}^* = \frac{b}{2}(b-y)(b+y)$$

$$A^*\bar{y}^* = \frac{b}{2}(b^2-y^2)$$

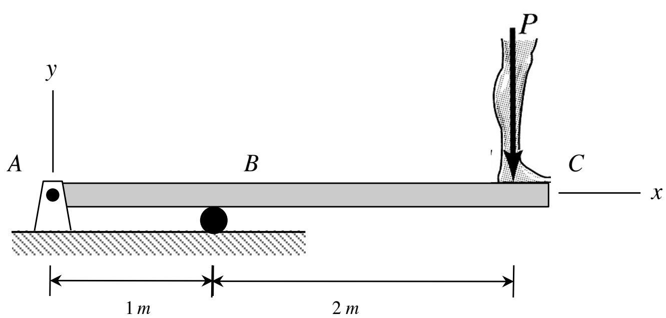
$$I_o = \frac{(2b)^3 \cdot b}{12} = \frac{2}{3}b^4$$

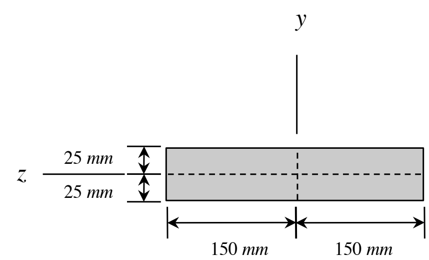
$$\tau = \frac{V_c \,(b^2-y^2)\,\frac{\cancel{b}}{2}}{\frac{2}{3}b^4\,\cancel{b}} = \frac{3}{4}V\,\frac{b^2-y^2}{b^4}$$

$\tau^{max} \Rightarrow$ at $y = 0 \rightarrow \tau^{max} = \frac{3}{4}\frac{V}{b^2} = \frac{3}{2}\frac{V}{Area}$

Example 10.10

A timber plank is to be used as a diving board. The diving board is held down at end A by a steel strap that is secured by anchor bolts and rests on a roller at location B. Calculate the maximum permissible load P_{max} such that the maximum normal stress in the diving board does not exceed $11\ MPa$.



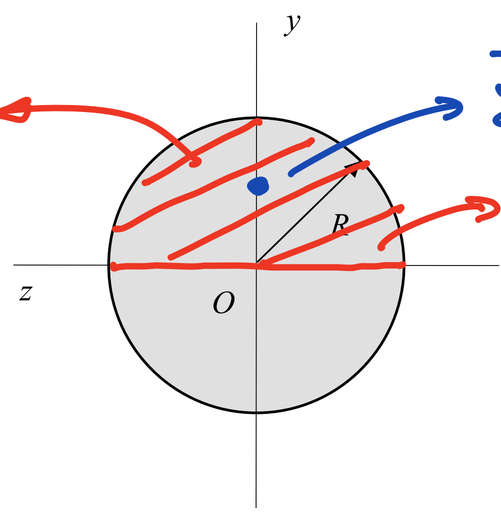


plank cross section

Example 10.11

Use the shear stress formula for a general shape cross section developed earlier in the chapter to determine an expression for the maximum shear stress along the symmetry axis y of the circular cross section beam shown below.

A^* $\quad$ $\bar{y}^*$



τ^{max}

τ at $y=0$

$$A^* = \frac{\pi R^2}{2}$$

$$\bar{y}^* = \frac{2\,(2R)}{3\pi}$$

$$I_0 = \frac{\pi R^4}{4}$$

$$t = 2R$$

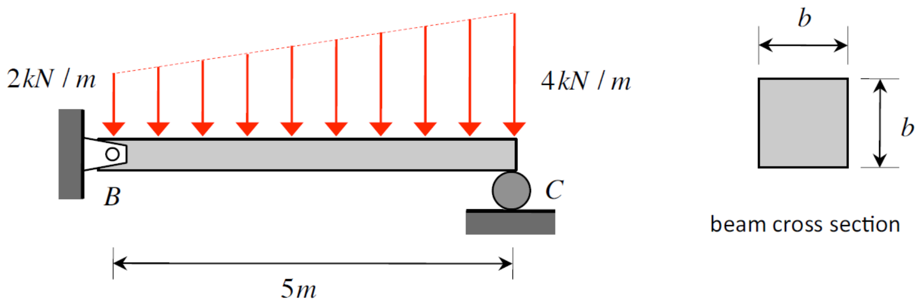
$$\tau^{max} = V \frac{A^* \bar{y}^*}{I t}$$

$$\tau^{max} = \frac{4}{3} \frac{V}{\pi R^2}$$

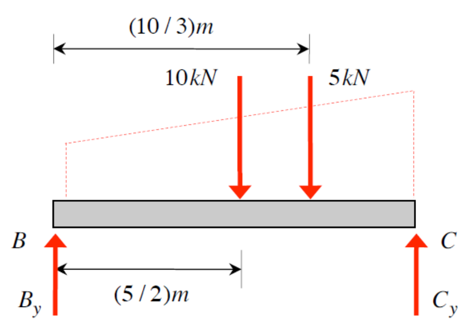
Example 10.13

A linearly-varying distributed load acts between B and C on the simply-supported beam shown below. The beam has a square cross section. It is known that the magnitude of allowable normal stress in the beam is $\sigma_{allow} = 10ksi$. Determine the minimum value for b such that the beam does not fail under this loading. Where is the critical stress location for this loading?

NOTE: $\sigma_{allow} = 10ksi = 68.947\,MPa$



Using the following FBD:



we have:

$$\sum M_B = -(10)(5/2) - (5)(10/3) + C_y(5) = 0 \quad \Rightarrow \quad C_y = \frac{25+50/3}{5} = \frac{25}{3}kN$$

$$\sum F_y = B_y - 10 - 5 + C_y = 0 \quad \Rightarrow \quad B_y = 10 + 5 - 25/3 = \frac{20}{3}kN$$

Using the fundamental equilibrium equations:

$$\frac{dV}{dx} = p(x) \quad \Rightarrow$$

$$V(x)=V(0)+\int_0^x p(x)\,dx=B_y-\int_0^x[2+2x/5]\,dx$$

$$=B_y-2x-\frac{x^2}{5}=\frac{20}{3}-2x-\frac{x^2}{5}\quad;\quad Check: V(5)=-\frac{25}{3}\overset{checks}{=}-C_y$$

$$\frac{dM}{dx}=V(x)\quad\Rightarrow$$

$$M(x)=M(0)+\int_0^x V(x)\,dx=0+\int_0^x\left[\frac{20}{3}-2x-\frac{x^2}{5}\right]dx$$

$$=\frac{20}{3}x-x^2-\frac{x^3}{15}\quad;\quad Check:\ M(5)\overset{checks}{=}0$$

Note that M(x) has a maximum when dM/dx = V(x) = 0:

$$0=\frac{x^2}{5}+2x-\frac{20}{3}\quad\Rightarrow$$

$$x_{\max}=\frac{1}{2(1/5)}\left[-2+\sqrt{2^2+(4)(20)/(15)}\right]=\frac{5}{2}\left[-2+\sqrt{29/3}\right]=2.638m$$

Therefore:

$$M_{\max}=\frac{20}{3}x_{\max}-x_{\max}^2-\frac{x_{\max}^3}{15}$$

$$=\frac{20}{3}(2.638)-(2.638)^2-\frac{(2.638)^3}{15}=9.404kN\cdot m$$

Using the normal stress equation for a beam in bending:

$$|\sigma|_{\max}=\frac{M_{\max}y_{\max}}{I}\quad;\quad I=\frac{1}{12}b^4\ \ and\ \ y_{\max}=\frac{b}{2}$$

$$=6\frac{M_{\max}}{b^3}\quad\Rightarrow$$

$$b_{\min}=\sqrt[3]{\frac{6M_{\max}}{\sigma_{allow}}}=\sqrt[3]{\frac{(6)(9.404)\times10^3\,N\cdot m}{(68.947)\times10^6\,N/m^2}}=0.0935m$$

Additional notes: