Problem 11.1 (10 Points)

The elbow shown below is fixed to the ground at the center of the coordinate system. Two loads 75 N and 150 N are applied at the free end in the y and z directions, respectively. If the elbow has a circular cross-section with a diameter of 20 cm, find:

(a) The internal reactions at a cross-section perpendicular to the y-axis at point A ($y = 1$ m). Classify the forces as either axial or shear forces, and the moments as either bending or torsion.

(b) The stresses induced (magnitude and direction) in the stress elements M and N on the cross-section at A, shown in Fig. 11.1 (b), due to each reaction calculated in part (a). Use the three-dimensional stress elements in Fig. 11.1 (c).

(c) Draw the Mohr's circle for the stress states at M, and N, and calculate the principal stresses and the absolute maximum shear stress $|\tau_{max}|$.

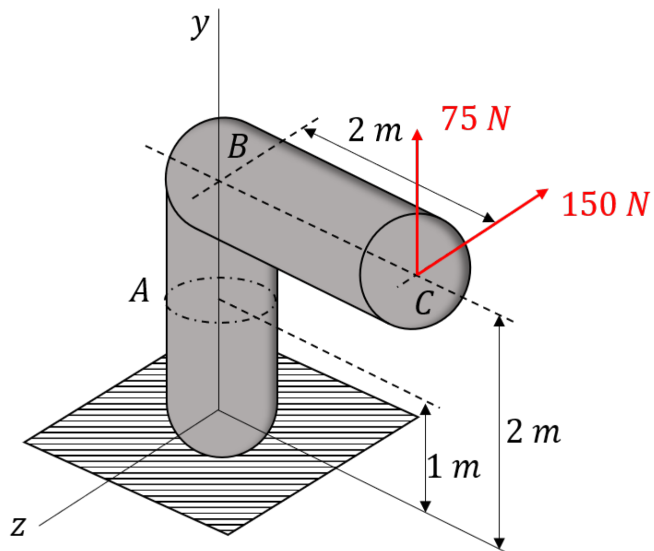


Fig. 11.1 (a)

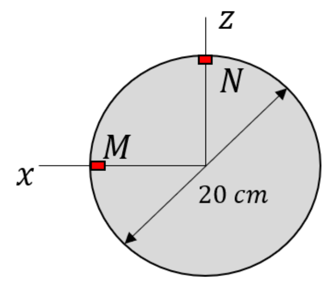


Fig. 11.1 (b)

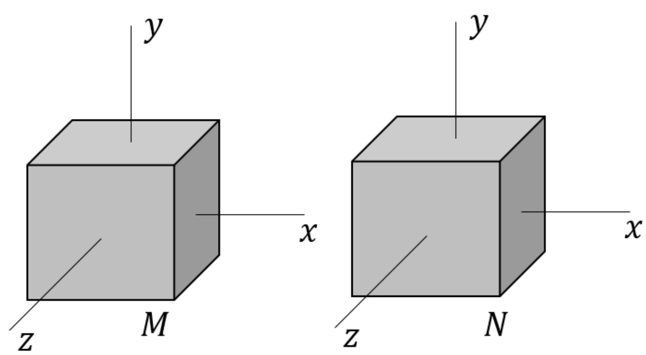
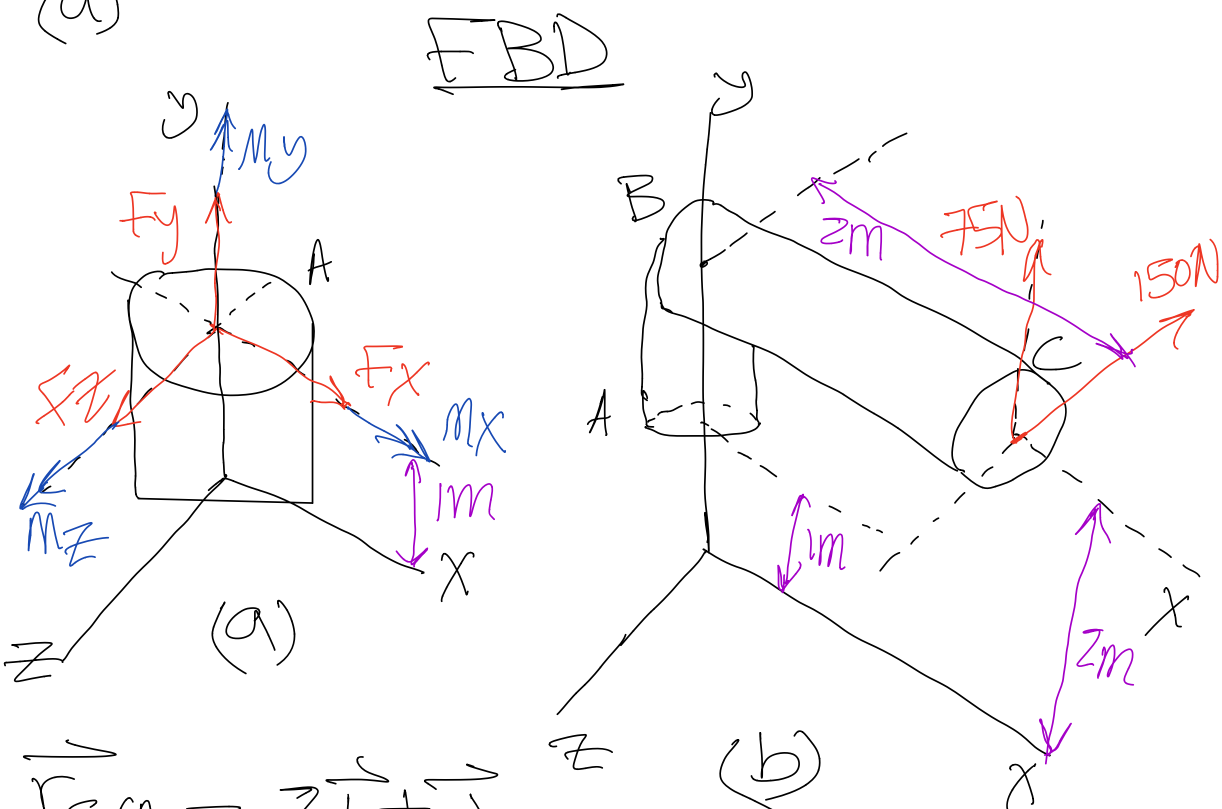


Fig. 11.1 (c)

Solution:

(a)



$$\vec{r}_{C/A} = 2\vec{i} + \vec{j}$$

from (a),

$$\vec{F}_A = F_x\vec{i} + F_y\vec{j} + \vec{F_z}$$

$$\vec{M}_A = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$$

From (b),

$$\sum\vec{F} = 0$$

$$-\vec{F}_A + 75\vec{j} - 150\vec{k} = 0$$

$$\therefore \vec{F}_A = 75\vec{j} - 150\vec{k} \quad (N)$$

$$\sum \vec{M} = 0$$

$$-\vec{M}_A + \vec{r}_{C/A} \times (75\vec{j} - 150\vec{k}) = 0$$

$$\Rightarrow \vec{M}_A = (2\vec{i} + \vec{j}) \times (75\vec{j} - 150\vec{k})$$

$$\therefore \vec{M}_A = -150\vec{i} + 300\vec{j} + 150\vec{k} \quad (Nm)$$

	Reaction at A	Type of reaction
1	$75\vec{j}$ N	Axial force
2	$-150\vec{k}$ N	shear force
3	$-150\vec{i}$ Nm	Bending moment
4	$300\vec{j}$ Nm	Torsion
5	$150\vec{k}$ Nm	Bending moment

(b)

For the circular cross section,

$$A = \frac{\pi (0.2)^2}{4} = 0.0314 \, m^2$$

$$I = \frac{\pi (0.2)^4}{64} = 7.854 \times 10^{-5} \, m^4$$

$$J = I_p = \frac{\pi (0.2)^4}{32} = 1.571 \times 10^{-4} \, m^4$$

1. $75\vec{j}$ N (axial force)

$$\sigma_{m,1} = \sigma_{N,1} = \sigma_y = \frac{F_y}{A} = \frac{75}{0.0314} = 2388.53 \, Pa$$

2. $-150\vec{k}$ N (shear force in z)

$$\tau_{M,1} = \tau_{yz} = \frac{4F_z}{3A} = \frac{4(-150)}{3(0.0314)}$$

$$= -6369.42 \text{ Pa}$$

$$\tau_{N,1} = 0$$

3. $-150\vec{i}$ Nm (Bending moment)

$$\sigma_{N,2} = \sigma_y = \left|\frac{M_x R}{I}\right| = \frac{(150)(0.1)}{7.854\times10^{-5}}$$

$$= 1.91\times10^{5} \text{ Pa}$$

$$\sigma_{M,2} = \sigma_y = 0$$

4. $300\vec{j}$ Nm (Torsion)

$$\tau_{M,2} = \tau_{yz} = -\left|\frac{M_y R}{J}\right| = \frac{-300(0.1)}{1.571\times10^{-4}}$$

$$= -1.91\times10^{5} \text{ Pa}$$

$$\tau_{N,2} = \tau_{yx} = \left|\frac{M_y R}{J}\right| = \frac{300(0.1)}{1.571\times10^{-4}} = 1.91\times10^{5} \text{ Pa}$$

5. $150\vec{k}$ Nm (Bending moment)

$$\sigma_{N,3} = \sigma_y = 0$$

$$\sigma_{M,3} = \sigma_y = \left|\frac{M_z R}{I}\right| = \frac{(150)(0.1)}{7.854\times10^{-5}}$$

$= 1.91 \times 10^5$ pa

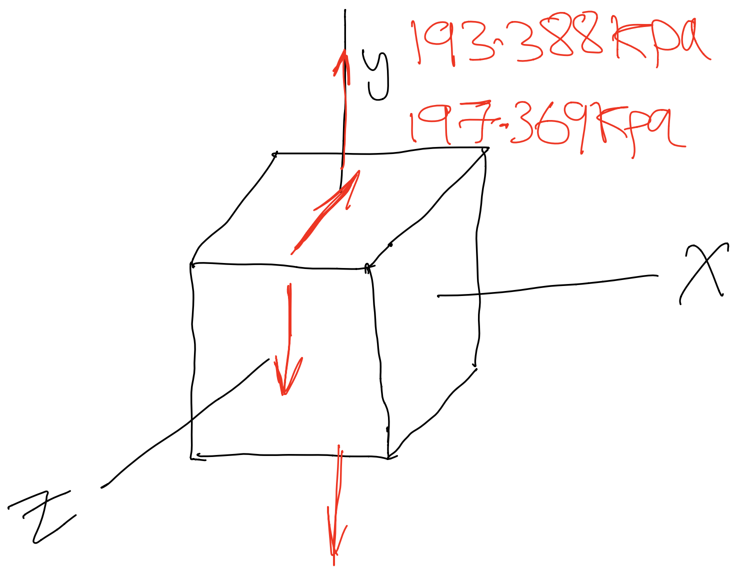
In summary,

The state of stress at M:

$\sigma_{M,y} = \sigma_{M,1} + \sigma_{M,2} + \sigma_{M,3}$

$= 193.388$ KPa

$\tau_{M,yz} = \tau_{M,1} + \tau_{M,2} = -197.369$ Kpa



The state of stress at N:

$\sigma_{N,y} = \sigma_{N,1} + \sigma_{N,2} + \sigma_{N,3} = 193.388$ Kpa

$\tau_{N,yx} = \tau_{N,1} + \tau_{N,2} = 191$ Kpa

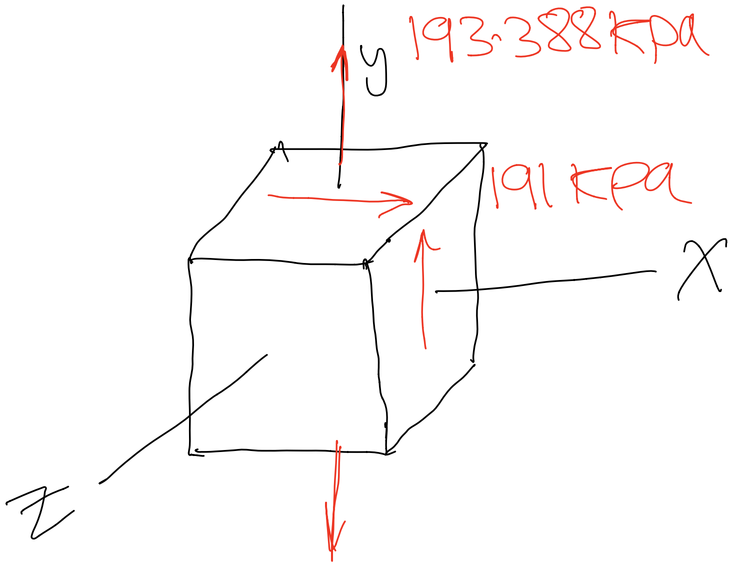


Table method :

	Reaction at A	stress comp. @ "M"	stress comp. @ "N"
1	$F_y = 75N$	$\sigma_y = \frac{F_y}{A} = 2388.53$ Pa	$\sigma_y = \frac{F_A}{A} = 2388.53$ Pa
2	$F_z = -150N$	$\tau_{yz} = \frac{4F_z}{3A} = -636942$ Pa	—
3	$M_x = -150Nm$	—	$\sigma_y = \left\lvert\frac{M_x R}{I}\right\rvert = 1.91\times10^5$ Pa
4	$M_y = 300 N.m$	$\tau_{yz} = -\left\lvert\frac{M_y R}{J}\right\rvert = -1.91\times10^5$ Pa	$\tau_{yx} = \left\lvert\frac{M_y R}{J}\right\rvert = 1.91\times10^5$ Pa
5	$M_z = 150 N.m$	$\sigma_y = \left\lvert\frac{M_z R}{I}\right\rvert = 1.91\times10^5$ Pa	—

(c) For M:

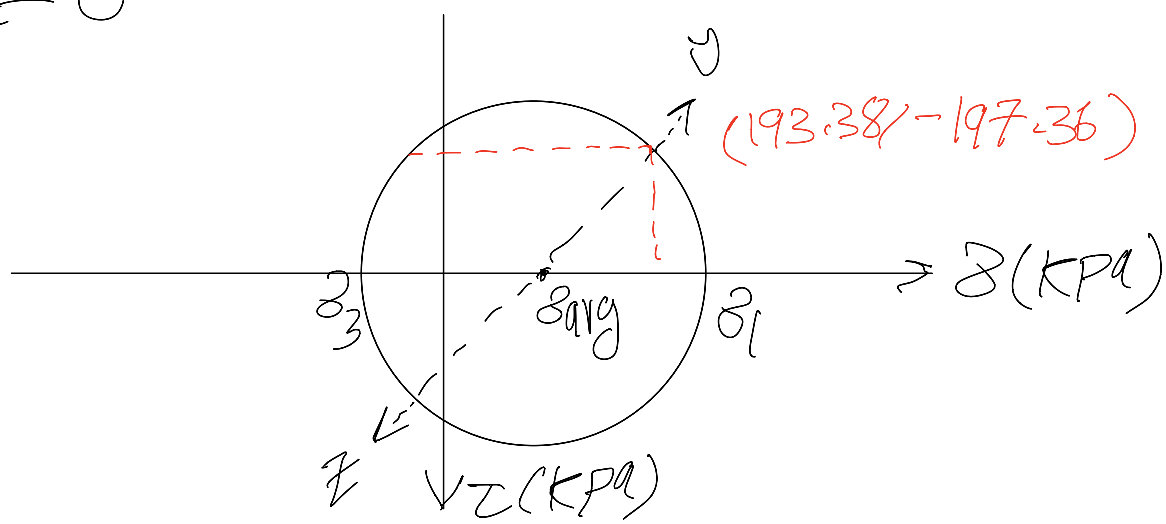
$$\sigma_{avg} = \frac{193.388}{2} = 96.694 \text{ Kpa}$$

$$R = \sqrt{(96.694)^2 + (197.369)^2} = 219.78 \text{ Kpa} = |\tau_{max}|$$

$$\sigma_1 = \sigma_{avg} + R = 316.47 \text{ Kpa} = \sigma_{max}$$

$$\sigma_3 = \sigma_{avg} - R = -123.086 \text{ Kpa} = \sigma_{min}$$

$$\sigma_2 = 0$$

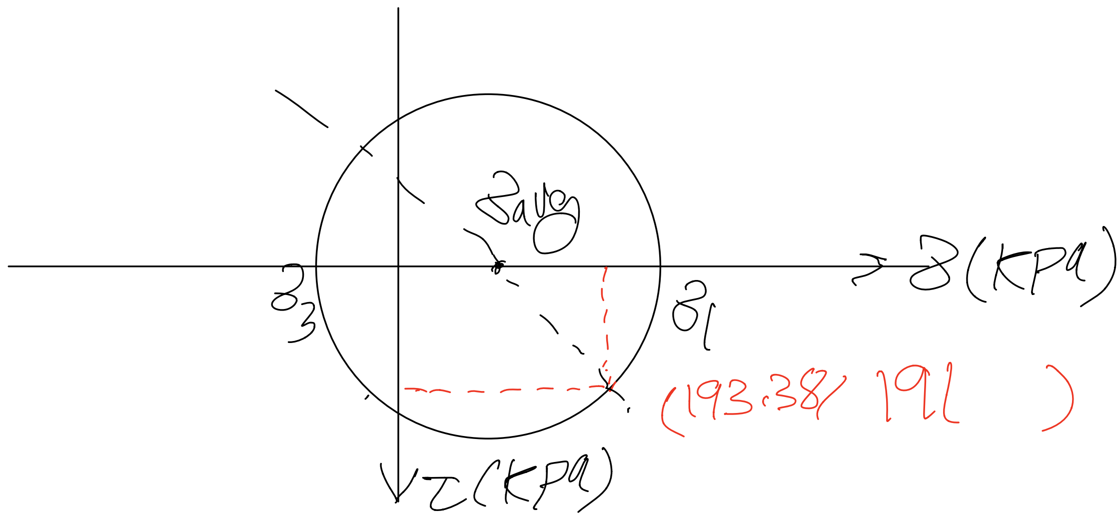


For N: $\sigma_{avg} = 96.694$ Kpa

$R = 214.08$ Kpa $= |\tau_{max}|$

$\sigma_1 = 310.775$ Kpa $= \sigma_{max}$

$\sigma_3 = -117.386$ Kpa $= \sigma_{min}$, $\sigma_2 = 0$



σavg
σ (KPa)
σ3
σ1
(193.38, 19l)
τ (KPa)

Problem 11.2 (10 Points)
For the given stress element, determine the following:

(a) The principal stresses, the absolute maximum shear stress $|\tau_{max}|$ and the von-Mises stress.
(b) The factor of safety using the maximum shear stress theory and the maximum distortion energy theory for a ductile material with yield strength 32 ksi.
(c) Whether or not failure is predicted using Mohr's failure criterion for a brittle material with ultimate compressive strength 90 ksi and ultimate tensile strength 30 ksi.

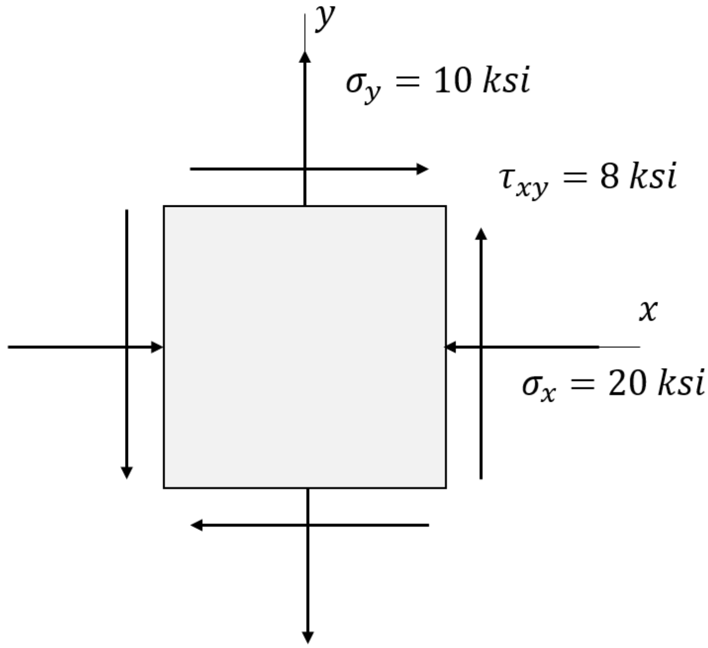


Fig. 11.2

Solution :

(a)

$$\sigma_x = -20 \text{ ksi}$$
$$\sigma_y = 10 \text{ ksi}$$
$$\tau_{xy} = 8 \text{ ksi}$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = -5 \text{ ksi}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(-15)^2 + 8^2} = 17 \text{ ksi}$$

Principal stress :

$$\sigma_1 = \sigma_{avg} + R = 12 \text{ ksi}$$
$$\sigma_2 = \sigma_{avg} - R = -22 \text{ ksi}$$

Absolute maximum shear stress :

$$|\tau_{max}| = R = 17 \text{ ksi}$$

von mises stress :

$$\sigma_M = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} = 29.866 \text{ ksi}$$

(b) $\sigma_Y = 32 \text{ ksi}$

For maximum shear stress theory

$$SF = \frac{\sigma_Y}{2|\tau_{max}|} = 0.94$$

For maximum distortion energy theory

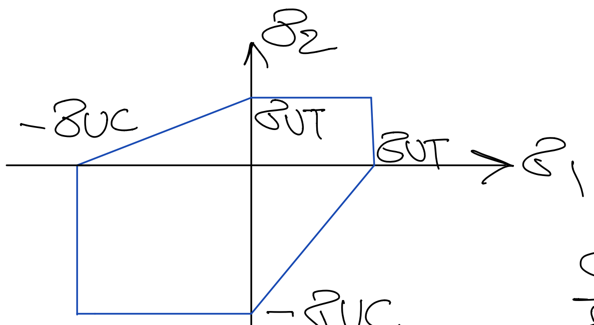
$$SF = \frac{\sigma_y}{\sigma_m} = 1.071$$

(c)

Mohr's criterion:

$\sigma_{UC} = 90$ ksi

$\sigma_{UT} = 30$ ksi



$\sigma_1 > 0$, $\sigma_2 < 0$

$\Rightarrow$ Fourth quadrant

$$\frac{\sigma_1}{\sigma_{UT}} = 0.4$$

$$\frac{\sigma_2}{\sigma_{UC}} = -0.244$$

Here $\frac{\sigma_1}{\sigma_{UT}} < \frac{\sigma_2}{\sigma_{UC}} + 1$

$\Rightarrow$ Failure not predicted.

Problem 11.3 (10 Points)
A horizontal rigid beam ACB is pinned to a wall at A and is supported by a pin-fixed column CD, as shown in the figure below. The beam is subjected to a linearly distributed load between B and C. The column CD has Young's modulus E and a rectangular cross-section, as shown. Consider supports C and D to act as pinned-fixed when buckling in the x-y plane and fixed-fixed when buckling in the y-z plane. Determine the maximum linearly distributed load q such that column CD does not buckle. For this problem, use the parameters: $L = 1.5\text{m}$, $b = 30\text{mm}$, $h = 45\text{mm}$, $E = 150\text{GPa}$. Use **Euler's buckling** theory to solve the problem.

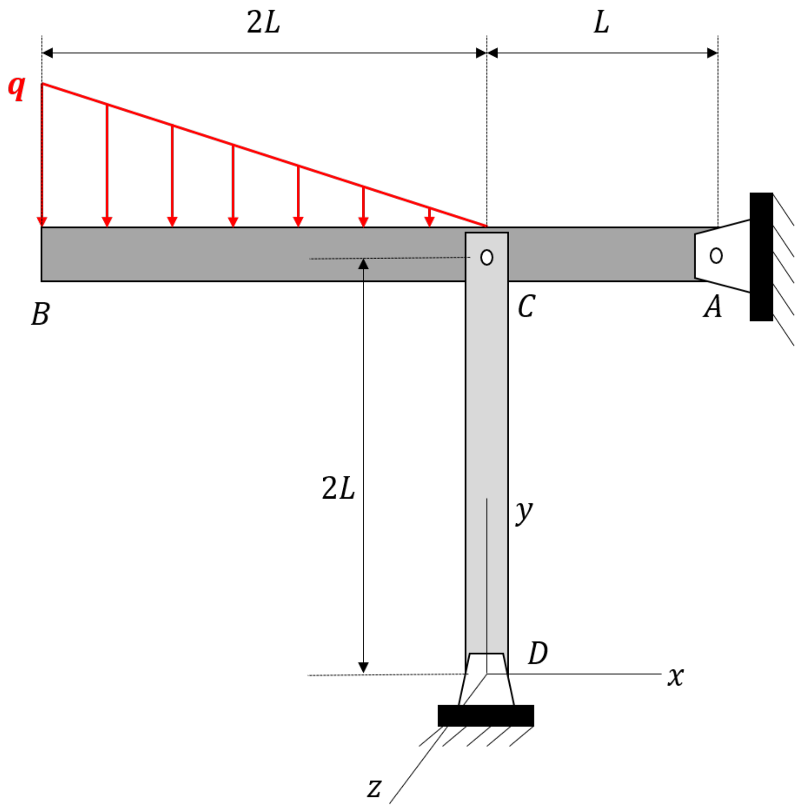


Fig. 11.3 (a)

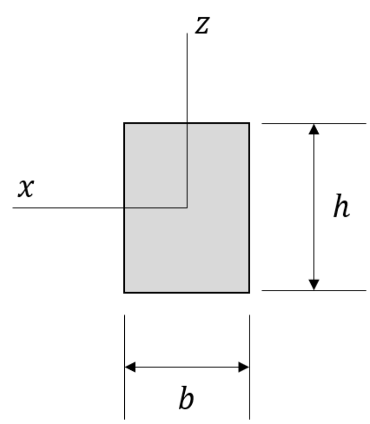
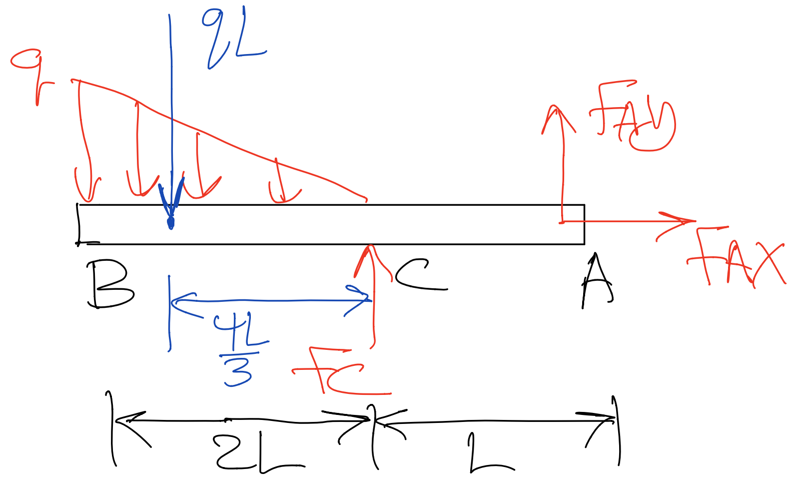


Fig. 11.3 (b)

Solution:

FBD



$$\circlearrowleft^{+} \sum M_A = 0$$

$$q\cancel{L}\left(\frac{7L}{3}\right) - F_C \cancel{L} = 0 \Rightarrow F_C = \frac{7}{3} qL$$

For buckling in x-y plane,

$$I_z = \frac{1}{12} h b^3$$

$$P_{cr} = \frac{\pi^2 E I_z}{(KL)^2} = \frac{\pi^2 \times (150 \times 10^9) \times \frac{1}{12} \times 0.045 \times (0.03)^3}{(0.7 \times 2 \times 1.5)^2}$$

$$= 33.989 \text{ KN}$$

and

$$P_C = \frac{7}{3} qL \Rightarrow q_{cr} = \frac{3 P_{cr}}{7L}$$

$$q_{cr} = 9.711 \text{ KN/m}$$

For buckling in x-z plane,

$$I_x = \frac{1}{12} b h^3$$

$$P_{cr} = \frac{\pi^2 E I_z}{(KL)^2} = \frac{\pi^2 \times (150 \times 10^9) \times \frac{1}{12} \times 0.03 \times (0.045)^3}{(0.5 \times 2 \times 1.5)^2}$$

$$= 149.894 \text{ KN}$$

$$P_{cr} = \frac{7}{3} qL \Rightarrow q_{cr} = \frac{3P_{cr}}{7L}$$

$$= 42.826 \text{ KN/m}$$

so, the maximum distributed load

$$q = 9.711 \text{ KN/m}$$

Problem 11.4 (5 points)

1. Consider the state of stress shown below in a *ductile* material. Let τ_{MSS} and τ_{MDE} be the values of the shear stress τ above that correspond to failure of the material using the maximum shear stress and maximum distortional energy theories, respectively. Circle the answer below that best describes the relative sizes of τ_{MSS} and τ_{MDE}. (2.5 points)

   (a) $\tau_{MSS} < \tau_{MDE}$
   (b) $\tau_{MSS} = \tau_{MDE}$
   (c) $\tau_{MSS} > \tau_{MDE}$

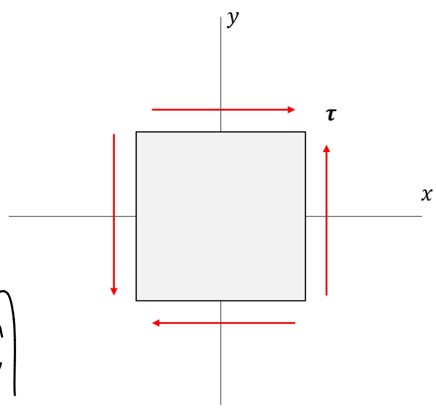


Fig. 11.4 (a)

$\sigma_{avg} = 0$

$R = \tau = |\tau_{max}|$

$\tau_{MSS} = \frac{\sigma_Y}{2}$

$= 0.5\sigma_Y$

$\sigma_1 = \tau_{MDE}$

$\sigma_2 = -\tau_{MDE}$

$\sigma_M = \sqrt{3}\,\tau_{MDE} = \sigma_Y$

$\tau_{MDE} = \frac{\sigma_Y}{\sqrt{3}}$

$= 0.577\sigma_Y$

2. Cylindrical columns A, B, C and D shown below are made of the same material (Young's modulus of E) and have the same circular cross-section of radius R. A compressive axial load P is applied to each column. The critical Euler's buckling loads P_{cr}^{A}, P_{cr}^{B}, P_{cr}^{C}, and P_{cr}^{D} for columns A, B, C, and D, respectively, are such that: (2.5 points)

   (a) $P_{cr}^{D} > P_{cr}^{C} > P_{cr}^{A} > P_{cr}^{B}$
   (b) $P_{cr}^{D} > P_{cr}^{A} > P_{cr}^{C} > P_{cr}^{B}$
   (c) $P_{cr}^{D} > P_{cr}^{B} > P_{cr}^{C} > P_{cr}^{A}$
   (d) $P_{cr}^{D} > P_{cr}^{C} > P_{cr}^{B} > P_{cr}^{A}$
   (e) None of the above.

(e in previous version)

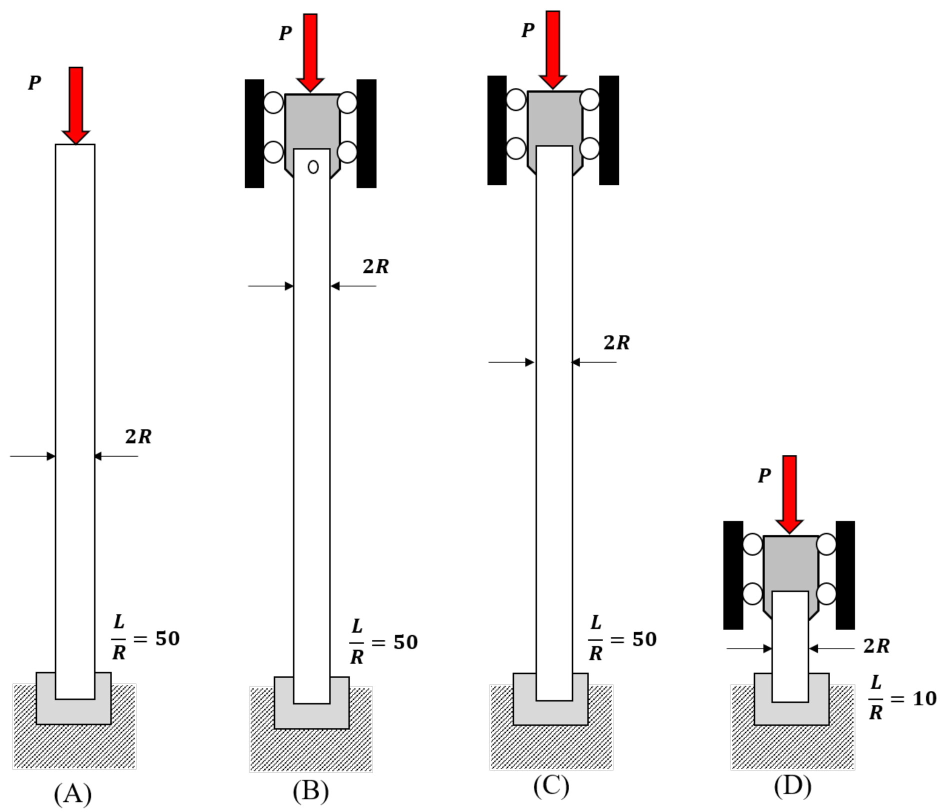


Fig. 11.4 (b)

Provide a written explanation for your answers.

$$P_{cr}^{A} = \frac{\pi^2 E\left(\frac{1}{4}\pi R^4\right)}{(2 \times 50R)^2} = \frac{\pi^2 E \frac{1}{4}\pi R^4}{10000\,R^2} = \frac{\pi^3 E R^2}{40000}$$

$$P_{cr}^{B} = \frac{\pi^2 E\left(\frac{1}{4}\pi R^4\right)}{(0.7 \times 50R)^2} = \frac{\pi^3 E R^2}{4900}$$

$$P_{cr}^{C} = \frac{\pi^2 E\left(\frac{1}{4}\pi R^4\right)}{(0.5 \times 50R)^2} = \frac{\pi^3 E R^2}{2500}$$

$$P_{cr}^{D} = \frac{\pi^2 E\left(\frac{1}{4}\pi R^4\right)}{(0.5 \times 10R)^2} = \frac{\pi^3 E R^2}{100}$$