ME 323 – Mechanics of Materials
Examination #2
November 5th, 2020

PURDUE UNIVERSITY | Mechanical Engineering

Name (Print) ____
(Last) **(First)**

PROBLEM # 1 (25 points)

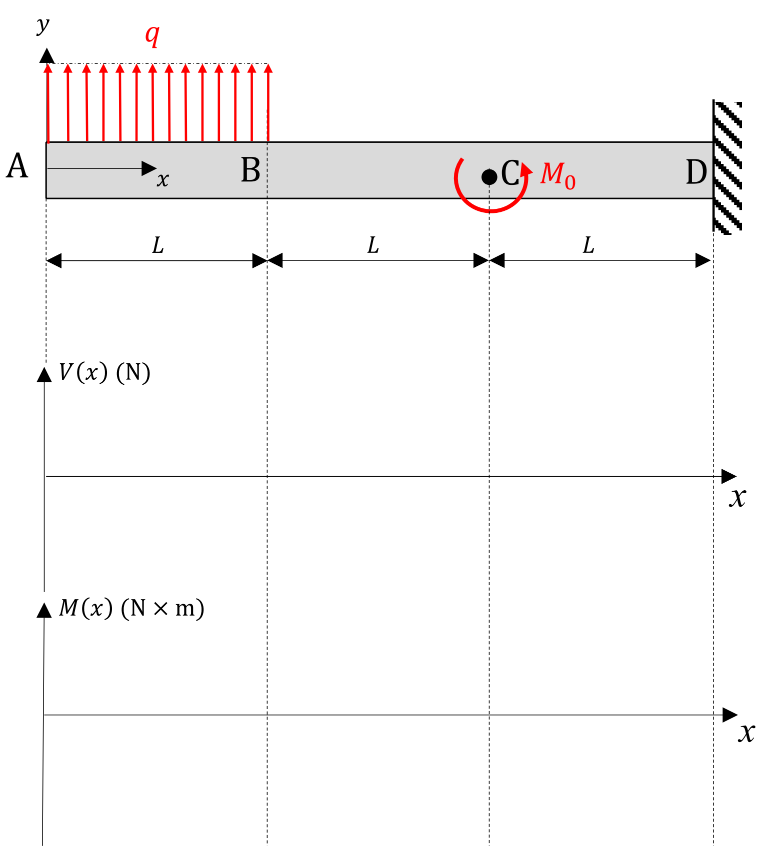
A cantilever beam ABCD of the length $3L$ is fixed to the end at D. The beam is subject to a constant distributed load q (N/m) over the section AB, and a moment M_0 of the magnitude $4qL^2$ (N×m) at C. The cross section of the beam has a triangular shape, as shown below.

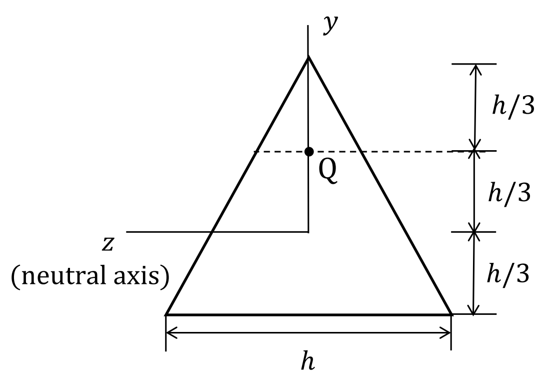
The second area moment for a triangle is $I_z = \frac{1}{36}h^4$.

a) Draw the shear force and bending moment diagrams. Mark the values at the sections A, B, C, and D.

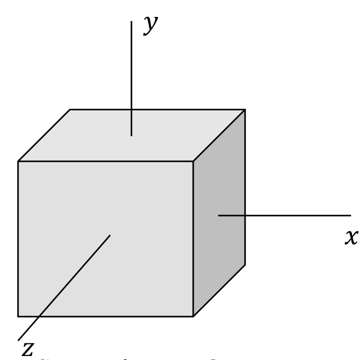
b) Determine the maximum compressive flexural stress (largest magnitude) and the maximum tensile flexural stress along the beam.

c) Determine the stress state at the point Q which is located on the cross section B. Sketch the stress state on the given stress element.





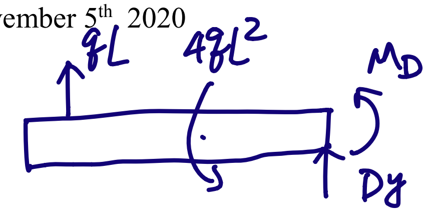
Cross section



Stress element Q

PURDUE UNIVERSITY | Mechanical Engineering

FBD:

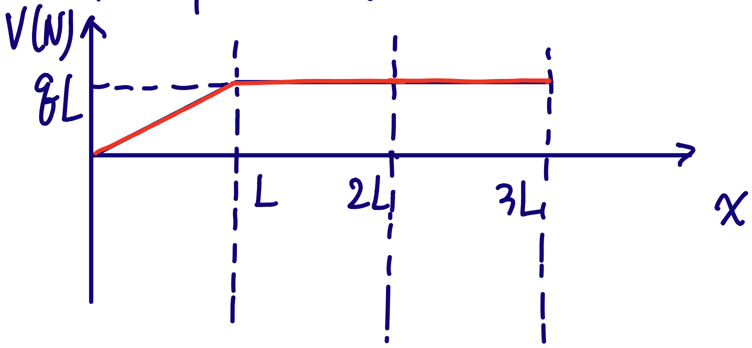


$$\Sigma F_y = qL + D_y = 0$$

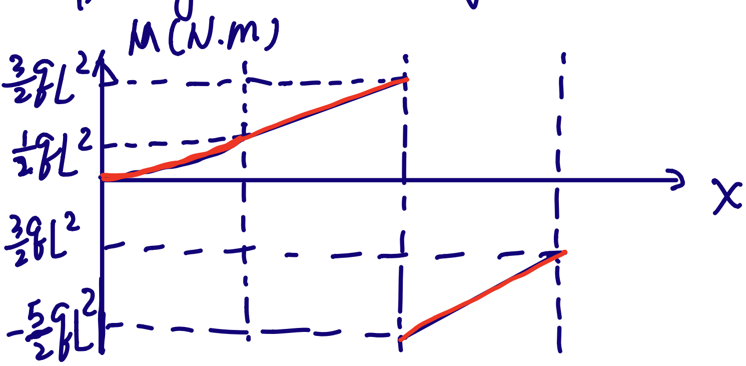
$$\Sigma M_D = -qL\cdot\frac{5}{2}L + 4qL^2 + M_D = 0$$

$$D_y = -qL,\quad M_D = -\frac{3}{2}qL^2$$

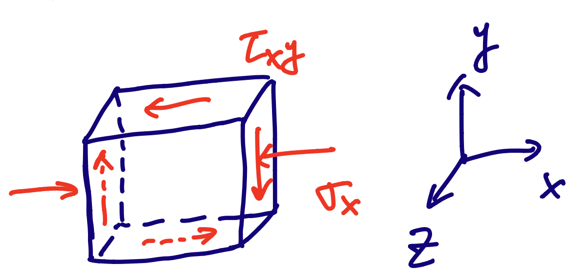
shear force diagram:



Bending moment diagram:



stress element Q:



(b): Maximum Compressive σ_x:

$$|(\sigma_x)_{max,c}| = \frac{\frac{3}{2}qL^2\cdot\frac{2}{3}h}{\frac{1}{36}h^4} = 36qL^2/h^3$$

Maximum tensile σ_x:

$$(\sigma_x)_{max,t} = \frac{\frac{5}{2}qL^2\cdot\frac{2}{3}h}{\frac{1}{36}h^4} = 60qL^2/h^3$$

(c): At Q:

$$\sigma_x = -\frac{My}{I} = -\frac{\frac{1}{2}qL^2\cdot\frac{h}{3}}{\frac{1}{36}h^4} = -6q^2/h^3$$

$$\tau = \frac{VQ}{It} = \frac{V\cdot(A^*\bar{y}^*)}{It}$$

$$V = qL,\quad A^* = \frac{1}{2}\cdot\frac{h}{3}\cdot\frac{h}{3} = \frac{h^2}{18}$$

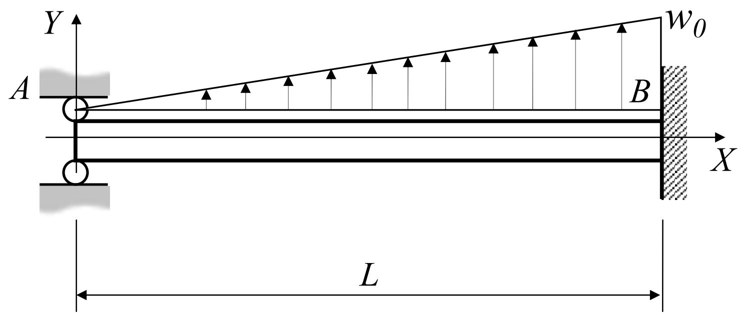
$$\bar{y}^* = \frac{1}{3}\cdot\frac{h}{3} + \frac{h}{3} = \frac{4}{9}h,\quad t = \frac{h}{3}$$

$$\tau = \frac{qL\cdot\frac{h^2}{18}\cdot\frac{4}{9}h}{\frac{1}{36}h^4\cdot\frac{1}{3}h} = \frac{8}{3}\frac{qL}{h^2}$$

(negative τ_{xy})

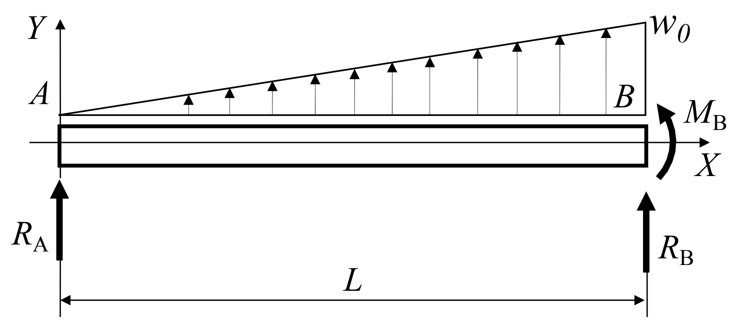
Name (Print) ____
(Last) **(First)**

PROBLEM # 2 (25 points)



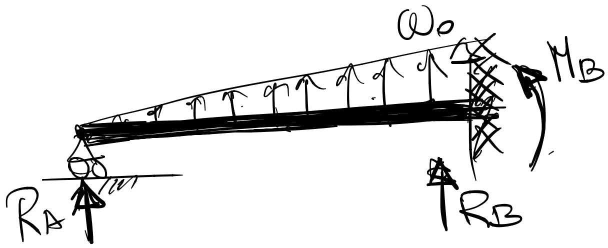
The linearly elastic beam shown in the figure supports a linearly-distributed load with maximum intensity w_0 at end B. The beam is homogeneous, with Young's modulus E, and has constant cross-section, with moment of inertia I.

(a) Using the following free body diagram, write the equations of equilibrium and identify whether the structure is statically determinate or indeterminate.



Using the second-order integration method:

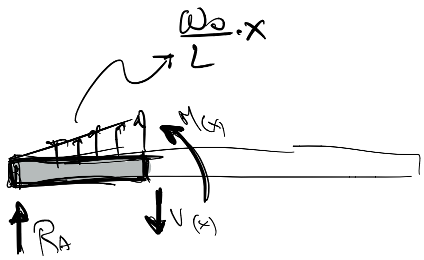
(b) Determine the bending moment $M(x)$ of the beam (as a function of the reaction at A, the external loads and the geometric parameters).
(c) Determine the slope $v'(x)$ and deflection $v(x)$ of the beam.
(d) Indicate the boundary conditions at supports A and B.
(e) Solve for the reaction at A, i.e., R_A.



$$\oplus\uparrow \quad \sum F_y = 0 = R_A + R_B + \tfrac{1}{2}\omega_0 L$$

$$\oplus\circlearrowright \quad \left(\sum M\right)_B = 0 = M_B + R_A \cdot L + \frac{\omega_0 L}{2}\cdot\frac{L}{3}$$

2 equations & 3 unknowns (R_A, R_B, M_B)
$\Rightarrow$ Statically Indeterminate



$$\Rightarrow M(x) = R_A x + \frac{\omega_0 x^2}{L}\cdot\frac{1}{2}\cdot\frac{x}{3}$$

Second order integration method

$$EI v'' = R_A x + \frac{1}{6L}\omega_0 x^3$$

$$EI v' = \frac{1}{2}R_A x^2 + \frac{1}{24L}\omega_0 x^4 + C_1$$

$$EI v = \frac{1}{6}R_A x^3 + \frac{1}{120L}\omega_0 x^5 + C_1 x + C_2$$

Boundary Conditions

$v(0) = 0 \Rightarrow C_2 = 0$

$v(L) = 0 \Rightarrow \frac{1}{6}R_A L^3 + \frac{\omega_0}{120}L^4 + C_1 L = 0$

$v'(L) = 0 \Rightarrow \frac{1}{2}R_A L^2 + \frac{1}{24}\omega_0 L^3 + C_1 = 0$

Solve for R_A

$$\begin{cases} \frac{1}{6}R_A L^2 + \frac{\omega_0 L^3}{120} + C_1 = 0 \\ \frac{1}{2}R_A L^2 + \frac{1}{24}\omega_0 L^3 + C_1 = 0 \end{cases}$$

2x2 system of equations

$$\left(\frac{1}{2} - \frac{1}{6}\right)R_A L^2 + \left(\frac{1}{24} - \frac{1}{120}\right)\omega_0 L^3 = 0 \Rightarrow$$

$$\boxed{R_A = -\frac{1}{30}\omega_0 L \Big/ \tfrac{1}{3} = -\frac{\omega_0 L}{10}}$$

ME 323 – Mechanics of Materials
Examination #1
November 5th, 2020

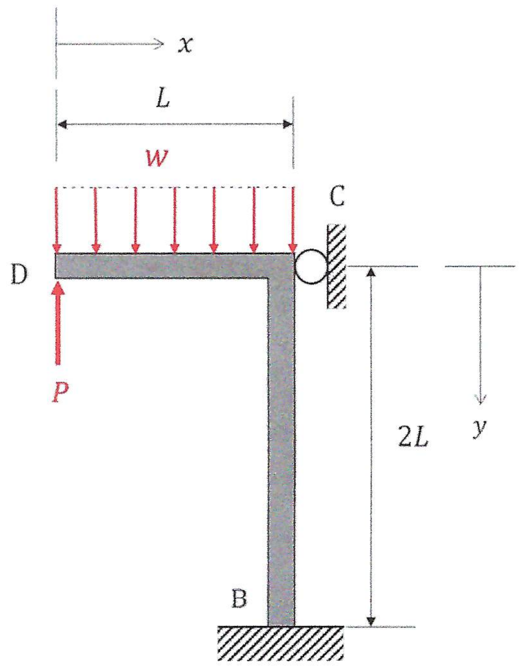
PURDUE UNIVERSITY® | Mechanical Engineering

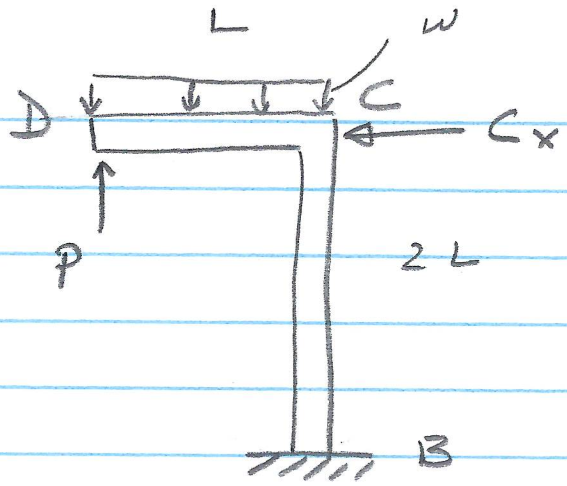
Name (Print) SOLUTION
(Last) **(First)**

PROBLEM # 3 (25 points)

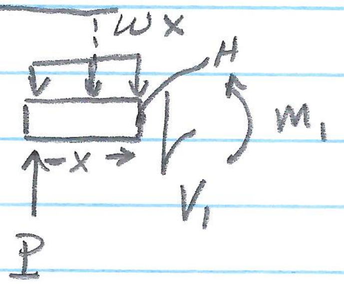
A cantilevered structure BCD is subjected to a vertical distributed load w (in the unit of load/length) between C and D and a force P at point D. The structure is made of a material with elastic modulus E, second moment of area I and cross-sectional area A. Assuming **the shear strain energy due to bending is negligible, use Castigliano's theorem** to determine:

1) the reaction at point C
2) the vertical (y-direction) deflection of point D





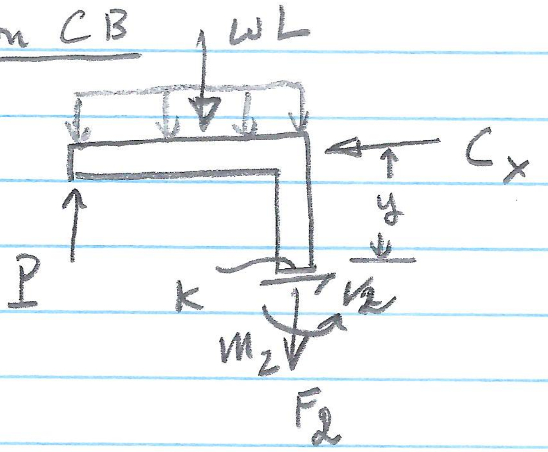
Section DC



$$\Sigma M_H = m_1 + wx\left(\frac{x}{2}\right) - Px = 0$$

$$\boxed{m_1(x) = Px - w\frac{x^2}{2}}$$

Section CB



$$\Sigma F_y = -F_2 - wL + P = 0$$

$$\boxed{F_2(y) = P - wL}$$

$$\Sigma M_K = M_2 + C_x y + wL\left(\frac{L}{2}\right) - PL = 0$$

$$\boxed{m_2(y) = PL - w\frac{L^2}{2} - C_x y}$$

$$U = U_1 + U_2 = \frac{1}{2EI}\int_0^L m_1^2\,dx + \frac{1}{2EI}\int_0^{2L} m_2^2\,dy$$

$$+ \frac{1}{2EA}\int_0^{2L} F_2^2\,dy$$

Deflection at C is zero:

$$\delta_c = \frac{\partial U}{\partial C_x} = \frac{1}{\cancel{2}EI}\int_0^L \cancel{2}m_1 \cancelto{0}{\frac{\partial m_1}{\partial C_x}}\,dx + \frac{1}{\cancel{2}EI}\int_0^{2L} \cancel{2}m_2 \frac{\partial m_2}{\partial C_x}\,dy$$

$$+ \frac{1}{\cancel{2}EA}\int_0^{2L} \cancel{2}F_2 \cancelto{0}{\frac{\partial F_2}{\partial C_x}}\,dy = 0$$

$$\frac{\partial m_1}{\partial C_x} = 0 \qquad \frac{\partial m_2}{\partial C_x} = -y \qquad \frac{\partial F_2}{\partial C_x} = 0$$

$$\delta_c = \int_0^{2L}\left[\left(PL - w\frac{L^2}{2}\right) - C_x y\right](-y)\,dy$$

$$= -\left(PL - w\frac{L^2}{2}\right)\frac{y^2}{2}\Bigg|_0^{2L} + C_x\frac{y^3}{3}\Bigg|_0^{2L} = 0$$

$$-\left(PL - \frac{wL^2}{2}\right)\frac{\overset{2}{\cancel{4}}\cancel{L^2}^{\,L}}{\cancel{2}} + C_x\frac{\overset{4}{\cancel{8}}L^{\cancel{3}}}{3} = 0$$

$$\boxed{C_x = \frac{3}{4}\left(P - \frac{wL}{2}\right)}$$

$$\delta_D = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^L M_1 \frac{\partial M_1}{\partial P}\,dx + \frac{1}{EI}\int_0^{2L} M_2 \frac{\partial M_2}{\partial P}\,dy$$

$$+ \frac{1}{EA}\int_0^{2L} F_2 \frac{\partial F_2}{\partial P}\,dy$$

Replace C_x in M_2

$$\boxed{M_2(y) = PL - w\frac{L^2}{2} - \left[\frac{3}{4}\left(P - \frac{wL}{2}\right)\right] y}$$

$$\frac{\partial M_1}{\partial P} = x \qquad \frac{\partial M_2}{\partial P} = \left(L - \frac{3}{4}y\right) \qquad \frac{\partial F_2}{\partial P} = 1$$

$$\delta_D = \frac{1}{EI}\int_0^L \left(Px - \frac{wx^2}{2}\right) x\,dx$$

$$+ \frac{1}{EI}\int_0^{2L} \left[\underbrace{\left(PL - \frac{wL^2}{2}\right)}_{A} - \underbrace{\frac{3}{4}\left(P - \frac{wL}{2}\right)}_{B} y\right]\left(L - \frac{3}{4}y\right) dy$$

$$+ \frac{1}{EA}\int_0^{2L} (P - wL)\,dy$$

$$\delta_D = \frac{1}{EI}\left[\frac{Px^3}{3}\Big|_0^L - w\frac{x^4}{8}\Big|_0^L \right]$$

$$+ \frac{1}{EI}\left[ALy\Big|_0^{2L} - \frac{3}{4}A\frac{y^2}{2}\Big|_0^{2L} - BL\frac{y^2}{2}\Big|_0^{2L} \right.$$

$$\left. + \frac{3}{4}B\frac{y^3}{3}\Big|_0^{2L} \right]$$

$$+ \frac{1}{EA}(P - wL)\,y\Big|_0^{2L}$$

$$= \frac{1}{EI}\left[\frac{PL^3}{3} - \frac{wL^4}{8} \right]$$

$$+ \frac{1}{EI}\left[\underbrace{AL(2L) - \frac{3}{4}A\frac{4L^2}{2}}_{S} \overbrace{- BL\frac{\cancel{4}^{2}L^2}{\cancel{2}} + \frac{\cancel{3}}{\cancel{4}}B\frac{\cancel{8}^{2}L^3}{\cancel{3}}}^{0} \right]$$

$$+ \frac{2L}{EA}(P - wL)$$

$$S = \left(PL - \frac{wL^2}{2}\right)\overbrace{\left(\frac{\cancel{8}^{16}L^2}{8} - \frac{12L^2}{8}\right)}^{L^2/2} - \frac{3}{4}\left(P - \frac{wL}{2}\right)\Big($$

$$= \frac{L^3}{2}\left(P - \frac{wL}{2}\right)$$

$$\delta_D = \frac{L^3}{EI}\left[\frac{P}{3} - \frac{wL}{8}\right] + \frac{L^3}{2EI}\left(P - \frac{wL}{2}\right) + \frac{2L}{EA}\left(P - wL\right)$$

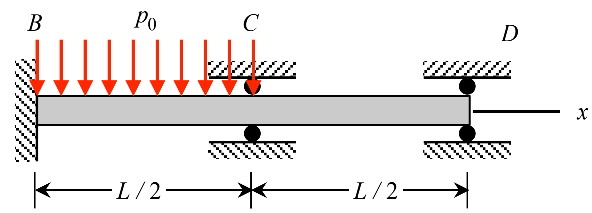
$$= \frac{L^3}{EI}\left[\frac{P}{3} + \frac{P}{2} - \frac{wL}{8} - \frac{wL}{4}\right]$$

$$\boxed{\delta_D = \frac{L^3}{2EI}\left(\frac{5P}{3} - \frac{3wL}{4}\right) + \frac{2L}{EA}\left(P - wL\right)}$$

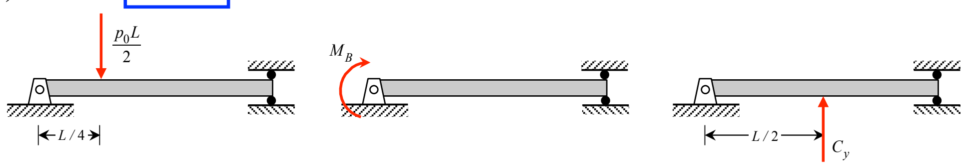
PROBLEM #4 (25 Points)

Part A – 5 points

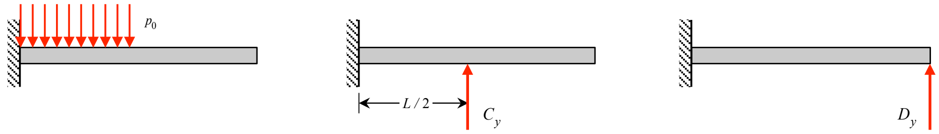
Consider the beam shown to the right. The superposition method is to be used to determine the reactions on the beam at locations C and D. Consider the following *True/False* questions regarding whether the loadings provided can be used in this analysis. No justification is needed for your answers.



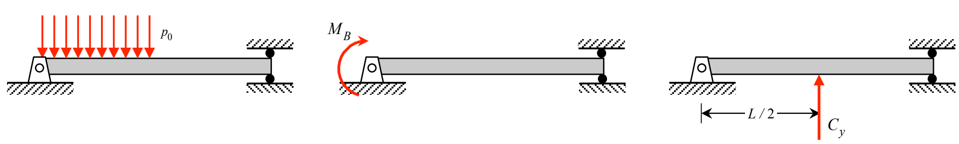
a) *TRUE* $\boxed{\textit{FALSE}}$ Cannot replace distributed loads by single-force equivalents when finding displacements.



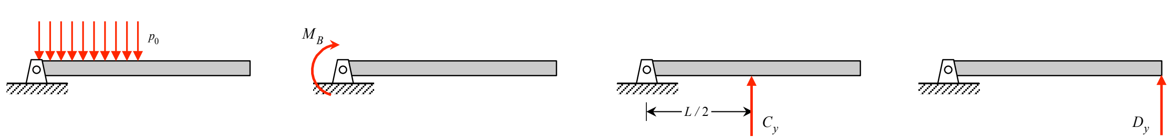
b) $\boxed{\textit{TRUE}}$ *FALSE*



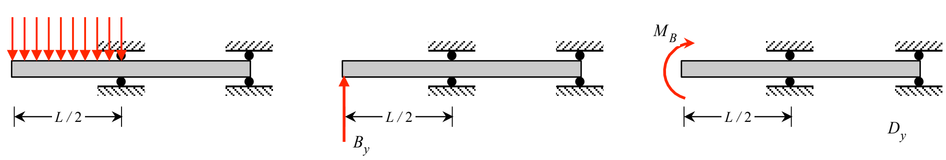
c) $\boxed{\textit{TRUE}}$ *FALSE*



d) *TRUE* $\boxed{\textit{FALSE}}$ Replacement "structures" cannot be in equilibrium.



e) $\boxed{\textit{TRUE}}$ *FALSE*



PROBLEM #4 ***(continued)***

Part B **– 8 points**

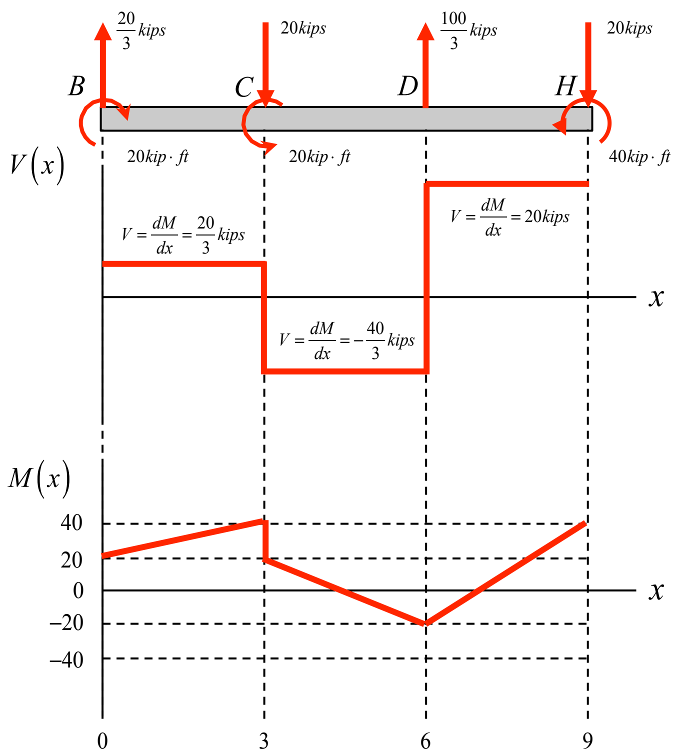
Neither the loading nor the boundary conditions on the beam shown below are provided in the figure. The bending moment diagram $M(x)$ for the beam is given below the beam, with $M(x)$ and x being provided in terms $kip \cdot ft$ and ft, respectively

For the bending moment diagram provided:

a) Draw the shear force diagram $V(x)$ on the axes provided.

b) Show the loading on the beam in the figure below.

No justification is needed for your answers.



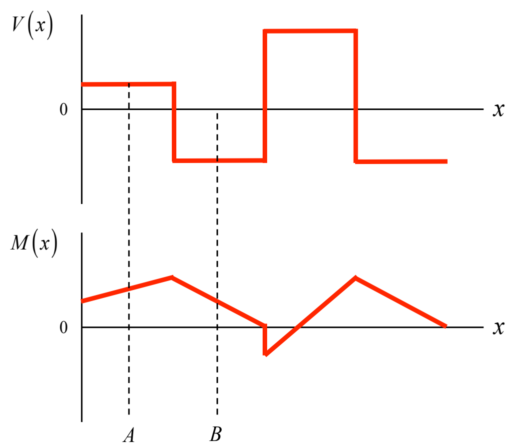
PROBLEM #4 ***(continued)***

Part C **– 8 points**

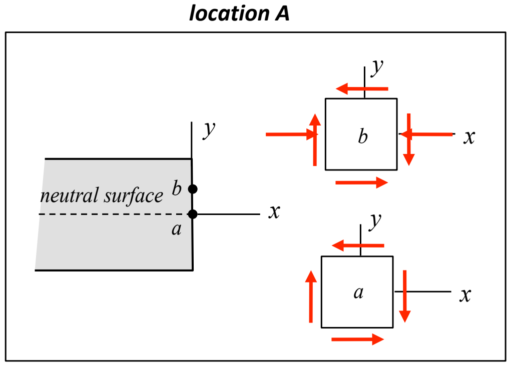
The shear force and bending moment diagrams for a loaded beam are shown below.

a) For location A along the length of the beam, show the *directions* of the normal and shear components of stress for points a and b on that cross-section on the stress elements provided.

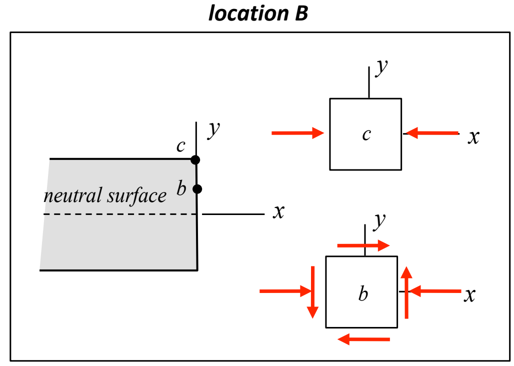
b) For location B along the length of the beam, show the *directions* of the normal and shear components of stress for points b and c on that cross-section on the stress elements provided.



location A



location B



At A:	At B:
M > 0 and V > 0.	M > 0 and V < 0.
Therefore:	Therefore:
$\sigma_a = 0$ and $\sigma_b < 0$	$\sigma_b < 0$ and $\sigma_c < 0$

PROBLEM #4 *(continued)*

***Part D* – 4 points**

The cross-sections for I-Beam 1 and Box-Beam 2 are shown below. Note that the wall thickness of the box beam is a constant value of b around its perimeter. Let I_1 and I_2 represent the centroidal second area moment (about the z-axis) for Beams 1 and 2, respectively. Each beam is experiencing the same shear force of V at the cross section. Let τ_{1B} and τ_{2B} be the shear stress at points B on Beams 1 and 2, respectively.

a) Circle the correct answer below in regard to the relative sizes of I_1 and I_2. You are *not* asked to provide numerical values for these second area moments, or justification for your answers.

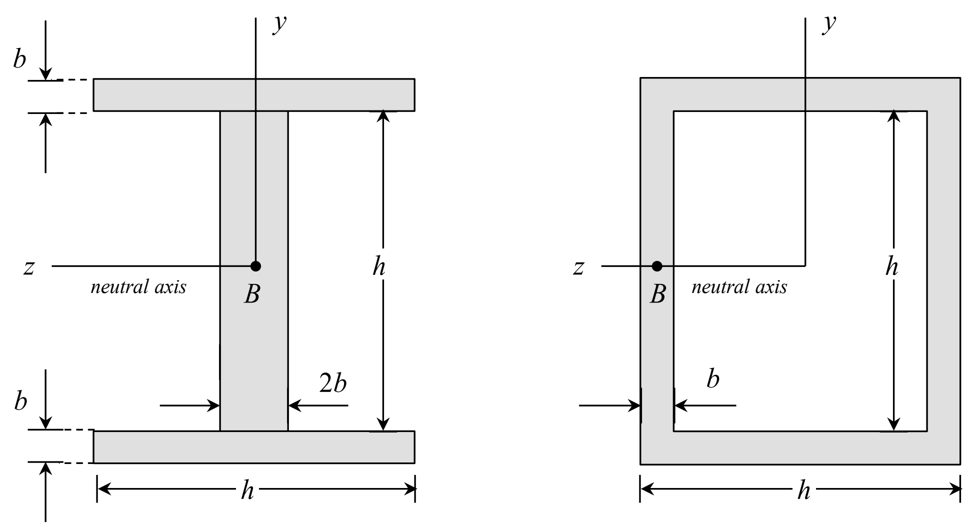
- $I_1 > I_2$
- $\boxed{I_1 = I_2}$
- $I_1 < I_2$

The vertical sections (the "webs") of the two cross-sections have the same second area moments. Likewise, the horizontal sections (the "flanges") have the same second area moments. Therefore, the second area moments are the same.

b) Circle the correct answer below in regard to the relative sizes of $|\tau_{1B}|$ and $|\tau_{2B}|$. You are *not* asked to provide numerical values for these stresses, or justification for your answers.

- $|\tau_{1B}| > |\tau_{2B}|$
- $\boxed{|\tau_{1B}| = |\tau_{2B}|}$
- $|\tau_{1B}| < |\tau_{2B}|$

The first area moments (Q) for the webs are the same, as well as for the flanges. The "thickness" of the cross sections are the same (2b). Therefore, the shear stresses are the same.



Beam 1 **Beam 2**