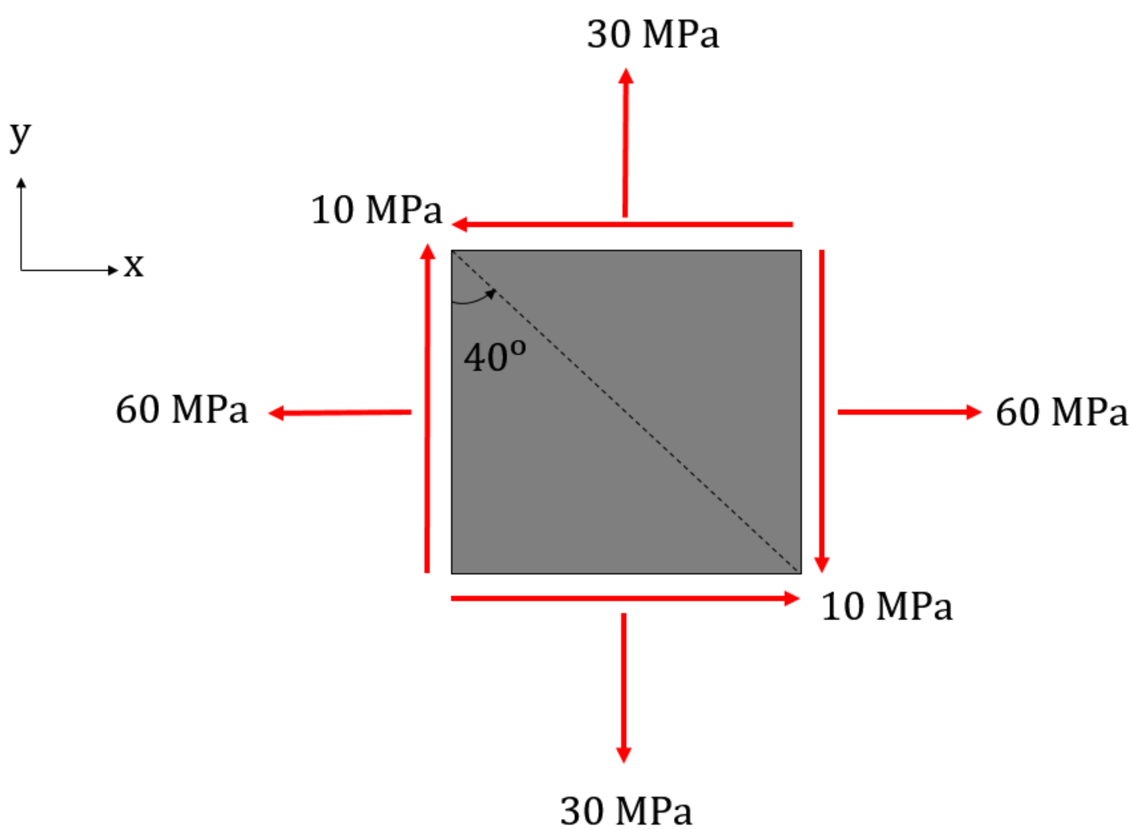
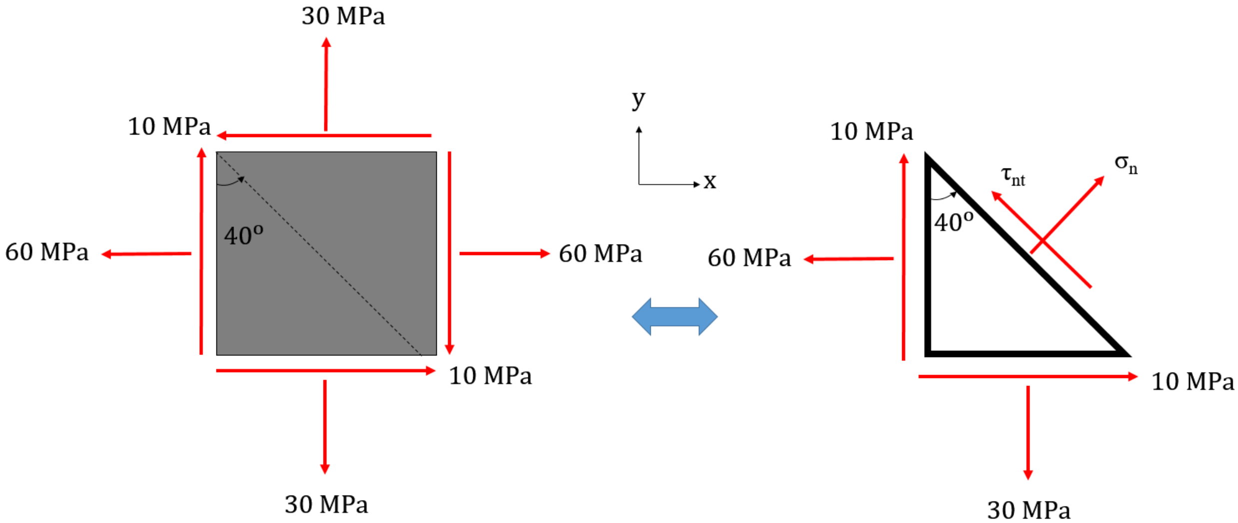
Problem 11.1 (10 points)

For the state of plane stress shown in the figure:

1. Draw the Mohr's circle and indicate the points that represent stresses on face *X* and on face *Y*.
2. Using the Mohr's circle, determine the normal and shear stress on the inclined plane shown in the figure and label this point as *N* on the Mohr's circle.



Solution:



The give state of plane stress has the following stresses:

$$\sigma_x = 60\ MPa$$

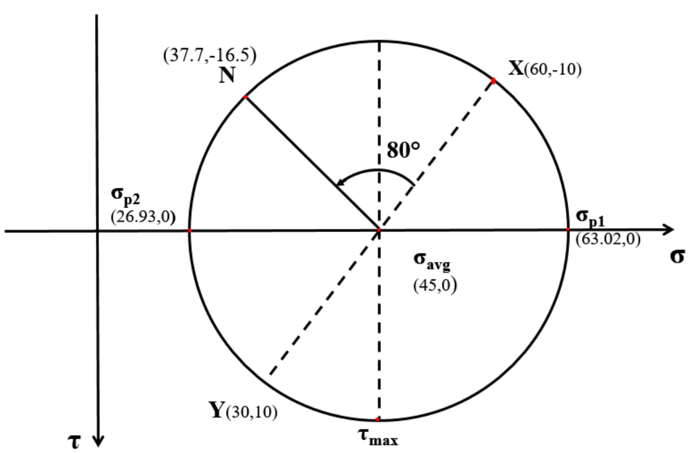
$$\sigma_y = 30\ MPa$$

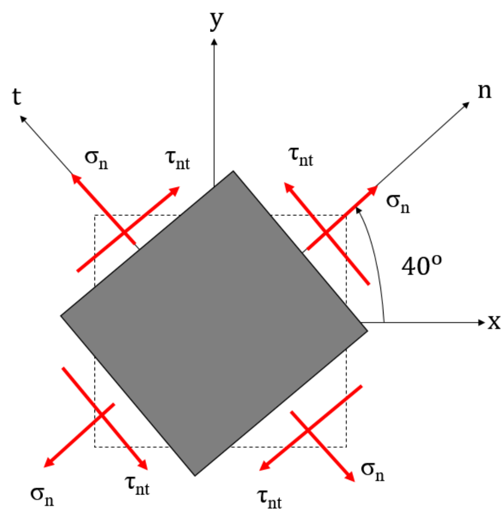
$$\tau_{xy} = -10\ MPa$$

To find the center of the Mohr's circle we find σ_{avg},

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = 45MPa$$

<u>**Mohr's circle**</u>**:**





The rotation of the inclined plane is = 40^o (C. C. W), the point 'N' on the Mohr's circle will be at an angle of 80^o (C. C. W) from point 'X'.

Coordinates of point N and the normal and shear stresses on the inclined plane are as follows:

Shear Stress: $\tau_{nt} = -16.5$ MPa

Normal Stress: $\sigma_n = 37.75$ MPa

Note: The rotation considered here is +40^o, however a rotation of $-50^o$is also valid (in this case the 'n' and 't' axis would be swapped.

Problem 11.2 (10 points)

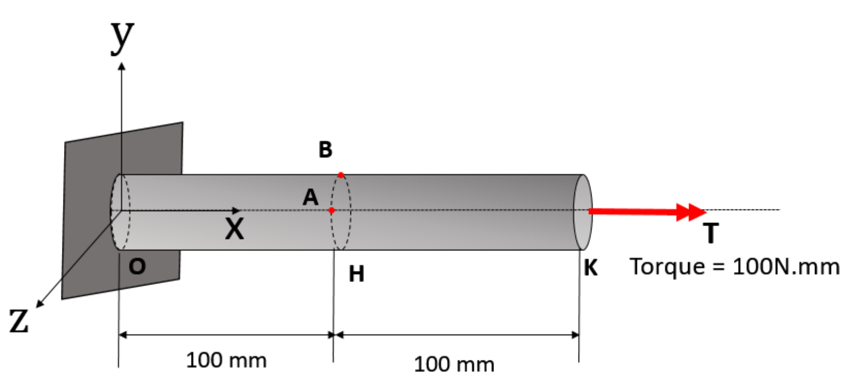
For the loading conditions shown in cases (a) – (b):

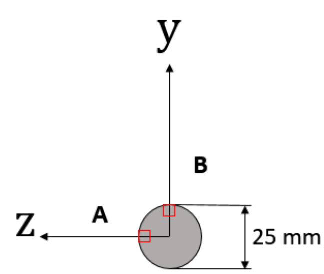
1. Determine the state of stress at points A and B
2. Represent the state of stress at points A and B in three-dimensional differential stress elements.

Using the Mohr's circle, determine:

3. The principal stresses and principal angles for the states of stress at A and B.
   <u>Note</u>: Identify first which is the plane corresponding to the state of plane stress (namely, xy-plane, xz-plane or yz-plane) for each point and loading condition.
4. The maximum in-plane shear stresses at points A and B.
5. The absolute maximum shear stress at points A and B.

Case (a):

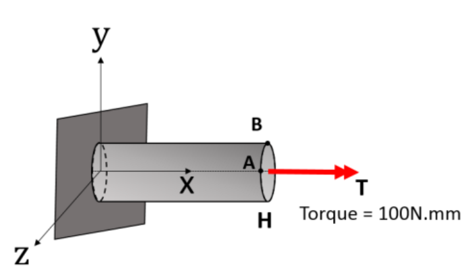




Cross-section at H

<u>Solution:</u> Case (a)

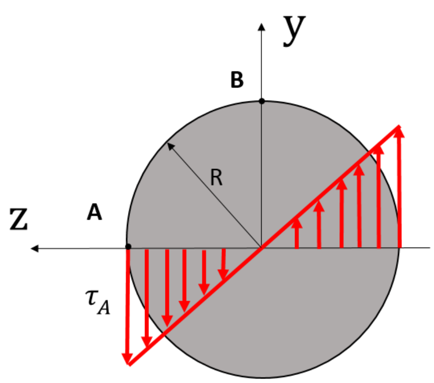
Making a cut at point H:



Internal resultant forces include only the torque.

POINT A

Stress distribution at point A:



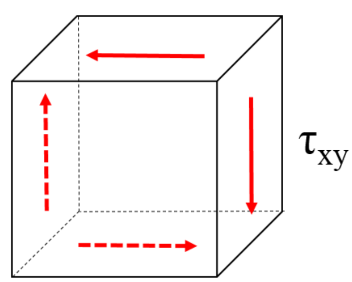
$$\tau_A = \frac{TR}{I_P} = \text{linear in radial position}$$

I_p = polar moment of area

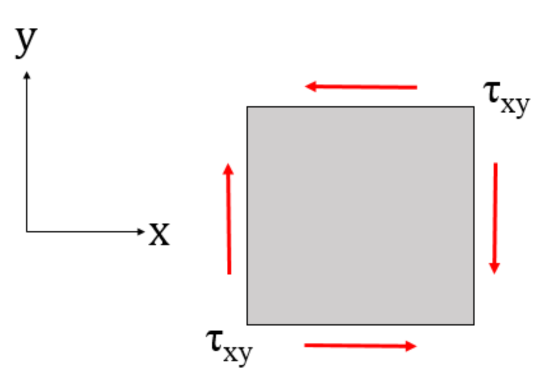
$$\tau_A = \frac{100Nm \times 12.5mm}{\frac{\pi}{32} \times 25^4 mm^4} = 0.03259 \frac{N}{mm^2}$$

There are no normal stresses acting on the point A, $\sigma_x = 0,\ \sigma_y = 0$ and the only shear stress acting is in the xy plane, $\tau_{xy} = 32.59\ \text{kPa}$

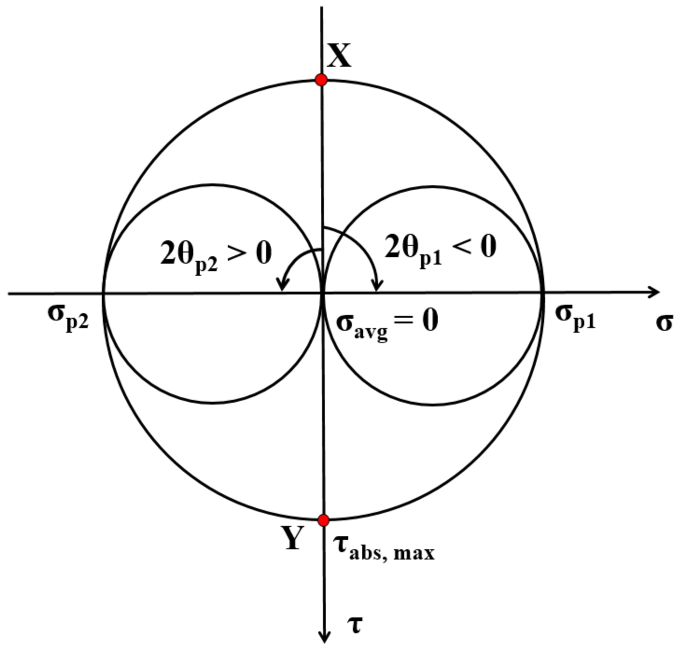
Three-dimensional differential stress element at A:



Since, $\sigma_z = 0,\ \tau_{yz} = \tau_{xz} = 0$, the xy plane is the plane corresponding to the state of plane stress.



Mohr's Circle:



Principal stress: $\sigma_{p_1} = 32.59 \text{ kPa}, \ \sigma_{p_2} = -32.59 \text{ kPa}$

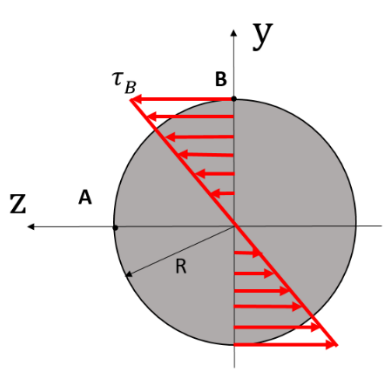
Principal angle: = $\theta_{p_1} = -45°, \theta_{p_2} = 45°$

Maximum in plane shear stresses: $\tau_{inplane,max} = 32.59 \text{ kPa}$

Absolute shear stress: $\tau_{max,abs} = 32.59 \text{ kPa}$

POINT B

Stress distribution at point B:



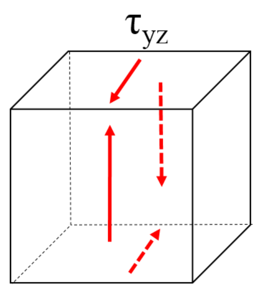
$$\tau_B = \frac{TR}{I_P} = \text{linear in radial position}$$

I_p = polar moment of area

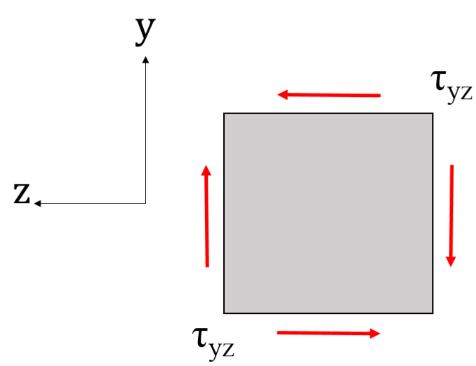
$$\tau_B = \frac{100\text{Nm} \times 12.5\text{mm}}{\frac{\pi}{32} \times 25^4\text{mm}^4} = 0.03259 \frac{\text{N}}{\text{mm}^2}$$

There are no normal stresses acting on the point B, $\sigma_x = 0$, $\sigma_y = 0$ and the only shear stress acting is in the xy plane, $\tau_{xy} = 32.59 \text{ kPa}$

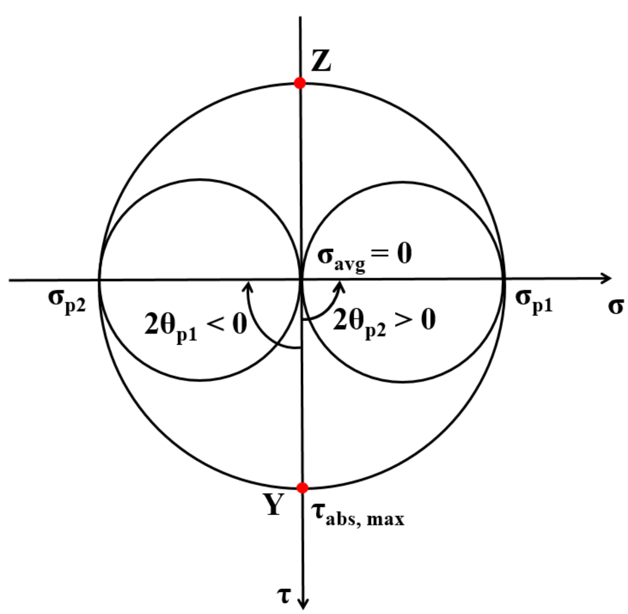
Three-dimensional differential stress element at B:



Since, $\sigma_z = 0$, $\tau_{yx} = \tau_{xz} = 0$, the yz plane is the plane corresponding to the state of plane stress.



Mohr's Circle:



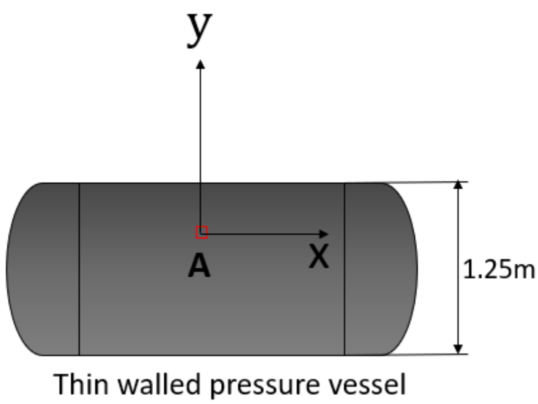
Principal stress: $\sigma_{p_1} = 32.59 \text{ kPa}, \ \sigma_{p_2} = -32.59 \text{ kPa}$

Principal angle: = $\theta_{p_1} = +45°, \theta_{p_2} = -45°$

Maximum in plane shear stresses: $\tau_{inplane,max} = 32.59 \text{ kPa}$

Absolute shear stress: $\tau_{max,abs} = 32.59 \text{ kPa}$

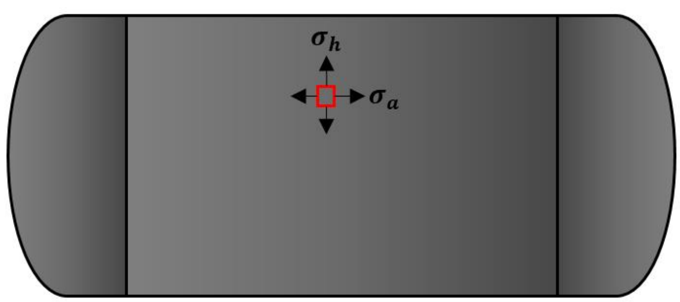
Case (b):



Internal pressure = 3MPa
Wall thickness(***t***) = 15mm

Notice that there is no point B for this loading condition

The element A will only experience hoop and axial stresses

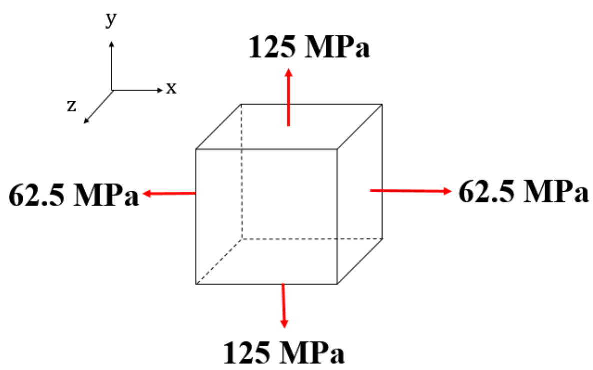


$$\text{Pressure} = \text{P} = 3 \times 10^6 \text{ Pa}, \text{thickness} = \text{t} = 15 \times 10^{-3} \text{ m}, \text{radius} = \text{r} = 1.25\text{m}$$

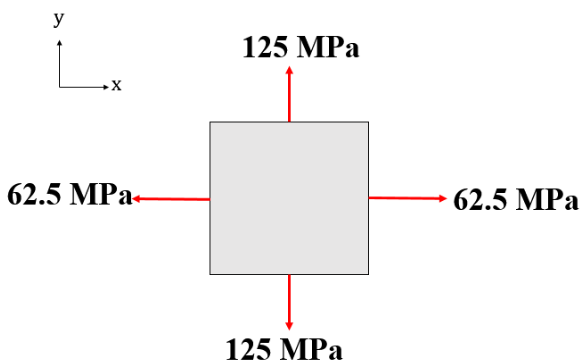
Axial stress = $\sigma_a = \frac{pr}{2t} = \frac{3\times10^6\times\frac{1.25}{2}}{2\times15\times10^{-3}} = 62.5 \text{ MPa} = 62.5 \times 10^6 \text{ Pa}$

Hoop stress = $\sigma_h = \frac{pr}{t} = \frac{3\times10^6\times\frac{1.25}{2}}{15\times10^{-3}} = 125 \text{ MPa} = 125 \times 10^6 \text{ Pa}$

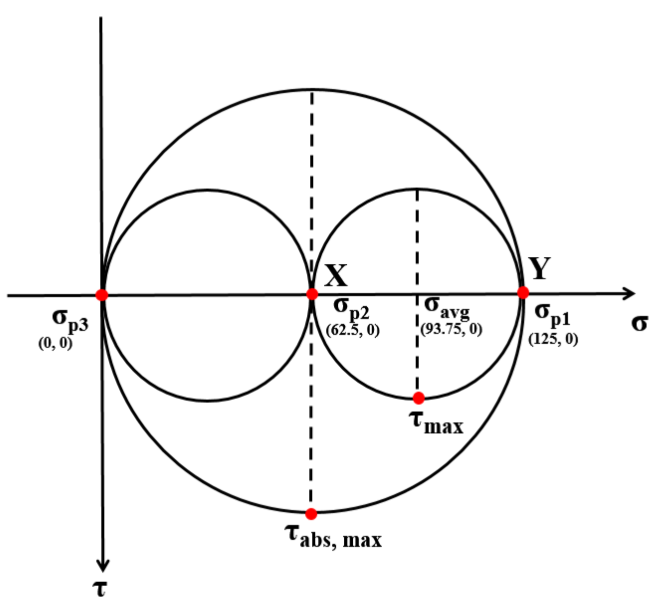
Three-dimensional differential stress element at A:



Since, $\sigma_z = 0, \ \tau_{yz} = \tau_{xz} = 0$, the xy plane is the plane corresponding to the state of plane stress.



Mohr's Circle:



$$\sigma_{avg} = \frac{125 + 62.5}{2} = 93.75\ MPa$$

Principal stress: $\sigma_{p_1} = 125$ MPa, $\sigma_{p_2} = 62.5$ MPa, $\sigma_3 = 0$ MPa

Principal angle: = $\theta_{p_1} = 90°$, $\theta_{p_2} = 0°$

Maximum in plane shear stresses: $\tau_{inplane,max} = 31.25$ MPa

Absolute shear stress: $\tau_{max,abs} = 62.5$ MPa

Problem 11.3 (10 points)

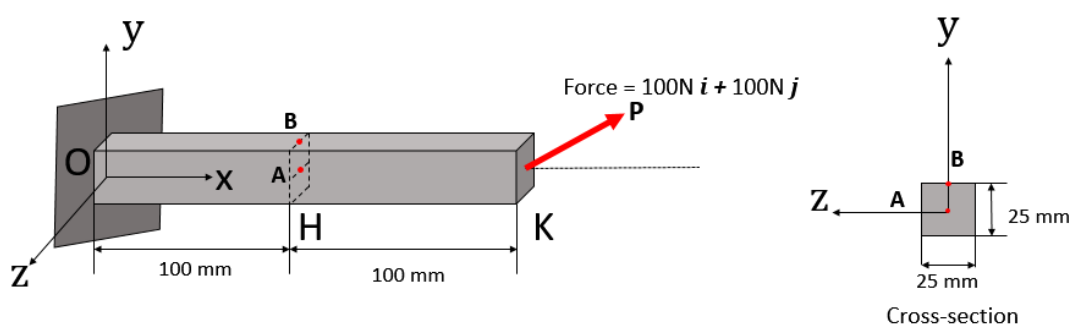
For the loading conditions shown in cases (c) – (d):

1. Determine the state of stress at points A and B
2. Represent the state of stress at points A and B in three-dimensional differential stress elements.

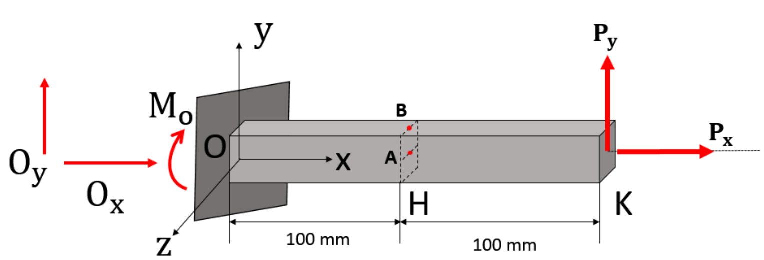
Using the Mohr's circle, determine:

3. The principal stresses and principal angles for the states of stress at A and B.
   <u>Note</u>: identify first which is the plane corresponding to the state of plane stress (namely, xy-plane, xz-plane or yz-plane) for each point and loading condition.
4. The maximum in-plane shear stresses at points A and B.
5. The absolute maximum shear stress at points A and B.

Case (c):



<u>FBD:</u>

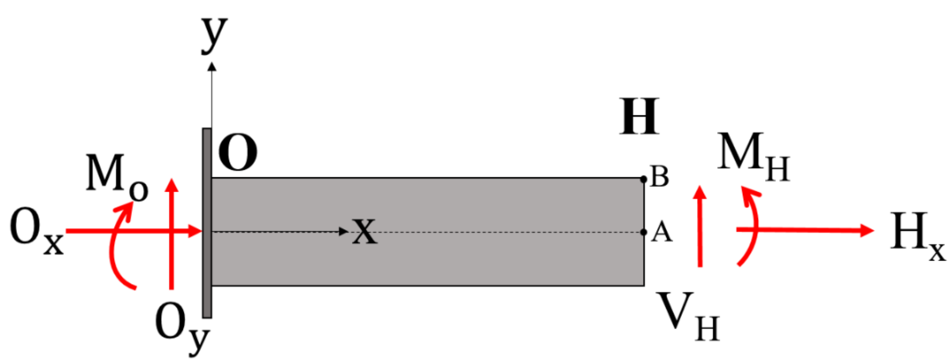


$$O_x = -P_x = -100 \text{ N}$$

$$O_x = -P_x = -100 \text{ N}$$

$$M = P_y \times 200 \text{ mm} = 20 \times 10^3 \text{ Nmm}$$

Making a cut at point H:



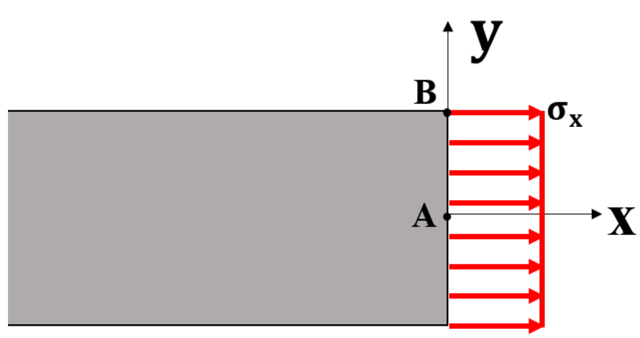
$$V_H = P_y = 100 \text{ N}$$

$$H_x = P_x = 100 \text{ N}$$

$$M_H = 100\text{N x } 100\text{mm} = 10^4 \text{ N.mm}$$

POINT A

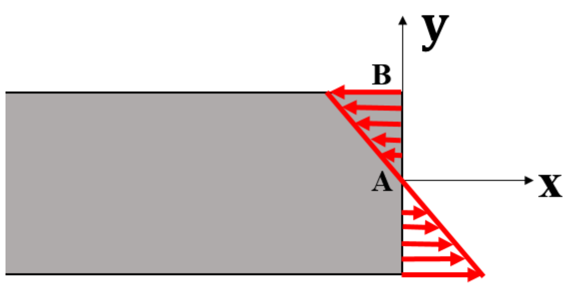
<u>Normal Stress Distribution due to axial loading:</u>



$$P_x = 100 \text{ N}$$

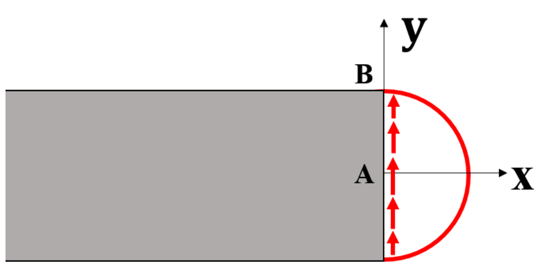
$$\sigma_x = \frac{100\text{N}}{25^2\text{mm}^2} = 0.16\text{N/mm}^2 = 0.16\text{MPa}$$

Normal Stress Distribution due to bending:



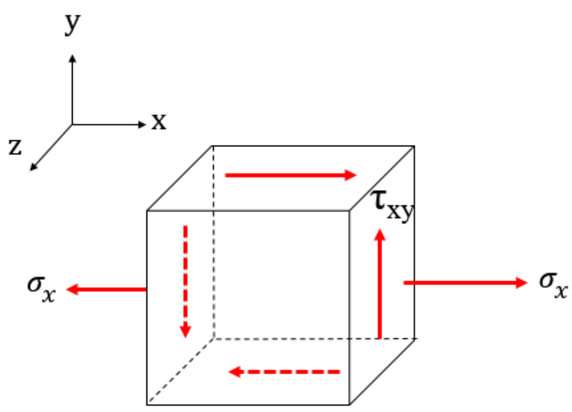
$$\sigma_x = \frac{M_H\, y}{I} = 0 \text{ MPa}$$

Shear Stress Distribution due to transverse loading:



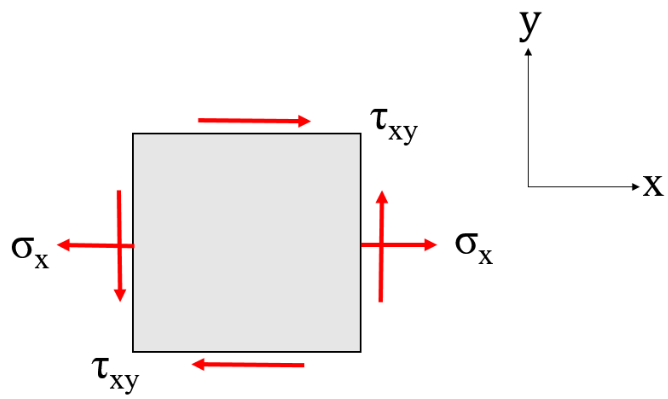
$$\tau_{xy} = \frac{3V}{2A} = \frac{3 \times 100\text{N}}{2 \times 625\text{ mm}^2} = 0.24 \text{ MPa}$$

Three-dimensional differential stress element at A:

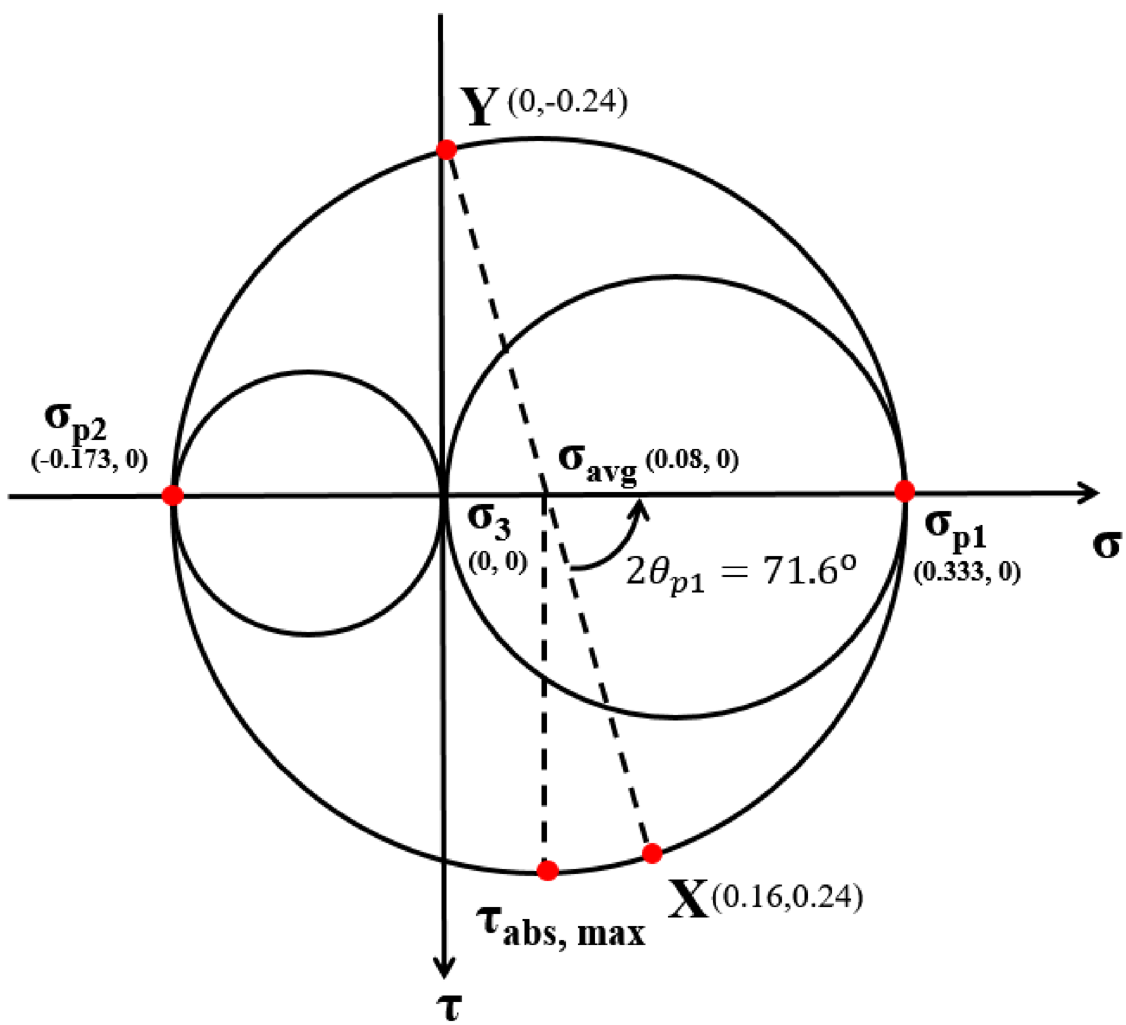


$$\boxed{\boldsymbol{\sigma_x = 0.16\text{MPa}, \tau_{xy} = 0.24\text{MPa}}}$$

Since, $\sigma_z = 0$, $\tau_{yz} = \tau_{xz} = 0$, the xy plane is the plane corresponding to the state of plane stress.



Mohr's circle:



$$\sigma_{avg} = \frac{0.16}{2} = 0.08 \text{ MPa}$$

Principal stress: $\sigma_{p_1} = 0.33 \text{ MPa}, \ \sigma_{p_2} = -0.17 \text{ MPa}, \sigma_3 = 0 \text{ MPa}$

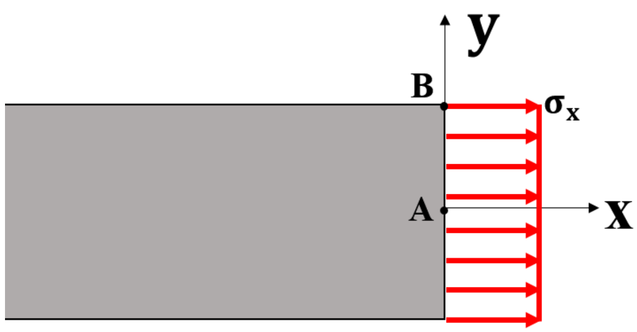
Principal angle: = $\theta_{p_1} = 35.78°, \theta_{p_2} = 125.78°$

Maximum in plane shear stresses: $\tau_{inplane,max} = 0.253 \text{ MPa}$

Absolute shear stress: $\tau_{max,abs} = 0.253 \text{ MPa}$

POINT B

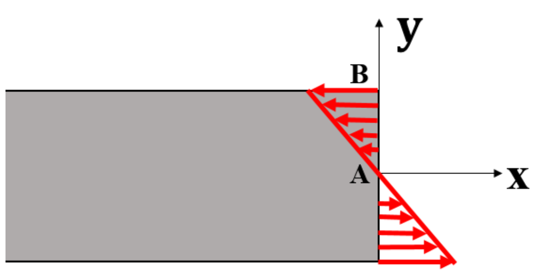
Normal Stress Distribution due to axial loading:



$$P_x = 100 \text{ N}$$

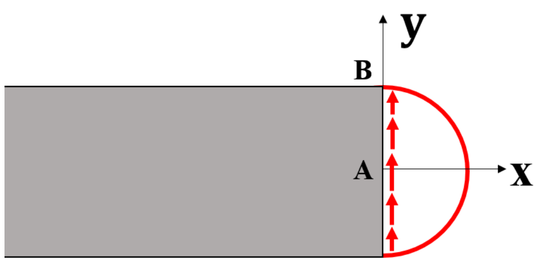
$$\sigma_x = \frac{100\text{N}}{25^2\text{mm}^2} = 0.16\text{N/mm}^2 = 0.16\text{MPa}$$

Normal Stress Distribution due to bending:



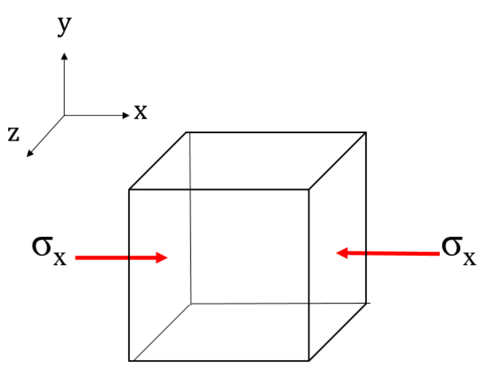
$$\sigma_x = \frac{M_H\, y}{I} = \frac{100\text{N x } 12.5\text{mm}}{\frac{25^4}{12}\text{mm}^4} = 3.84 \text{ MPa (compressive)}$$

Shear Stress Distribution due to transverse loading:



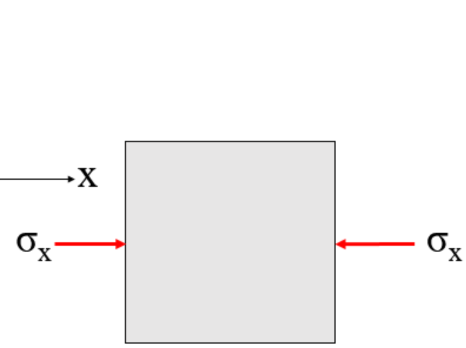
$$\tau_{xy} = \frac{3V}{2A} = 0 \text{ MPa}$$

Three-dimensional differential stress element at B:

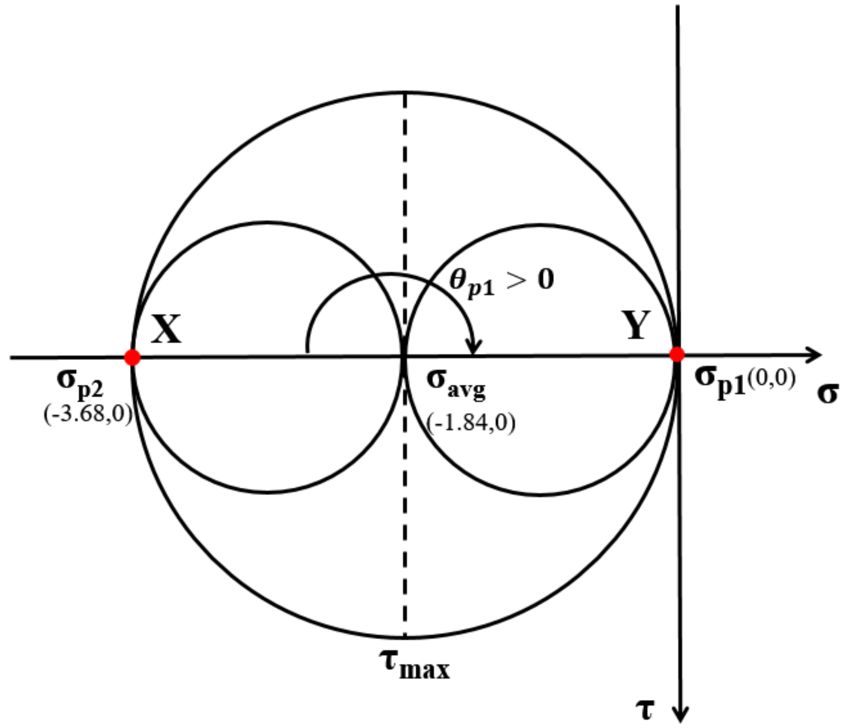


$$\boxed{\sigma_x = \ 3.84\text{MPa} - 0.16\text{MPa} = 3.68\text{MPa}}$$

Since, $\sigma_z = 0$, $\tau_{yz} = \tau_{xz} = 0$, the xy plane is the plane corresponding to the state of plane stress.



Mohr's circle:



$$\sigma_{avg} = \frac{-3.68}{2} = -1.84 \text{ MPa}$$

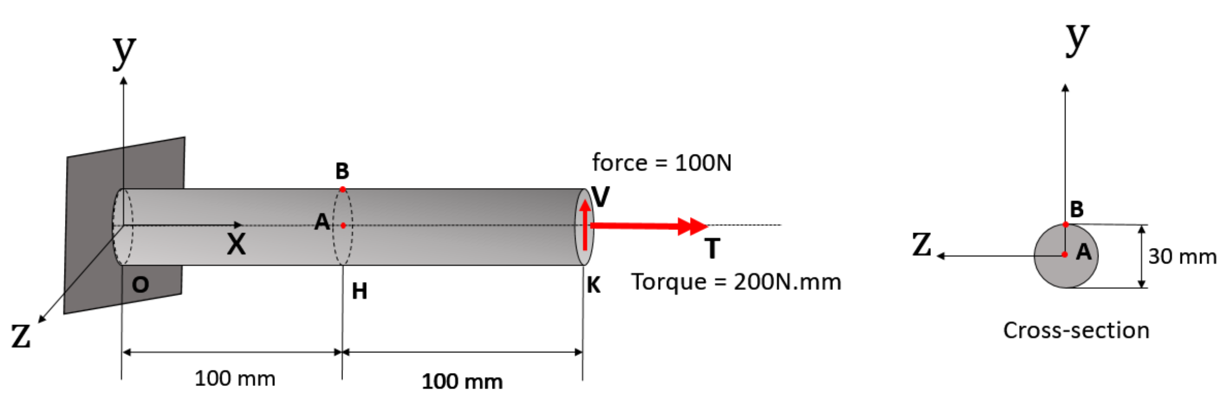
Principal stress: $\sigma_{p_1} = 0$ MPa, $\sigma_{p_2} = -3.68$ MPa, $\sigma_3 = 0$ MPa

Principal angle: = $\theta_{p_1} = 90°$, $\theta_{p_2} = 0°$

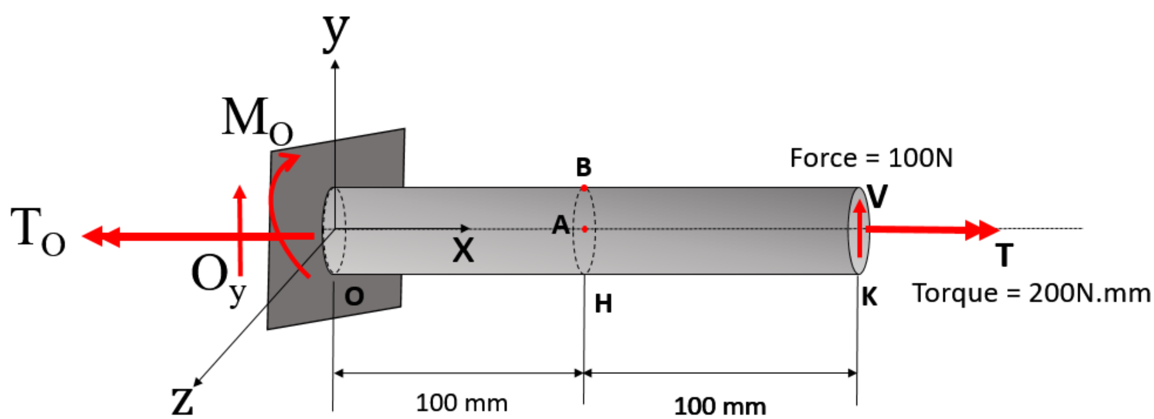
Maximum in plane shear stresses: $\tau_{inplane,max} = 1.84$ MPa

Absolute shear stress: $\tau_{max,abs} = 1.84$ MPa

Case (d):



FBD:

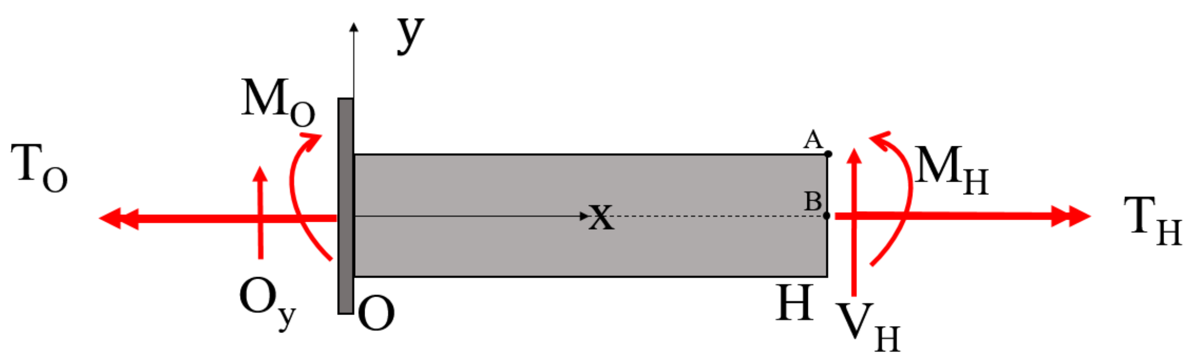


$$O_y = -100 \text{ N}$$

$$M_o = 100 \text{ N} \times 200 \text{ mm} = 20 \times 10^3 \text{ N.mm}$$

$$T_o = 200 \text{ N.mm}$$

Making a cut at H and finding the internal resultant force, moment and torque we have:



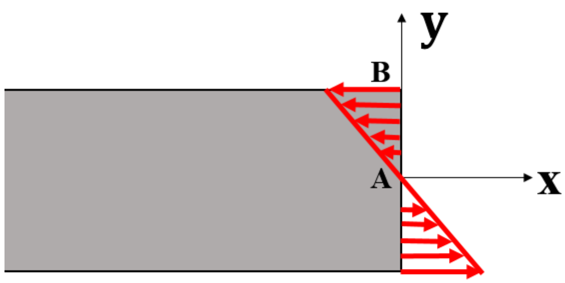
$$V_H = -100 \text{ N}$$

$$M_H = 10^4 \text{ N.mm}$$

$$T_H = 200 \text{ N.mm}$$

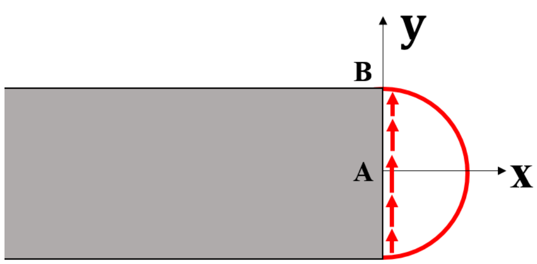
POINT A

Normal Stress Distribution due to bending at A:



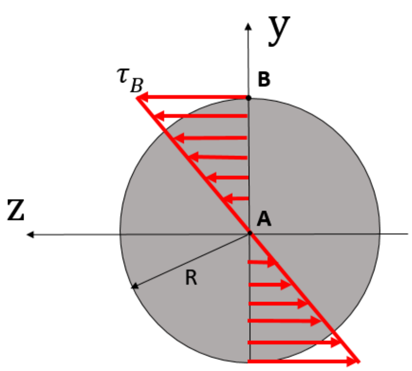
$$\sigma_x = \frac{M_H\, y}{I} = 0\ \text{MPa}$$

Shear Stress Distribution due to transverse loading at A:



$$\tau_{xy} = \frac{4V}{3A} = \frac{4 \times 100\text{N}}{3 \times \frac{\pi}{4}\ \text{x}\ (30\ \text{mm})^2} = 0.188\ \text{MPa}$$

Shear stress distribution due to torsional loading at A:

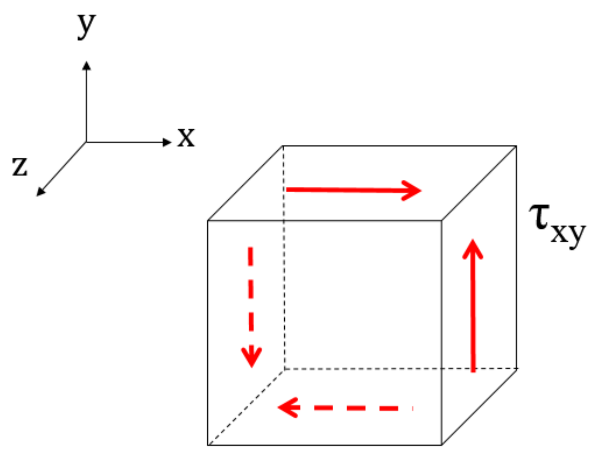


$$\tau_{xy} = \frac{T_H R}{I_P} = \text{linear in radial position}$$

I_p = polar moment of area

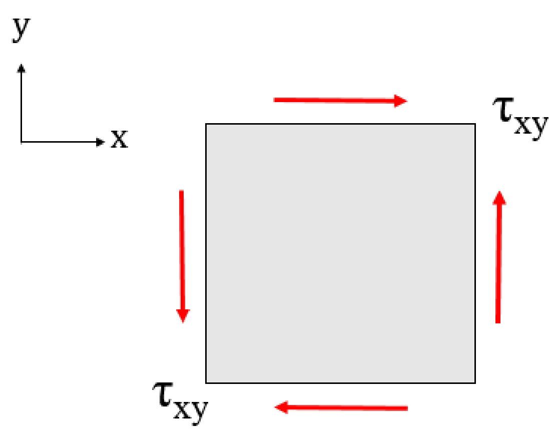
$$\tau_{xy} = 0\text{MPa}$$

Three-dimensional differential stress element at A:

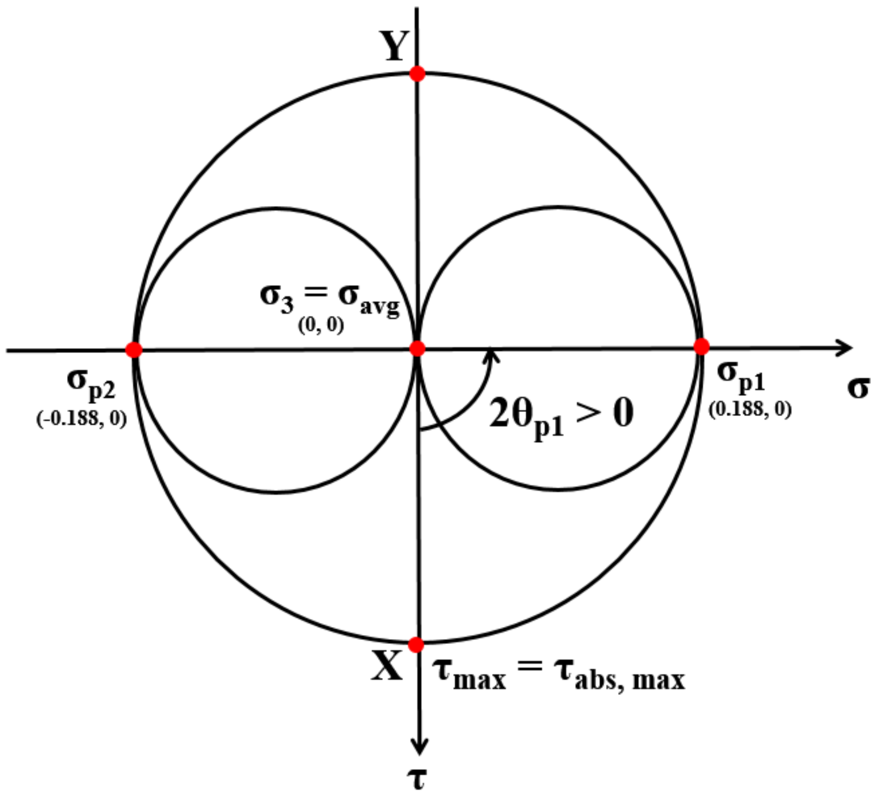


$$\boxed{\tau_{xy} = \mathbf{0.18862\ MPa = 188.62\ kPa}}$$

Since, $\sigma_z = 0$, $\tau_{yz} = \tau_{xz} = 0$, the xy plane is the plane corresponding to the state of plane stress.



Mohr's circle:



$$\sigma_{avg} = 0 \text{ MPa}$$

Principal stress: $\sigma_{p_1} = 0.188 \text{ MPa}$, $\sigma_{p_2} = -0.188 \text{ MPa}, \sigma_3 = 0 \text{ MPa}$

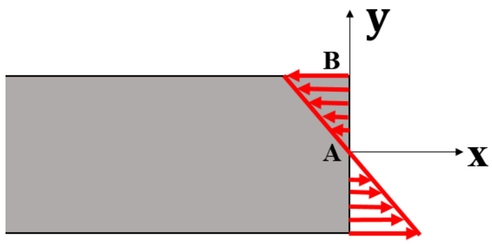
Principal angle: = $\theta_{p_1} = 45°$, $\theta_{p_2} = 135°$

Maximum in plane shear stresses: $\tau_{inplane,max} = 0.188 \text{ MPa}$

Absolute shear stress: $\tau_{max,abs} = 0.188 \text{ MPa}$

POINT B

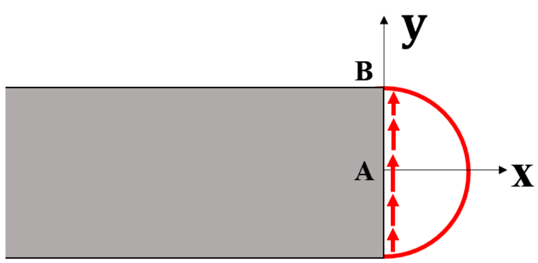
Normal Stress Distribution due to bending at B:



$$\sigma_x = \frac{M_H\, y}{I} \qquad = \frac{10^4 \text{N.mm x } 15\text{mm}}{\frac{\pi}{4} \text{x } 15^4 \text{mm}^4}$$

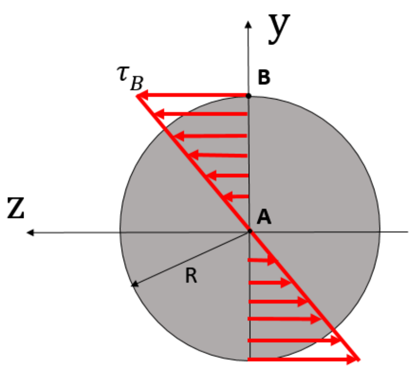
$$\sigma_x = \ \ 3.77 \text{ MPa (compressive)}$$

Shear Stress Distribution due to transverse loading at B:



$$\tau_{xy} = 0 \text{ MPa}$$

Stress distribution due to torsional loading at point B:



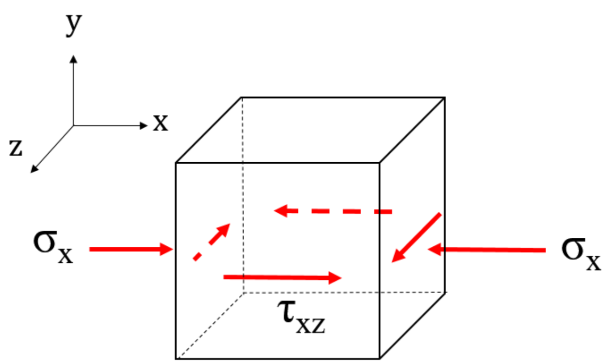
$$\tau_B = \frac{T_H R}{I_P} = \text{linear in radial position}$$

$$I_p = \text{polar moment of area}$$

$$\tau_B = \frac{200.\text{Nmm} \times 15\text{mm}}{\frac{\pi}{32} \times 30^4 \text{mm}^4}$$

$$\tau_B = 0.0377 \frac{\text{N}}{\text{mm}^2} = 0.0377\text{MPa}$$

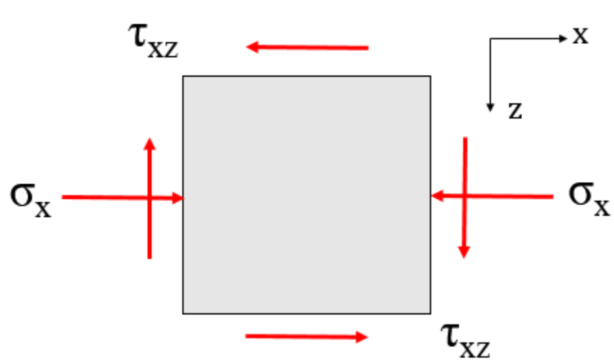
Three-dimensional differential stress element at A:

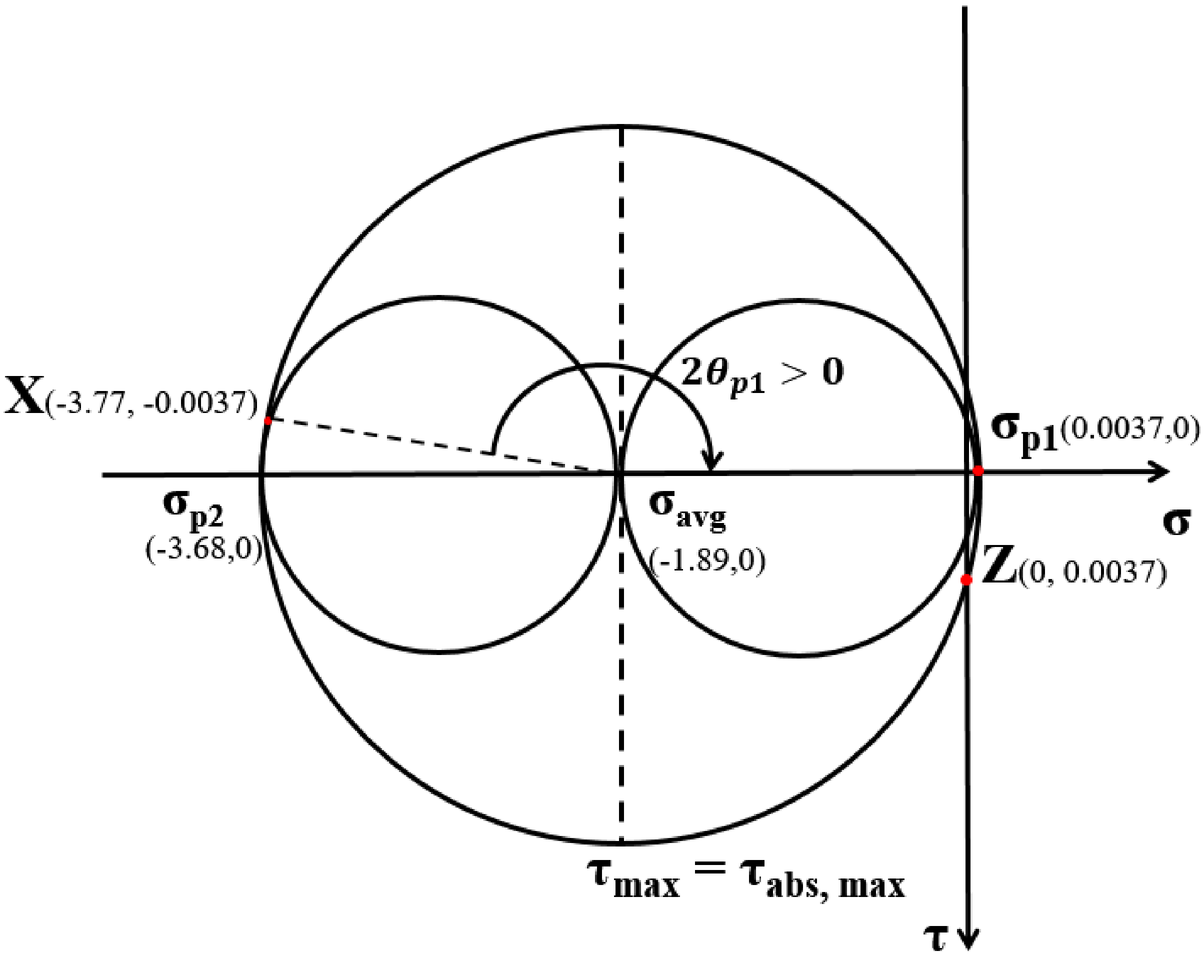


$$\sigma_x = 3.77 \text{ MPa}$$

$$\tau_{xz} = 0.0377 \text{ MPa} = 37.7 \text{ kPa}$$

Since, $\sigma_y = 0$, $\tau_{yz} = \tau_{xy} = 0$, the xz plane is the plane corresponding to the state of plane stress.





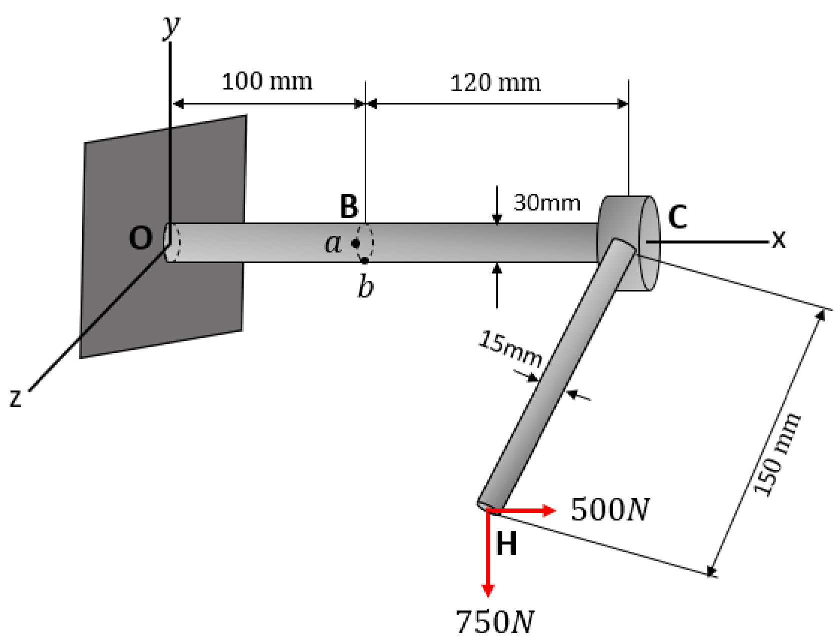
Principal stress: $\sigma_{p_1} = 0.0037MPa$, $\sigma_{p_2} = -3.77$ MPa, $\sigma_3 = 0$ MPa

Principal angle: = $\theta_{p_1} = 90.57^o$, $\theta_{p_2} = 0.57°$

Maximum in plane shear stresses: $\tau_{inplane,max} = 1.8854$ MPa

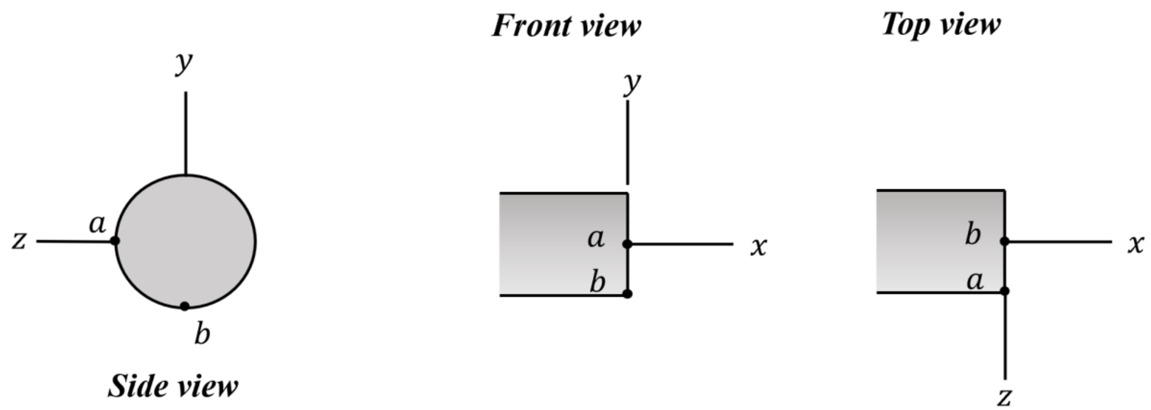
Absolute shear stress: $\tau_{max,abs} = 1.8854$ MPa

Problem 11.4 (10 points)



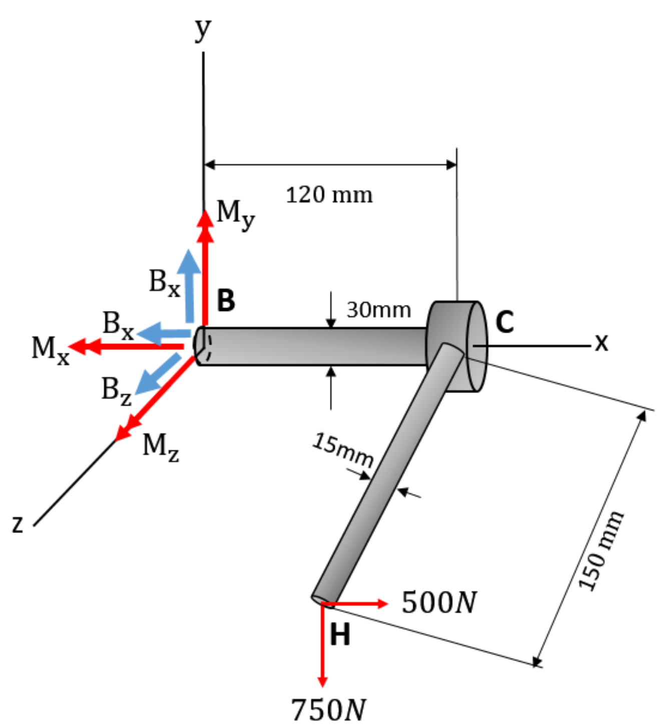
Consider the elastic structure shown in the figure, where a force equal to 500 N ***i*** - 750 N ***j*** is applied at the end of the segment CH parallel to the z-axis.

1. Determine the internal resultants at cross section B (i.e., axial force, two shear forces, torque, and two bending moments).



2. Show the stress distribution due to each internal resultant on the appropriate view of the cross B (i.e., side view, front view or top view).
3. Determine the state of stress on points *a* and *b* on cross section B.
4. Represent the state of stress at points *a* and *b* in three-dimensional differential stress elements.
5. Determine the principal stresses and the absolute maximum shear stress at point *b*.

<u>FBD</u>:



$$\mathbf{M_B} = \mathrm{M_x}\,\boldsymbol{i} + \mathrm{M_y}\,\boldsymbol{j} + \mathrm{M_z}\,\boldsymbol{z}$$

Using force balance we get:

$$\mathrm{B_x} = 500\ \mathrm{N}$$
$$\mathrm{B_y} = 750\ \mathrm{N}$$

$$B_z = 0 \text{ N}$$

<u>Moment balance about point B:</u>

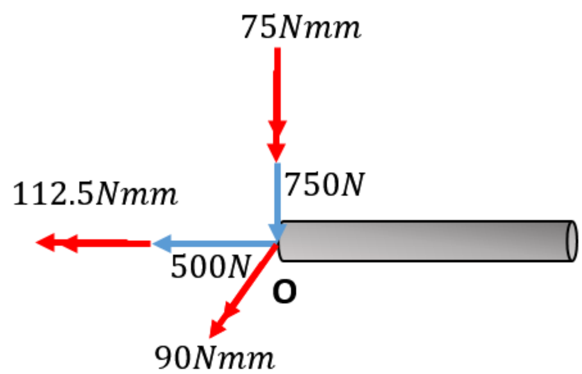
Coordinates of point H w.r.t point B = $\boldsymbol{r}_{H/B} = \;\; 120 \text{ mm } \boldsymbol{i} + \; 0 \text{ mm } \boldsymbol{j} + 150 \text{ mm } \boldsymbol{z}$

Force = **F** = $500\text{N } \boldsymbol{i} - 750\text{N } \boldsymbol{j} + 0 \text{ mm } \boldsymbol{z}$

$$\mathbf{M_B} + \boldsymbol{r}_{H/B} \text{ x } \mathbf{F} = 0$$

$$(\text{M}_\text{x}\, \boldsymbol{i} + \; \text{M}_\text{y}\, \boldsymbol{j} + \text{M}_\text{z}\, \boldsymbol{z}) + \boldsymbol{r}_{\boldsymbol{H/B}} \text{ x } \mathbf{F} = 0$$

$$\text{M}_\text{x} = \; -112500 \text{ N.mm} = \; -112.5 \text{ N.m}$$
$$\text{M}_\text{y} = \; -75000 \text{ N.mm} = \; -75 \text{ N.m}$$
$$\text{M}_\text{z} = \; -90000 \text{ N.mm} = \; -75 \text{ N.m}$$



The reactions are as follows

Torque $= \text{M}_\text{x} = -112.5$ N.m

Axial force $= \text{B}_\text{x} = 500\text{N}$

Shear force 1 $= \text{B}_\text{y} = 750\text{N}$

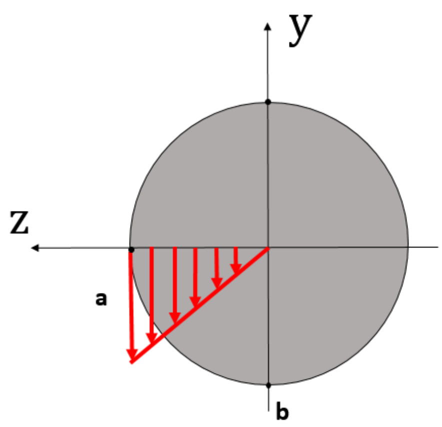
Shear force 2 $= \text{B}_\text{z} = 0$

Bending moment 1 (about y axis) $= \text{M}_\text{y} = -75$ N.m

Bending moment 2 (about z axis) $= \text{M}_\text{z} = \; 90$ N.m

POINT 'a'

Stress distribution due to torsional loading (M_x) at point 'a' :



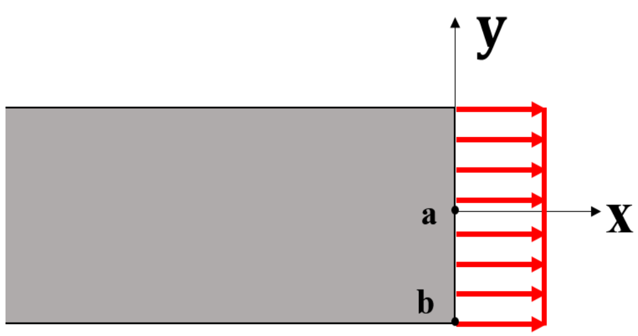
$$\tau_{xy1} = \frac{T_H R}{I_P} = \text{linear in radial position}$$

I_p = polar moment of area

$$\tau_{xy1} = \frac{112.5\ \text{N.m} \times 0.015\text{m}}{\frac{\pi}{2} \times (0.015)^4 \text{mm}^4}$$

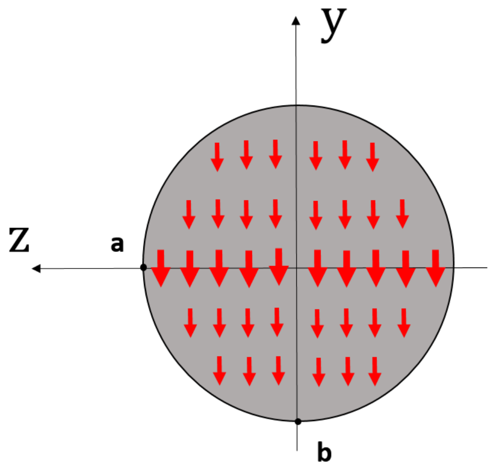
$$\tau_{xy1} = 21.22 \frac{\text{N}}{\text{m}^2} = 21.22\text{Pa}$$

Stress distribution due to axial loading (B_x) at point 'a':



$$\sigma_{x1} = \frac{B_x}{A} = \frac{500}{\pi(0.015)^2} \frac{\text{N}}{\text{m}^2} = 707.355\ \text{kPa}$$

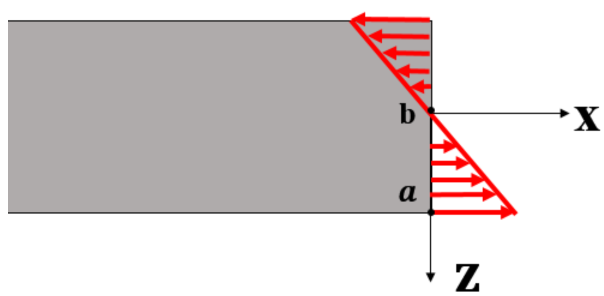
Stress distribution due to Shear force 1 (B_y) loading at point 'a':



$$\tau_{xy2} = \frac{4V}{3A} = \frac{B_y}{3\frac{\pi}{4}(R)^2}$$

$$\tau_{xy2} = \frac{4V}{3A} = \frac{750N}{3\frac{\pi}{4}(0.015m)^2} = 1.414\ \text{MPa}$$

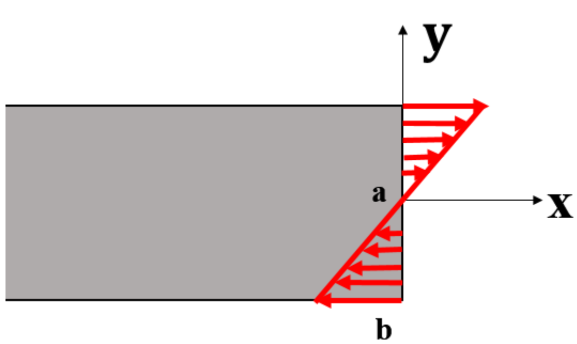
Normal Stress Distribution due to bending moment 1 (M_y) at point '*a*':



$$\sigma_{x2} = \frac{M_y\, z}{I} = \frac{75\ \text{N.m x}\ (0.015\text{m})^3}{\frac{\pi}{4}\ \text{x}\ (0.015\ \text{mm})^4}$$

$$\sigma_{x2} = 28.29\ \text{MPa (tensile)}$$

Normal Stress Distribution due to bending moment 2 (M_z) at point '*b*':



$$\sigma_{x2} = 0\ \text{MPa}$$

State of stress at point'a': $\boldsymbol{\sigma_x} = \sigma_{x1} + \sigma_{x2} = 28.99\ \text{MPa}$

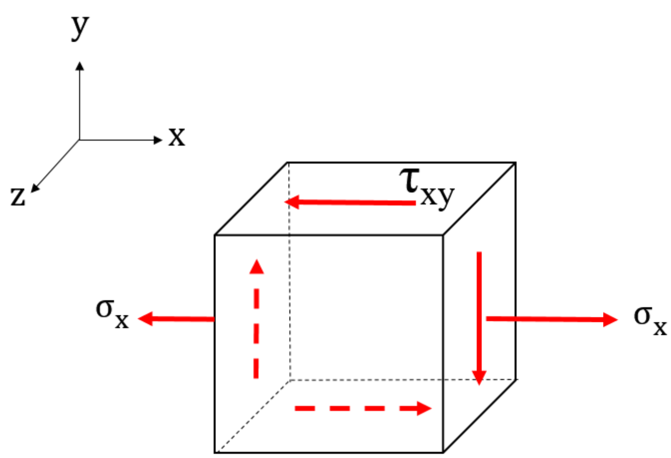
$\boldsymbol{\sigma_y} = 0\ \text{Mpa}$

$\boldsymbol{\sigma_y} = 0\ \text{Mpa}$

$\boldsymbol{\tau_{xy}} = \tau_{xy1} + \tau_{xy2} = 22.63\ \text{MPa}$

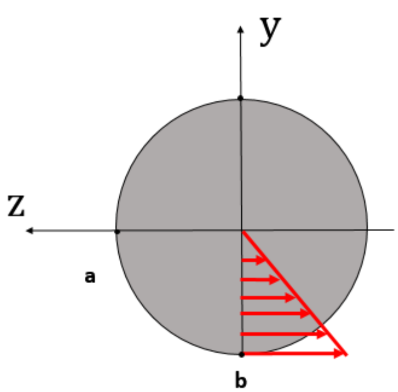
$\boldsymbol{\tau_{yz}} = 0\ \text{Mpa}$

$\boldsymbol{\tau_{zx}} = 0\ \text{Mpa}$



POINT 'b'

Stress distribution due to torsional loading ($\mathbf{M_x}$) at point 'b' :



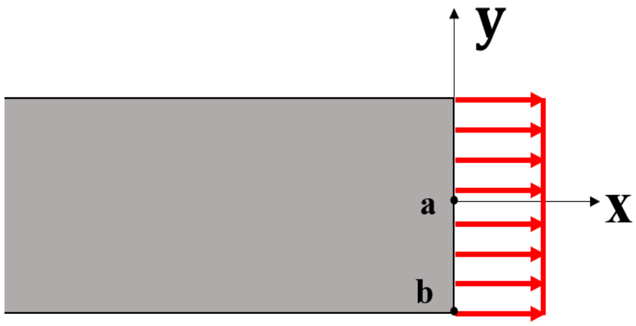
$$\tau_{xz1} = \frac{T_H R}{I_P} = \text{linear in radial position}$$

I_p = polar moment of area

$$\tau_{xz1} = \frac{112.5\ \text{N.m} \times 0.015\text{m}}{\frac{\pi}{2} \times (0.015)^4\text{m}^4}$$

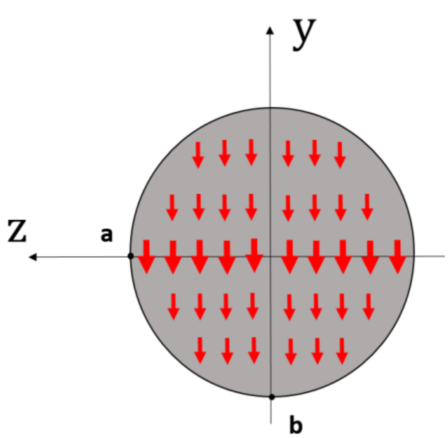
$$\tau_{xz1} = 21.22\text{MPa}$$

Stress distribution due to axial loading (B_x) at point 'b':



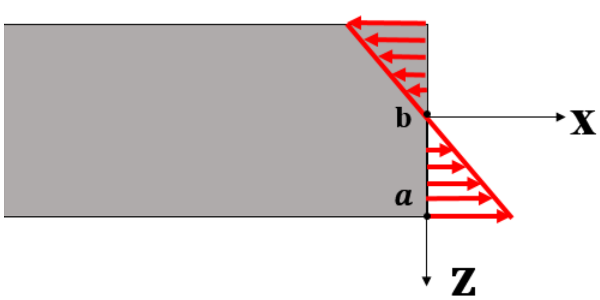
$$\sigma_{x1} = \frac{B_x}{A} = \frac{500}{\pi(0.015)^2}\frac{N}{m^2} = 707.355\ kPa$$

Stress distribution due to Shear force 1 (B_y) loading at point 'b':



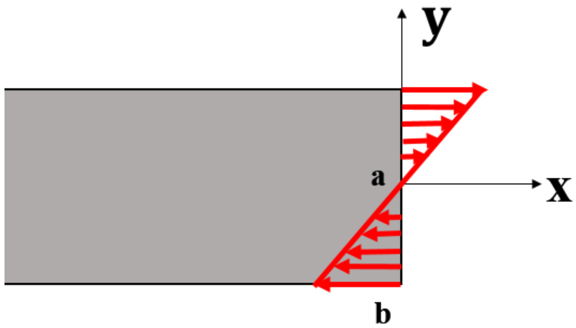
$$\tau_{xy} = 0MPa$$

Normal Stress Distribution due to bending moment 1 (M_y) at point 'a':



$$\sigma_{x1} = 0\ MPa$$

Normal Stress Distribution due to bending moment 2 (M_z) at point '*b*':



$$\sigma_{x2} = \frac{M_z\,y}{I} = \frac{90\ \text{N.m x}\ (0.015\text{m})^3}{\frac{\pi}{4}\ \text{x}\ (0.015\ \text{mm})^4}$$

$$\sigma_{x2} = 33.95\ \text{MPa (compressive)}$$

State of stress at point'*b*':

$\boldsymbol{\sigma_x = \sigma_{x1} + \sigma_{x2} = 33.24\ \text{MPa (compressive)}}$

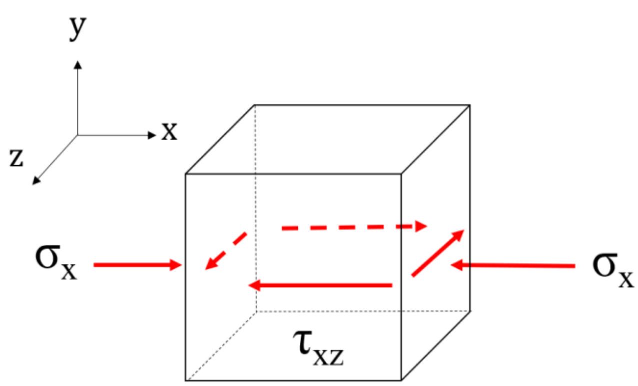
$\boldsymbol{\sigma_y = 0\ \text{Mpa}}$

$\boldsymbol{\sigma_y\ = 0\ \text{Mpa}}$

$\boldsymbol{\tau_{xy} = 0\ \text{MPa}}$

$\boldsymbol{\tau_{yz} = 0\ \text{Mpa}}$

$\boldsymbol{\tau_{zx} = 21.22\ \text{Mpa}}$



Principal stress: $\sigma_{p_1} = 10.33$ MPa, $\sigma_{p_2} = -43.57$ MPa, $\sigma_3 = 0$ MPa

Maximum in plane shear stresses: $\tau_{inplane,max} = 26.95$ MPa

Absolute shear stress: $\tau_{max,abs} = 26.95$ MPa