Lecture 40-41: Failure analysis (static failure theories)

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Fall 2019
Lecture Book: Ch. 15
Motivation

• We have spent the last few classes finding the state of stress at various points in a body due to combined loading
  • We have seen various combinations of normal stresses and shear stresses

• Mohr’s circle gives us a way to compare different states of stress
  • For any state of stress, we can identify three important parameters: the two in-plane principal stresses and the absolute maximum shear stress

• Now: how can we use this information to predict whether a point in a body will fail?
  • First, we need to define what “failure” means...this depends on the type of material!
Failure theories overview

- Use the results of “simple” tests to formulate hypotheses
  - Usual hypothesis: the mechanism that causes failure in a tensile test is the same mechanism that causes failure in more complex stress states
- We have different failure theories for brittle and ductile materials
- “All models are wrong, but some are useful”
The tensile test

Lecture Book: Ch. 15, pg. 2

Free-body diagram

Mohr’s circle at any point on any cross section

What is the maximum normal stress? On which plane does this stress exist?

\[ \sigma_{p1} = \sigma = \frac{P}{A} = \sigma_x \]
\[ \Theta = 0^\circ \]

What is the maximum shear stress? On which planes does this stress exist?

\[ \tau_{\text{max}} = R = \sigma/2 \]
\[ \Theta = \pm 45^\circ \]
Brittle failure: Maximum normal stress theory

**Hypothesis:** A brittle material fractures when the maximum principal stress equals or exceeds the ultimate normal stress when fracture occurs in a tensile test.

Define this “ultimate normal stress” as the **ultimate strength**

Assumption: The ultimate strength in tension and compression is the same.

\[ \text{tensile: } \frac{P}{A} = \sigma_U \]

\[ \sigma_p \geq \sigma_U \]

\[ \text{comp: } \frac{P}{A} = -\sigma_U \Rightarrow \tau_2 \leq -\sigma_U \]

*Lecture Book: Ch. 15, pg. 11*
Brittle failure: Maximum normal stress theory

We can visualize the failure boundary in principal stress space.

\[ \sigma_{p1} = \sigma_{\text{ug}} + R \]
\[ \sigma_{p2} = \sigma_{\text{ug}} - R \]

Failure criteria:

\[ \sigma_{p1} \geq \sigma_U \]
\[ \sigma_{p2} \leq -\sigma_U \]

Factor of safety:

\[ FS = \frac{\sigma_U}{\sigma_{p1}} \]
\[ FS = \frac{\sigma_U}{\sigma_{p2}} \]

The "real" FS is whichever of these is smaller.

Lecture Book: Ch. 15, pg. 11
Brittle failure: Mohr’s theory

Modification to maximum normal stress theory based on the observation that many materials are stronger in compression than they are in tension, i.e. $\sigma_{UT} < \sigma_{UC}$, and the maximum normal stress theory is non-conservative when the principal stresses have different signs.

$$\sigma_{UC} > \sigma_{UT}$$

Lecture Book: Ch. 15, pg. 12
Brittle failure: Mohr’s theory

For a general state of plane stress, there are three possible situations

**Case 1:** \( \sigma_{p1} > \sigma_{p2} > 0 \)

**Case 2:** \( \sigma_{p1} > 0 > \sigma_{p2} \)

**Case 3:** \( 0 > \sigma_{p1} > \sigma_{p2} \)

Lecture Book: Ch. 15, pg. 6

\[
\frac{\sigma_{p1} - \sigma_{p2}}{\sigma_{UT}} > 1
\]

Lecture Book: Ch. 15, pg. 12
Brittle failure: Mohr’s theory

We can visualize the failure boundary in principal stress space.

**Failure criteria**

**Case 1:** \( \sigma_{p1} > \sigma_{p2} > 0 \)

\[
\sigma_{p1} \geq \sigma_{UT}
\]

\[
\sigma_3 = 0
\]

\[
FS = \frac{\sigma_{UT}}{\sigma_{p1}}
\]

**Case 2:** \( \sigma_{p1} > 0 > \sigma_{p2} \)

\[
\frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}} \geq 1
\]

\[
\sigma_3 = \sigma_{p2}
\]

\[
FS = \frac{1}{\frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}}}
\]

**Case 3:** \( 0 > \sigma_{p1} > \sigma_{p2} \)

\[
\sigma_{p2} \leq -\sigma_{UC}
\]

\[
\sigma_3 = \sigma_{p2}
\]

\[
FS = \frac{\sigma_{UC}}{\sigma_{p2}}
\]

Lecture Book: Ch. 15, pg. 12
Re-order the principal stresses: $\sigma_1 > \sigma_2 > \sigma_3$

\[
\frac{\sigma_1}{\sigma_{ut}} - \frac{\sigma_3}{\sigma_{uc}} \geq 1
\]
Brittle failure: Summary

<table>
<thead>
<tr>
<th>Maximum normal stress theory</th>
<th>Mohr’s failure theory</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Failure criterion</strong></td>
<td>Failure criteria (3 possible cases based on the signs of the principal stresses)</td>
</tr>
<tr>
<td>( \sigma_{p1} \geq \sigma_U ) or ( \sigma_{p2} \leq -\sigma_U )</td>
<td>( \sigma_{p1} &gt; \sigma_{p2} &gt; 0: \sigma_{p1} \geq \sigma_{UT} )</td>
</tr>
<tr>
<td><strong>Factor of safety</strong></td>
<td>( 0 &gt; \sigma_{p1} &gt; \sigma_{p2} : \sigma_{p2} \leq -\sigma_{UC} )</td>
</tr>
<tr>
<td>( FS = \left</td>
<td>\frac{\sigma_U}{\sigma_{p1}} \right</td>
</tr>
<tr>
<td>(whichever is smaller is the real factor of safety)</td>
<td>Factor of safety</td>
</tr>
<tr>
<td>( \sigma_{p1} &gt; \sigma_{p2} &gt; 0: FS = \frac{\sigma_{UT}}{\sigma_{p1}} )</td>
<td>( 0 &gt; \sigma_{p1} &gt; \sigma_{p2} : FS = \left</td>
</tr>
</tbody>
</table>

\[ \sigma_{p1} > 0 > \sigma_{p2} : FS = \frac{1}{\frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}} - \frac{\sigma_{p1} \sigma_{UT} - \sigma_{p2} \sigma_{UC}}{\sigma_{UT} \sigma_{UC}}} \]

**Re-ordered**: \( \sigma_1 > \sigma_2 > \sigma_3 \) \( \Rightarrow FS = \frac{\sigma_{UT} - \sigma_{UC}}{\sigma_{UT} \sigma_{UC}} \)
The state of stress shown exists at a location in a component made of a brittle material with $\sigma_{UC} = 850$ MPa and $\sigma_{UT} = 170$ MPa. According to Mohr’s theory, has the material failed?

\[ \sigma_x = 0 \]
\[ \sigma_{avg} = \frac{120 + 0}{2} = 60 \text{ MPa} \]
\[ \sigma_y = +120 \text{ MPa} \]
\[ \tau_{xy} = +80 \text{ MPa} \]

\[ R = \sqrt{\left(\frac{120}{2}\right)^2 + (80)^2} = \sqrt{60^2 + 80^2} = 100 \text{ MPa} \]

\[ \sigma_1 = \sigma_{avg} + R = 160 \text{ MPa} \]
\[ \sigma_2 = 0 \]
\[ \sigma_3 = \sigma_{avg} - R = -40 \text{ MPa} \]

$\sigma_1 = 160$ MPa.
\[
\sigma_{p1} = \sigma_{ut} \quad \sigma_2 \quad \sigma_{p2} \leq \sigma_{uc} \quad \sigma_{p3} = 0
\]

\[
\begin{align*}
\frac{\sigma_{p1}}{\sigma_{ut}} - \frac{\sigma_{p2}}{\sigma_{uc}} &= 1 \\
\Rightarrow \quad \frac{160 \text{ MPa}}{150 \text{ MPa}} &= \frac{(-40 \text{ MPa})}{850 \text{ MPa}}
\end{align*}
\]

\[\sigma_1 > \sigma_2 > \sigma_3 \quad 0.99 < 1 \Rightarrow \text{Safe}\]

What value of \( \sigma_{p2} \) would cause failure?

\[
\frac{\sigma_{p1}}{\sigma_{ut}} - \frac{\sigma_{p2}}{\sigma_{uc}} = 1 \rightarrow \sigma_{p2}^\text{fail} = -50 \text{ MPa}
\]
Ductile failure: Maximum shear stress theory

• On the microscale, permanent (plastic) deformation occurs by “slip”

• Failure in a tensile test of a ductile material often looks very similar

    *Aluminum – failure due to normal or shear stress?*
Hypothesis: for any stress state, yielding of a ductile material occurs when the absolute maximum shear stress equals or exceeds the maximum shear stress when yielding occurs in a tensile test.

Uniaxial tension → $\tau_{\text{max,abs}} = R = \frac{\sigma}{2}$

Failure → $\sigma_{pl} = \sigma = \sigma_y$

Failure criterion:

$\tau_{\text{max}} \geq \frac{\sigma_y}{2}$

Factor of safety:

$FS = \frac{\sigma_y}{2\tau_{\text{max}}}$

Lecture Book: Ch. 15, pg. 5
Ductile failure: Maximum shear stress theory

For a general state of plane stress, there are three possible situations

**Case 1:** $\sigma_{p1} > \sigma_{p2} > 0$

**Case 2:** $\sigma_{p1} > 0 > \sigma_{p2}$

**Case 3:** $0 > \sigma_{p1} > \sigma_{p2}$

Lecture Book: Ch. 15, pg. 6
Reorder the principal stresses so that $\sigma_1 > \sigma_2 > \sigma_3$.

$\sigma_{\text{max}} = \left| \frac{\sigma_1 - \sigma_3}{2} \right| \quad \Rightarrow \quad \sigma_{\text{max}} \geq \frac{\sigma_Y}{2}$ is the failure criterion.
Ductile failure: Maximum shear stress theory

We can visualize the failure boundary in principal stress space.

![Diagram of principal stress space with failure boundaries]

**Failure criteria**

Case 1: \( \sigma_{p1} > \sigma_{p2} > 0 \)

\[ \frac{\sigma_1}{\sigma_Y} = \frac{\sigma_{p1}}{\sigma_{p2}} \]

\[ \frac{\sigma_2}{\sigma_Y} = \frac{\sigma_{p2}}{\sigma_{p2}} \]

Case 2: \( \sigma_{p1} > 0 > \sigma_{p2} \)

\[ \frac{\sigma_1}{\sigma_Y} = \frac{\sigma_{p1}}{\sigma_{p1}} \]

\[ \frac{\sigma_2}{\sigma_Y} = \frac{0}{\sigma_{p2}} \]

\[ \frac{\sigma_3}{\sigma_Y} = \frac{\sigma_{p2}}{\sigma_{p2}} \]

Case 3: \( 0 > \sigma_{p1} > \sigma_{p2} \)

\[ \frac{\sigma_1}{\sigma_Y} = \frac{0}{\sigma_{p1}} \]

\[ \frac{\sigma_2}{\sigma_Y} = \frac{\sigma_{p1}}{\sigma_{p2}} \]

\[ \frac{\sigma_3}{\sigma_Y} = \frac{\sigma_{p2}}{\sigma_{p2}} \]

**Factor of safety**

\[ FS = \frac{\sigma_Y}{2c_{max}} \]

\[ = \frac{\sigma_Y}{|\sigma_1 - \sigma_3|} \]
von Mises proposed a different hypothesis: yielding occurs when the \textit{distortion energy density} equals or exceeds the distortion energy density when yielding occurs in a tensile test.  

Evidence: a material subjected to purely hydrostatic stress ($\sigma_{p1} = \sigma_{p2} = \sigma_{p3}$) never yields.

Total elastic strain energy density \[ \bar{u} = \frac{1}{2E} \left[ \sigma_{p1}^2 + \sigma_{p2}^2 - 2\nu \sigma_{p1}\sigma_{p2} \right] \]

\[ \bar{u}_v = \frac{1}{2G} \left( \sigma_{p1} + \sigma_{p2} \right) \]

\[ \bar{u}_d = \frac{1}{6G} \left( \sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2 \right) \]

For yielding in the tensile test:

\[ \bar{u}_{d,\text{yield}} = \frac{1}{6G} \sigma_Y^2 \]

\[ \sigma_{p2} = \sigma_{p2} = 0 \]

\[ \tau_{\text{max}} \]

\[ \tau \]

\[ \sigma \]

\[ \sigma_{p1} \]

\[ \tau_{\text{abs}} \]

So, our failure criterion for any plane stress state is:

\[ \frac{1}{6G} \left( \sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2 \right) = \frac{1}{6G} \sigma_Y^2 \]

\[ \text{Von Mises stress: } \sigma_M = \sqrt{\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2} \]

\[ \Rightarrow \text{failure criterion: } \sigma_M \geq \sigma_Y \]
Ductile failure: Maximum distortional energy theory

In principal stress space, the maximum distortional energy failure boundary is an ellipse.

\[
\sigma_{p1} = -\sigma_{p2} = \sigma_Y, \quad \sigma_{p1} = \sigma_{p2} = \frac{\sigma_Y}{2}
\]

Failure criterion

\[
\sigma_M \geq \sigma_Y
\]

where

\[
\sigma_M = \sqrt{\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2}
\]

Factor of safety

\[
FS = \frac{\sigma_Y}{\sigma_m}
\]
Ductile failure: Summary

Maximum shear stress theory
Failure criterion: $\tau_{\text{max}}^{\text{abs}} \geq \frac{\sigma_Y}{2}$

3 possible cases for $\tau_{\text{max}}^{\text{abs}}$ based on signs of principal stresses

$\sigma_{p1} > \sigma_{p2} > 0$: $\sigma_{p1} \geq \sigma_Y$, $0 > \sigma_{p1} > \sigma_{p2}$: $|\sigma_{p2}| \geq \sigma_Y$

$\sigma_{p1} > 0 > \sigma_{p2}$: $\sigma_{p1} - \sigma_{p2} \geq \sigma_Y$

Or, if you re-order the principal stresses so $\sigma_1 > \sigma_2 > \sigma_3$,

$\tau_{\text{max}}^{\text{abs}} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{\sigma_Y}{2}$ is the failure criterion for all cases

Factor of safety: $FS = \frac{\sigma_Y}{2\tau_{\text{max}}^{\text{abs}}} = \frac{\sigma_Y}{\sigma_1 - \sigma_3}$

Maximum distortional energy (von Mises) theory
Failure criterion (based on the von Mises stress):

$\sigma_M = \sqrt{\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2} \geq \sigma_Y$

Factor of safety: $FS = \frac{\sigma_Y}{\sigma_M}$
Example 15.1

The state of stress shown is in a component made of a ductile material with a yield strength of $\sigma_y = 250 \text{ MPa}$. Does the maximum shear stress theory predict failure for the material? Does the maximum distortion energy predict failure for the material?

\[
\sigma_x = +125 \text{ MPa} \\
\sigma_y = -40 \text{ MPa} \\
\tau_{xy} = +105 \text{ MPa}
\]

\[
\sigma_{\text{max}} = \frac{125 + (-40)}{2} = 42.5 \text{ MPa}
\]

\[
\sigma_p = \sqrt{(125 - (-40))^2 + (105)^2} = 133.5 \text{ MPa}
\]

$\sigma_p = 176 \text{ MPa}$

$\sigma_{p2} = -91 \text{ MPa}$

\[\sigma_p = 0\]
MDE / von Mises

\[
\sigma_m = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1 \sigma_2 - \sigma_2 \sigma_3 - \sigma_3 \sigma_1} = 235.2 \text{ MPa}
\]

\[
\sigma_m = 146 \text{ MPa}
\]

\[
\sigma_3 = -91 \text{ MPa}
\]

\[
\sigma_2 = 0
\]

\[
\sigma_y = 250 \text{ MPa}
\]

\[
\sigma_m < \sigma_y \Rightarrow \text{ safe by MDE!}
\]

MSS

\[
\sum_{i=1}^{3} \sigma_i = \left| \frac{\sigma_1 - \sigma_3}{2} \right| = 132.5 \text{ MPa} > \frac{\sigma_y}{2} = 125 \text{ MPa}
\]

FS = \frac{\sigma_y}{2\sigma_m} = \frac{\sigma_y}{\sigma_1 - \sigma_3} = 0.94 < 1 \Rightarrow \text{ fail by MSS}
Revisit Example 14.12

Wind blowing on a sign produces a resultant force $P$ in the $-y$ direction at the point shown. The support pole weighs $W_p$ and the sign weighs $W_s$. The pole is a pipe with outer and inner diameters $d_o$ and $d_i$, respectively.

What are the factors of safety for points $a$ and $b$ according to the maximum distortion energy theory if the pole is made from an aluminum alloy with a yield strength of 20 ksi? $\sigma_y = 20$ ksi

Point $a$

$\sigma_x^a = \frac{-W_p - W_s}{A} + \frac{PLd_o}{2I_{zz}} = 9433$ psi

$\tau_{xz}^a = \frac{(PL)d}{2I_p} = 871$ psi

$\sigma_m^a = 9.6$ ksi $\Rightarrow \text{FS}^a = \frac{20}{9.6} = 2.1$
Point b

\[ \sigma_x^b = \frac{-W_P - W_s}{A} - \frac{W_{sh} d_0}{2 I_{yy}} = -3025 \text{ psi} \]

\[ \tau_{xy}^b = \frac{-(P h) d_0}{2 I_p} - \frac{2 P}{A} = -938 \text{ psi} \]

\[ \sigma_m^b = 3.4 \text{ ksi} \implies \text{FS}^b = \frac{20}{3.4} = 5.9 \]

The "real" factor of safety is the smaller one (point a is closer to failure).
Determine the principal stresses and the maximum shear stress at point A (i.e., the point on top of the wrench handle). The diameter of the circular cross section is 12.5 mm.

If the wrench is made of a ductile material with a yield strength of 300 MPa, what value of the force will cause yielding at point A according to the maximum shear stress theory? How about the maximum distortion energy theory?

\[ \sigma_Y = 300 \text{ MPa} \]

\[ F = -P \hat{k} \]

\[ P = (100 \hat{i} + 400 \hat{j}) \text{ N} \]
\[ \vec{R}_x \hat{e} + \vec{R}_y \hat{j} + \vec{R}_z \hat{k} = \vec{F} = -p \hat{k} \]

\[ \begin{align*}
V_z &= -p \\
T_x \hat{i} + M_y \hat{j} + N_z \hat{k} &= \hat{r} \times \vec{F} = -400p \hat{i} + 100p \hat{j} + 0 \hat{k} \\
T_x &= -400p \\
M_y &= 100p
\end{align*} \]

**Stress dist**

**Stress at a**

\[ T_{xy} = \frac{+(400p)(d_{12})}{I_p} \]

\[ \sigma_x = -\frac{(100p)(d_{12})}{I} \]
Combine stresses

\[
\sigma_x^0 = \frac{-3200P}{\pi d^3}
\]

\[
\tau_{xy} = \frac{6400P}{\pi d^3}
\]

Principal stresses and failure

\[
\sigma_{\text{avg}} = \frac{-1600P}{\pi d^3}
\]

\[
R = \frac{6594P}{\pi d^3}
\]

\[
\sigma_1 = \sigma_{\text{avg}} + R = \frac{4994P}{\pi d^3}
\]

\[
\sigma_2 = 0
\]

\[
\sigma_3 = \sigma_{\text{avg}} - R = \frac{-8197P}{\pi d^3}
\]

Note: the units work out in all of these equations since there is a "mm" dimension embedded in the numerator.
\[ \frac{M_{SS}}{2} = \frac{|0_1 - 0_3|}{2} = \frac{65479}{\pi d^3} = \frac{\sigma_y}{2} \Rightarrow P_{mss} = 139.5 \text{ N} \]

\[ \frac{M_{DE}}{\sigma_m = \sigma_y} \Rightarrow P_{mde} = 159.5 \text{ N} \]

\[ \sigma_m = \frac{11538 P}{\pi d^3} \]