Lectures 24-26: Energy methods – Castigliano's theorems

Lecture Book: Chapter 16

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PURDUE ENGINEERING

Course outline

- Part 1: Overview of concepts
    - Static equilibrium
    - Stress and strain
    - Mechanical properties of materials
- Part 2: Types of deformation
    - Axial deformation
    - Torsion
    - Bending
- Part 3: Combined loading
    - Energy methods – deformation due to combined loading
    - Stress due to combined loading
    - Failure theories

Objectives

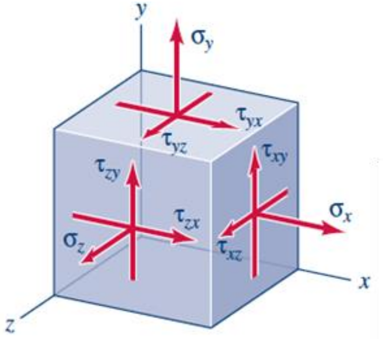
- Calculate the strain energy stored in elastic bodies
- Calculate the work done by external forces and moments
- Use the strain energy to calculate displacements and rotations
  - Work-energy relationship
  - Castigliano’s theorem

Strain energy

Lecture Book: Ch. 16, pg. 6-7

Strain energy density: energy per unit volume stored in an elastically-deformed body

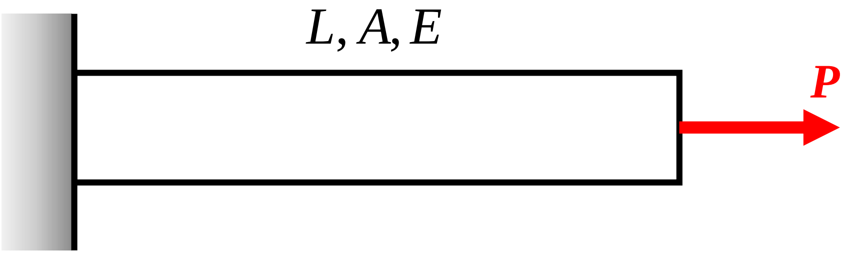
$$\bar{u} = \frac{1}{2}\left[\sigma_x\left(\varepsilon_x - \alpha\Delta T\right) + \sigma_y\left(\varepsilon_y - \alpha\Delta T\right) + \sigma_x\left(\varepsilon_z - \alpha\Delta T\right) + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right]$$



Total strain energy: integrate over the volume of the body: $U = \int_V \bar{u}dV$

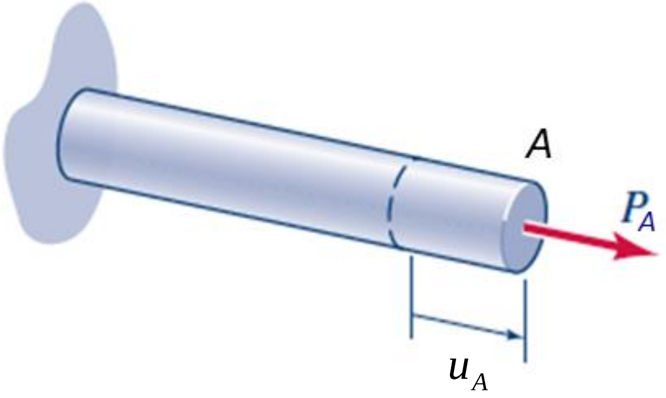
Goal: use what we know about stress and strain from each type of deformation to determine the *strain energy*

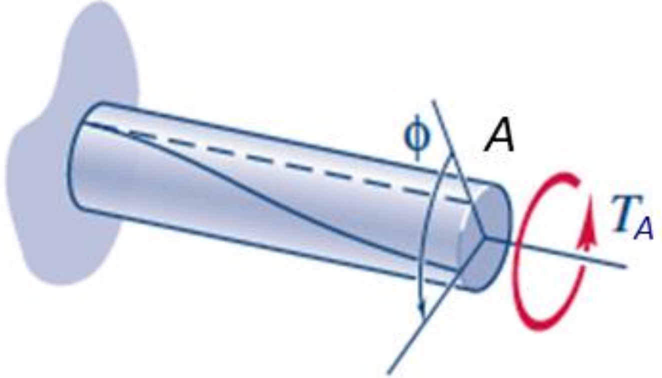
Example: axially loaded rod

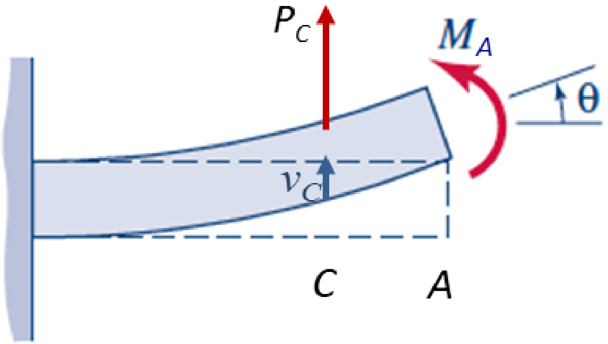


Work done by external loads

Lecture Book: Ch. 16, pg. 6







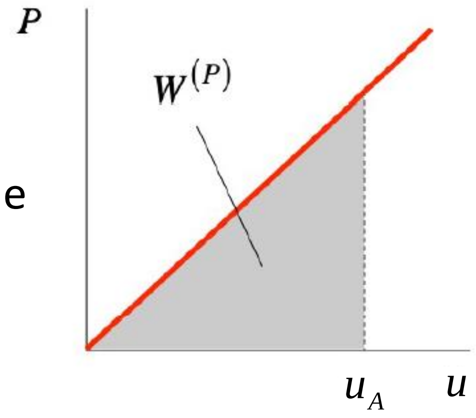
$$W^{(P_A)} = \int_0^{u_A} P_A du = \frac{1}{2} P_A u_A$$

$$W^{(T_A)} = \int_0^{\phi_A} T_A d\phi = \frac{1}{2} T_A \phi_A$$

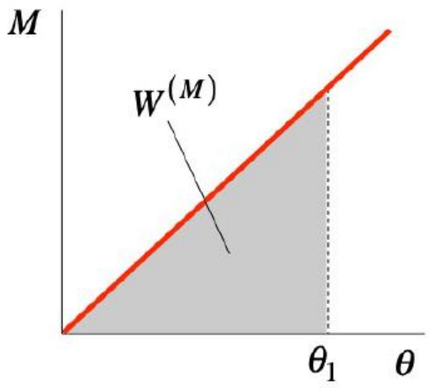
$$W^{(M_A)} = \int_0^{\theta_A} M_A d\theta = \frac{1}{2} M_A \theta_A$$

$$W^{(P_C)} = \int_0^{v_C} P_C dv = \frac{1}{2} P_C v_C$$

Linear elastic deformation due to a force:



Linear elastic deformation due to a moment or torque:



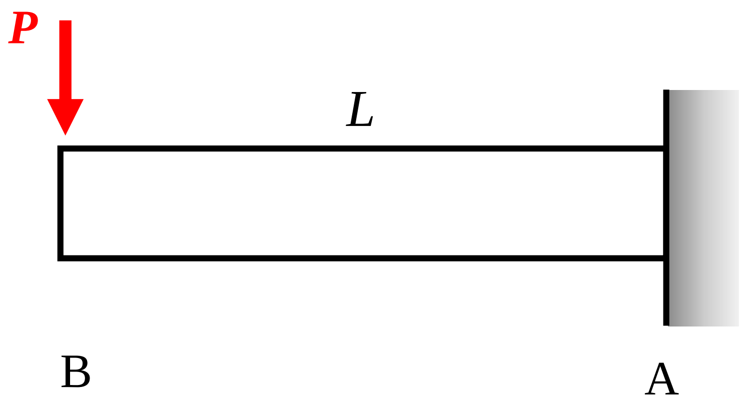
Lecture Book: Ch. 16, pg. 10-11

Member loading type	***Strain energy: load-based***	***Strain energy: displacement-based***
axial	$U=\frac{1}{2}\int_0^L \frac{F^2 dx}{EA}$	$U=\frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx$
torsion	$U=\frac{1}{2}\int_0^L \frac{T^2}{GI_p}dx$	$U=\frac{1}{2}\int_0^L GI_p\left(\frac{d\phi}{dx}\right)^2 dx$
bending - flexural	$U_\sigma=\frac{1}{2}\int_0^L \frac{M^2}{EI}dx$	$U_\sigma=\frac{1}{2}\int_0^L EI\left(\frac{d^2u}{dx^2}\right)^2 dx$
bending - shear	$U_\tau=\frac{1}{2}\int_0^L \frac{f_s V^2}{GA}dx$	

Work-energy relation: all of the work done on an elastic body is stored as strain energy: $W=U$

Example

Find the vertical deflection at point B using the work-energy relationship.
The beam has Young's modulus E and second area moment I.
Neglect shear strain energy due to bending.



Castigliano’s theorem: Determinate structures

<u>Castigliano’s Second Theorem</u> **Lecture Book: Ch. 16, pg. 11**

Consider a determinate linear elastic body acted on by:

N_P forces P_i,

N_M moments M_i, and

N_T torques T_i

Of all possible equilibrium configurations of the body, the actual configuration is the one for which:

$$\Delta_i = \frac{\partial U}{\partial P_i} \qquad i = 1, 2, \ldots, N_P$$

$$\theta_i = \frac{\partial U}{\partial M_i} \qquad i = 1, 2, \ldots, N_M$$

$$\phi_i = \frac{\partial U}{\partial T_i} \qquad i = 1, 2, \ldots, N_T$$

where $(\Delta_i, \theta_i, \phi_i)$ are the displacements, slopes (or in-plane rotations), and angles of twist corresponding to and in the direction of the applied loads (P_i, M_i, T_i)

Castigliano's theorem: Indeterminate structures

Castigliano's Second Theorem **Lecture Book: Ch. 16, pg. 11**

Consider an **indeterminate** linear elastic body acted on by N_P forces P_i, N_M moments M_i, and N_T torques T_i

Now, there are also N_R *redundant* reaction forces R_i in the strain energy function.

Of all possible equilibrium configurations of the body, the actual configuration is the one for which:

$$\Delta_i = \frac{\partial U}{\partial P_i} \qquad i = 1, 2, \ldots, N_P$$

$$\theta_i = \frac{\partial U}{\partial M_i} \qquad i = 1, 2, \ldots, N_M$$

$$\phi_i = \frac{\partial U}{\partial T_i} \qquad i = 1, 2, \ldots, N_T$$

AND

$$0 = \frac{\partial U}{\partial R_i} \qquad i = 1, 2, \ldots, N_R$$

for all redundant reactions R_i

where $(\Delta_i, \theta_i, \phi_i)$ are the displacements, slopes (or in-plane rotations), and angles of twist corresponding to and in the direction of the applied loads (P_i, M_i, T_i)

Bonus 3-D example

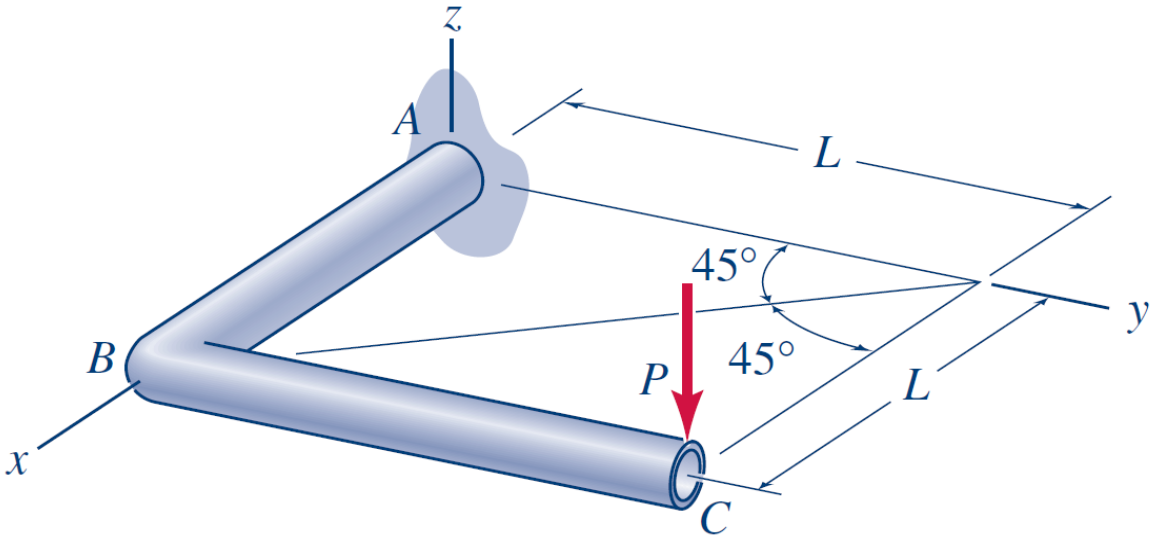
Determine the vertical displacement at C

Points A, B, and C lie in the x-y plane

Young's modulus E, the shear modulus G, and the second area moment I are constant

The polar area moment is $I_p = 2I$

Neglect shear strain energy due to bending



$$\frac{\partial U}{\partial P} = \frac{2PL^3}{3EI} + \frac{PL^3}{2GI} \Rightarrow (v_z)_C = \frac{2PL^3}{3EI} + \frac{PL^3}{2GI} \text{ downward}$$

Summary for Castigliano's theorem

1. Determine what "dummy loads" are needed
2. FBD and equilibrium for the entire structure. *Include all dummy loads!*
3. Indeterminate → choose redundant reactions R_i and get all reactions in terms of redundant reactions
4. Divide the beam into sections
   a) Section changes: external supports, changes in geometry, or changes in load
5. Cut each section; use equilibrium to find the internal resultants
   a) Sign convention is not important for internal resultants since they are squared in the energy
6. Internal resultants → strain energy in each segment → total strain energy ($U = U_1 + U_2 + U_3 + \dots$)
   a) <u>Sanity warning: Do not expand the squared terms in the integrals!!!</u>
7. For indeterminate problems: find all the reactions
   a) Take $0 = \partial U / \partial R_i$, set all dummy loads to 0, and solve together with the equilibrium equations
8. Find displacements and rotations: $\Delta_i = \partial U / \partial P_i$
   a) Set all dummy loads to 0 after taking the derivative