Objectives for beams

• Two weeks ago (end of Exam 1 material)
  • Calculate the internal shear force and bending moment in beams
  • Visualize these internal resultants using shear force and bending moment diagrams

• Last week (start of Exam 2 material)
  • Calculate the flexural stress distribution due to bending moments
  • Calculate the transverse shear stress distribution due to shear forces

• Today and next week
  • Calculate deflections and rotations of beams
  • Use the deflections to solve statically indeterminate problems
    • These are significantly more complex than indeterminate axial loading and torsion problems
  • Most of my examples will not be out of the Lecture Book
Deflections of beams: Overview

Recall the equilibrium equations for the internal shear force and bending moment:

\[
\frac{dV}{dx} = p(x) \quad \frac{dM}{dx} = V(x)
\]

In our derivation of the flexural stress, we also found the moment-curvature equation:

\[
\varepsilon = -\frac{y}{\rho}, \quad \sigma = -E \frac{y}{\rho}
\]

[\Rightarrow M = \frac{EI}{\rho}]

If the beam is long and thin, this equation is accurate even when the beam is not in pure bending.
Derivation of the governing equation

Goal: relate the **moment-curvature equation** to the **angle of rotation** $\theta$ and **deflection** $v$

As always, assume small rotations

$$\frac{1}{\rho} = \frac{M}{EI}$$

$\theta$ measures the angle of a tangent line to the **deflection curve** $v(x)$:

$$\theta \approx \tan \theta = \frac{dv}{dx}$$

The radius of curvature $\rho$ is also related to $v(x)$:

$$\frac{1}{\rho} \approx \frac{d^2v}{dx^2} \Rightarrow M(x) = EI \frac{d^2v}{dx^2}$$
Derivation of the governing equation

Combine with relationships between bending moment, shear force, and distributed load:

**Moment-curvature:**

\[ M(x) = EI \frac{d^2 v}{dx^2} \]

\[ \frac{dM}{dx} = V(x) = \frac{d}{dx} \left( EI \frac{d^2 v}{dx^2} \right) \]

**Load-deflection:**

\[ \frac{dV}{dx} = p(x) = \frac{d^2}{dx^2} \left( EI \frac{d^2 v}{dx^2} \right) \]

\[ p(x) = EI \frac{d^4 v}{dx^4} \]

**Constant cross section and material properties**

\[ V(x) = EI \frac{d^3 v}{dx^3} \]

Conclusion: we can integrate the moment-curvature equation twice or the load-deflection equation four times to find the deflection \( v(x) \).
Fourth- and second-order methods

Fourth-order method
Start with the governing equation and integrate four times

\[ EIv''''(x) = p(x) \]

Integrate

\[ EIv'''(x) = V(x) \]

Integrate

\[ EIv''(x) = M(x) \]

Integrate

\[ EIv'(x) = EI\theta(x) \]

Integrate

\[ EIv(x) \]

Second-order method
Cut the beam and use equilibrium to find \( M(x) \).
Then, integrate the moment-curvature equation twice

\[ EIv''(x) = M(x) \]

Integrate

\[ EIv'(x) = EI\theta(x) \]

Integrate

\[ EIv(x) \]
Example 1: Fourth-order method

Determine the deflection curve.

\[ v(x) = \frac{1}{6EI} \left( -P_B x^3 + 3P_B Lx^2 + 3M_B x^2 \right) \]

\[ v'(x) = \frac{1}{2EI} \left( -P_B x^2 + 2P_B Lx^2 + 2M_B x^2 \right) \]
### Boundary conditions

<table>
<thead>
<tr>
<th>BC type</th>
<th>Geometric BCs (2\textsuperscript{nd} and 4\textsuperscript{th} order method)</th>
<th>Natural BCs (4\textsuperscript{th} order method)</th>
</tr>
</thead>
<tbody>
<tr>
<td>fixed</td>
<td>$v = 0$ $v' = 0$</td>
<td>none</td>
</tr>
<tr>
<td>fixed rotation</td>
<td>$v' = 0$</td>
<td>$V = 0$</td>
</tr>
<tr>
<td>simple support</td>
<td>$v = 0$</td>
<td>$M = 0$</td>
</tr>
<tr>
<td>(pin or roller)</td>
<td>$v = 0$</td>
<td>$M = 0$</td>
</tr>
<tr>
<td>free</td>
<td></td>
<td>$V = 0$, $M = 0$</td>
</tr>
</tbody>
</table>
## Boundary conditions (cont.)

<table>
<thead>
<tr>
<th>BC type</th>
<th>Geometric BCs (2\textsuperscript{nd} and 4\textsuperscript{th} order method)</th>
<th>Natural BCs (4\textsuperscript{th} order method)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>external force</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>at $x = 0$:</td>
<td>$P$</td>
<td>$V(0^+) = +P$</td>
</tr>
<tr>
<td></td>
<td><strong>none</strong></td>
<td>$M(0) = 0$</td>
</tr>
<tr>
<td></td>
<td>$P$</td>
<td>$V(L^-) = -P$</td>
</tr>
<tr>
<td></td>
<td><strong>none</strong></td>
<td>$M(L) = 0$</td>
</tr>
<tr>
<td>at $x = L$:</td>
<td>$P$</td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>none</strong></td>
<td></td>
</tr>
<tr>
<td><strong>external moment</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>at $x = 0$:</td>
<td>$M_0$</td>
<td>$V(0) = 0$</td>
</tr>
<tr>
<td></td>
<td><strong>none</strong></td>
<td>$M(0^+) = -M_0$</td>
</tr>
<tr>
<td>at $x = L$:</td>
<td>$M_0$</td>
<td>$V(L) = 0$</td>
</tr>
<tr>
<td></td>
<td><strong>none</strong></td>
<td>$M(L^-) = +M_0$</td>
</tr>
</tbody>
</table>
Example 2: Second-order method

Determine the deflection curve. The beam has constant $EI$.

$$v(x) = \frac{w_0 L^4}{24 EI} \left[ -\left( \frac{x}{L} \right)^4 + 4 \left( \frac{x}{L} \right) - 3 \right]$$

$$v'(x) = \frac{w_0 L^3}{6 EI} \left[ -\left( \frac{x}{L} \right)^3 + 1 \right]$$
Example 3: Lecture Book, Example 11.4

Determine the deflection curve. The beam has constant $EI$.

\[ v(x) = \frac{P_0}{6EI} \begin{cases} 
\left(1 - \frac{a}{L}\right)x^3 + \left(3a - 2L - \frac{a^2}{L}\right)ax, & 0 \leq x \leq a \\
-\frac{a}{L}x^3 + 3ax^2 - \left(2aL + \frac{a^3}{L}\right)x + a^3, & a \leq x \leq L 
\end{cases} \]

\[ v'(x) = \frac{P_0}{6EI} \begin{cases} 
3\left(1 - \frac{a}{L}\right)x^2 + \left(3a - 2L - \frac{a^2}{L}\right)a, & 0 \leq x \leq a \\
-\frac{3a}{L}x^2 + 6ax - \left(2L + \frac{a^2}{L}\right)a, & a \leq x \leq L 
\end{cases} \]
## Continuity conditions

<table>
<thead>
<tr>
<th>CC type</th>
<th>Geometric CCs (2\textsuperscript{nd} and 4\textsuperscript{th} order method)</th>
<th>Natural CCs (4\textsuperscript{th} order method)</th>
</tr>
</thead>
</table>
| roller  | $v_2 = v_1 = 0$  
$v_2' = v_1'$ | $M_1 = M_2$ |
| discontinuity in load function | $v_2 = v_1$  
$v_2' = v_1'$ | $V_2 = V_1$  
$M_2 = M_1$ |
| point force  | $v_2 = v_1$  
$v_2' = v_1'$ | $V_2 = V_1 + P_0$  
$M_2 = M_1$ |
| point moment  | $v_2 = v_1$  
$v_2' = v_1'$ | $V_2 = V_1$  
$M_2 = M_1 - M_0$ |
| pin with force  | $v_2 = v_1$ | $V_2 = V_1 + P_0$  
$M_2 = M_1 = 0$ |
Determine the reactions at A and B and the deflection curve. The beam has constant $EI$.

\[ v \left( \frac{2L}{3} \right) = \frac{-M_0 L^2}{27EI} \]

Note: $M_0$ is an applied moment. It is not a reaction at B. Since B is a roller, it only provides a reaction force in the y direction.

\[ v(x) = \frac{M_0 L^2}{4EI} \left[ \left( \frac{x}{L} \right)^3 - \left( \frac{x}{L} \right)^2 \right] \]

\[ v'(x) = \frac{M_0 L}{4EI} \left[ 3 \left( \frac{x}{L} \right)^2 - 2 \left( \frac{x}{L} \right) \right] \]
Example 5

Determine the deflection curve. The beam has constant $EI$.

\[ v \left( \frac{3L}{7} \right) = \frac{-PL^3}{98EI} \]

\[ v(x) = \begin{cases} 
20 \left( \frac{x}{L} \right)^3 - 12 \left( \frac{x}{L} \right)^2 & , \quad 0 \leq x \leq L/3 \\
-7 \left( \frac{x}{L} \right)^3 + 15 \left( \frac{x}{L} \right)^2 - 9 \left( \frac{x}{L} \right) + 1 & , \quad L/3 \leq x \leq L
\end{cases} \]

\[ v'(x) = \begin{cases} 
20 \left( \frac{x}{L} \right)^2 - 8 \left( \frac{x}{L} \right) & , \quad 0 \leq x \leq L/3 \\
-7 \left( \frac{x}{L} \right)^2 + 10 \left( \frac{x}{L} \right) - 3 & , \quad L/3 \leq x \leq L
\end{cases} \]
Example 6

Determine the deflection curve. The beam has constant $EI$. 
Example 6

\[ v(x) = \frac{w_0 L^4}{336EI} \begin{cases} 
-14 \left( \frac{x}{L} \right)^4 + 13 \left( \frac{x}{L} \right)^3 - 3 \left( \frac{x}{L} \right)^2 \\
-14 \left( \frac{x}{L} \right)^4 + 45 \left( \frac{x}{L} \right)^3 - 51 \left( \frac{x}{L} \right)^2 + 24 \left( \frac{x}{L} \right) - 4, & 0 \leq x \leq L/2 \\
-14 \left( \frac{x}{L} \right)^4 + 45 \left( \frac{x}{L} \right)^3 - 51 \left( \frac{x}{L} \right)^2 + 24 \left( \frac{x}{L} \right) - 4, & L/2 \leq x \leq L 
\end{cases} \]

\[ v'(x) = \frac{w_0 L^3}{336EI} \begin{cases} 
-56 \left( \frac{x}{L} \right)^3 + 39 \left( \frac{x}{L} \right)^2 - 6 \left( \frac{x}{L} \right) \\
-56 \left( \frac{x}{L} \right)^3 + 135 \left( \frac{x}{L} \right)^2 - 102 \left( \frac{x}{L} \right) + 24, & 0 \leq x \leq L/2 \\
-56 \left( \frac{x}{L} \right)^3 + 135 \left( \frac{x}{L} \right)^2 - 102 \left( \frac{x}{L} \right) + 24, & L/2 \leq x \leq L 
\end{cases} \]
Example 7: Lecture Book, Example 11.6
Procedure: 2\textsuperscript{nd}-order method

1. FBD of the entire beam
2. Equilibrium for reaction forces and moments
3. Find the internal moment $M(x)$ in each segment
4. Integrate the moment-curvature equation for each segment: $EIv''(x) = M(x)$
5. Apply boundary and continuity conditions for $v(x)$ and $v'(x)$
6. Solve for unknowns and check units!
7. Calculate $v(x)$ and $v'(x)$ at any required points (typically maxima, minima, endpoints)
   1. Check free ends and points where $v'(x) = 0$ to find the maximum deflection
Procedure: 4\textsuperscript{th}-order method

1. FBD of the entire beam and equilibrium for reaction forces and moments (not required, but useful for checking your solutions for $V(x)$ and $M(x)$)

2. Write down the load function $p(x)$ in each segment. If there is no distributed load, $p(x) = 0$

3. Integrate the \textbf{load-deflection equation} for each segment: $EIv''''(x) = p(x)$

4. Apply boundary and continuity conditions for $v(x)$, $v'(x)$, $V(x)$, and $M(x)$
   1. Remember that $EIv''''(x) = V(x)$ and $EIv''''(x) = M(x)$

5. Solve for unknowns and check units!

6. Calculate $v(x)$ and $v'(x)$ at any required points (typically maxima, minima, endpoints)