Lectures 20-23: Beams—Deflections

Lecture Book: Chapter 11

Joshua Pribe

Fall 2019

PURDUE ENGINEERING

Objectives for beams

- ~~Two weeks ago (end of Exam 1 material)~~
    - ~~Calculate the internal shear force and bending moment in beams~~
    - ~~Visualize these internal resultants using shear force and bending moment diagrams~~
- ~~Last week (start of Exam 2 material)~~
    - ~~Calculate the flexural stress distribution due to bending moments~~
    - ~~Calculate the transverse shear stress distribution due to shear forces~~
- Today and next week
    - Calculate deflections and rotations of beams
    - Use the deflections to solve statically indeterminate problems
        - These are *significantly* more complex than indeterminate axial loading and torsion problems
    - Most of my examples will not be out of the Lecture Book

Deflections of beams: Overview

Recall the equilibrium equations for the internal shear force and bending moment:

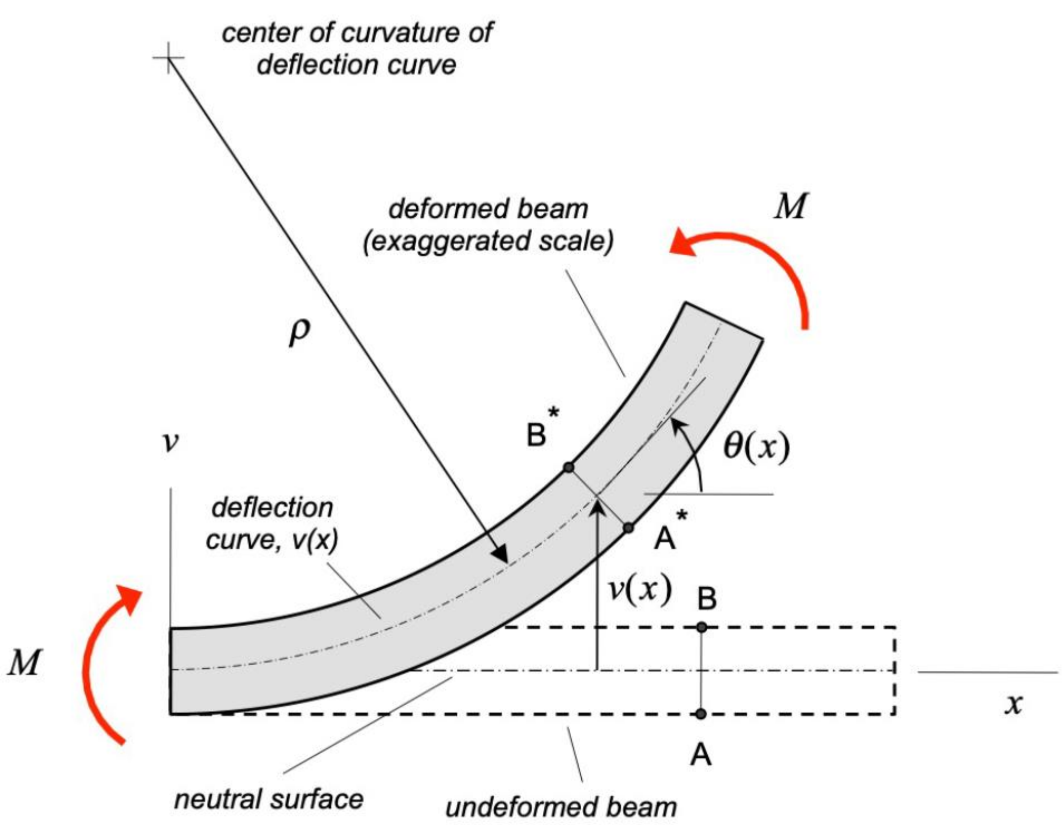
$$\frac{dV}{dx} = p(x) \qquad \frac{dM}{dx} = V(x)$$

Lecture Book: Chapter 11, Page 2

In our derivation of the flexural stress,
we also found the **moment-curvature equation**:

$$\varepsilon = -\frac{y}{\rho}, \quad \sigma = -E\frac{y}{\rho}$$

$$\Rightarrow M = \frac{EI}{\rho}$$



If the beam is long and thin, this equation is accurate even when the beam is not in pure bending

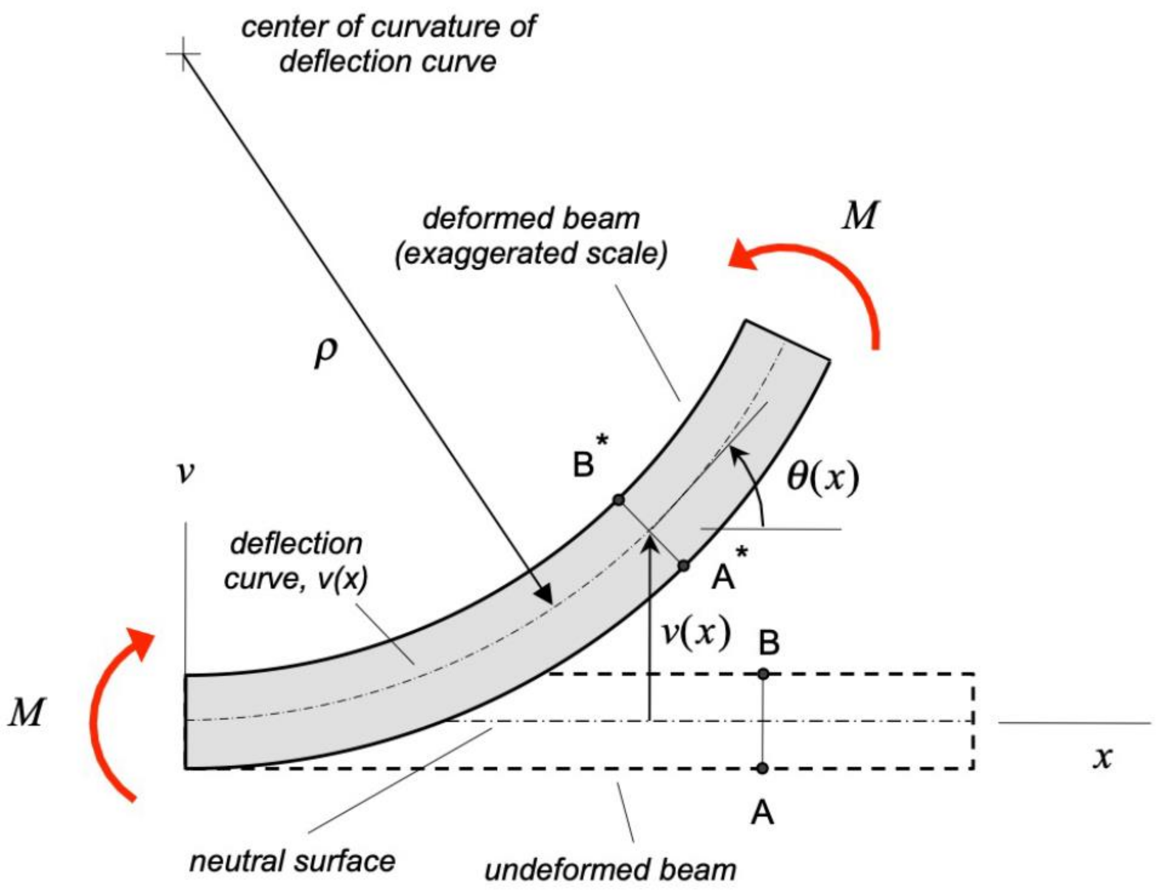
Derivation of the governing equation

Goal: relate the **moment-curvature equation** to the **angle of rotation θ** and **deflection v**

As always, assume small rotations

$$\frac{1}{\rho} = \frac{M}{EI}$$

Lecture Book: Chapter 11, Page 2



θ measures the angle of a tangent line to the **deflection curve $v(x)$**:

$$\theta \approx \tan\theta = \frac{dv}{dx}$$

The radius of curvature ρ is also related to $v(x)$:

$$\frac{1}{\rho} \approx \frac{d^2v}{dx^2} \Rightarrow M(x) = EI\,\frac{d^2v}{dx^2}$$

Derivation of the governing equation

Combine with relationships between bending moment, shear force, and distributed load:

Moment-curvature: $$M(x) = EI\frac{d^2v}{dx^2}$$

$$\frac{dM}{dx} = V(x) = \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right)$$

Load-deflection: $$\frac{dV}{dx} = p(x) = \frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)$$

Constant cross section and material properties

$$V(x) = EI\frac{d^3v}{dx^3}$$

$$p(x) = EI\frac{d^4v}{dx^4}$$

Conclusion: we can integrate the moment-curvature equation twice or the load-deflection equation four times to find the deflection $v(x)$.

Fourth- and second-order methods

<u>Fourth-order method</u>

Start with the governing equation and integrate four times

$$EIv''''(x) = p(x)$$

Integrate

$$EIv'''(x) = V(x)$$

Integrate

$$EIv''(x) = M(x)$$

Integrate

$$EIv'(x) = EI\theta(x)$$

Integrate

$$EIv(x)$$

<u>Second-order method</u>

Cut the beam and use equilibrium to find $M(x)$. Then, integrate the moment-curvature equation twice

$$EIv''(x) = M(x)$$

Integrate

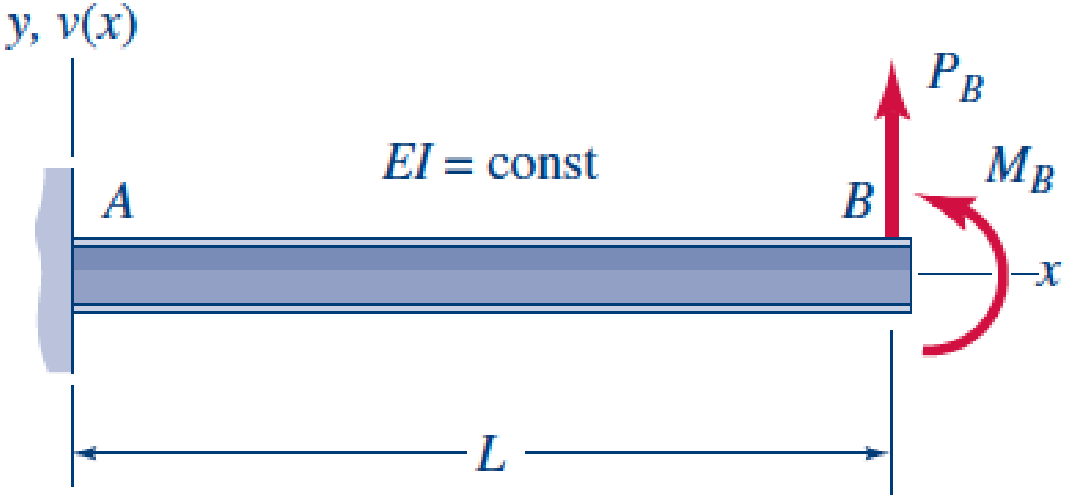
$$EIv'(x) = EI\theta(x)$$

Integrate

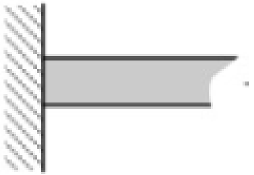
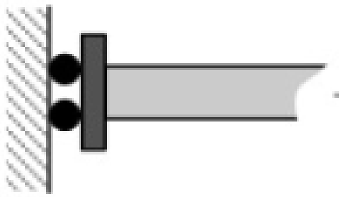
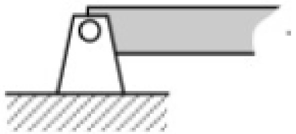
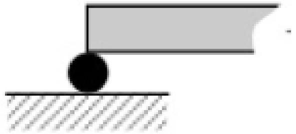
$$EIv(x)$$

Example 1: Fourth-order method

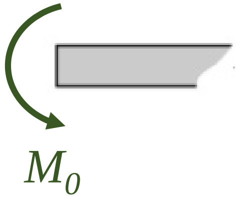
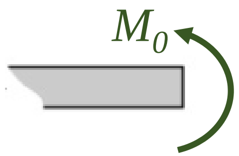
Determine the deflection curve.



Boundary conditions

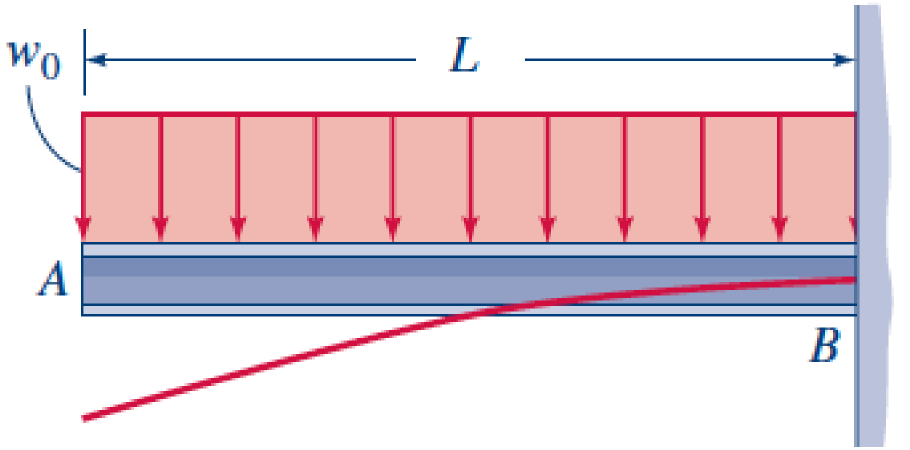
BC type		Geometric BCs (2nd *and* 4th order method)	Natural BCs (4th order method)
fixed		$v = 0$ $v' = 0$	*none*
fixed rotation		$v' = 0$	$V = 0$
simple support (pin or roller)	*pin joint*  *roller* 	$v = 0$ $v = 0$	$M = 0$ $M = 0$
free		*none*	$V = 0$ $M = 0$

Boundary conditions (cont.)

BC type		Geometric BCs (2nd *and* 4th order method)	Natural BCs (4th order method)
external force	at x = 0: 	*none*	$V(0^+) = +P$ $M(0) = 0$
	at $x = L$: 	*none*	$V(L^-) = -P$ $M(L) = 0$
external moment	at x = 0: 	*none*	$V(0) = 0$ $M(0^+) = -M_0$
	at $x = L$: 	*none*	$V(L) = 0$ $M(L^-) = +M_0$

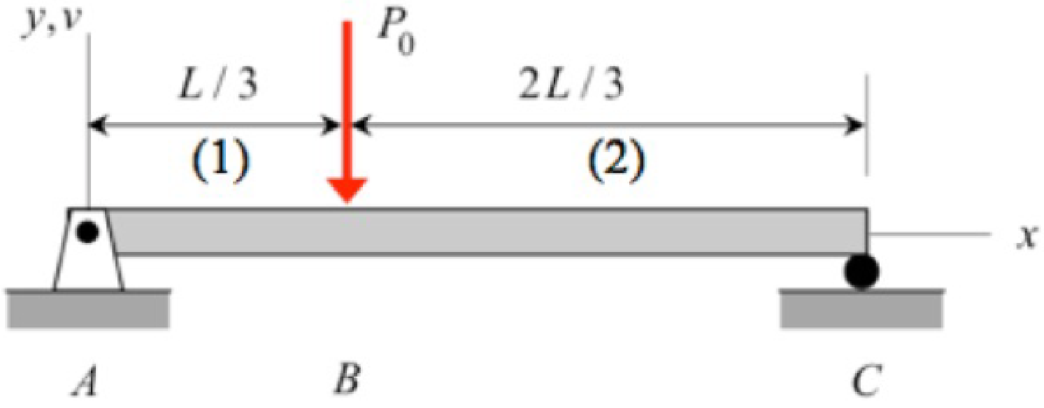
Example 2: Second-order method

Determine the deflection curve. The beam has constant EI.

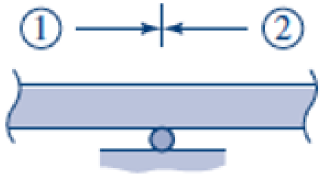
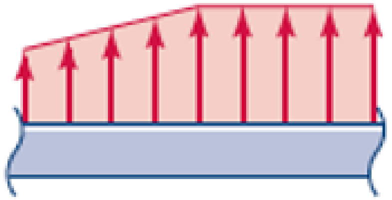
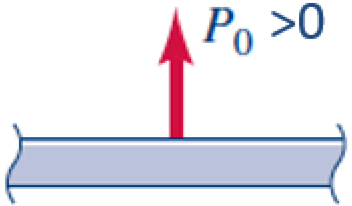
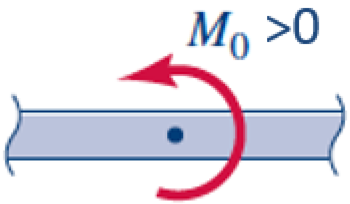
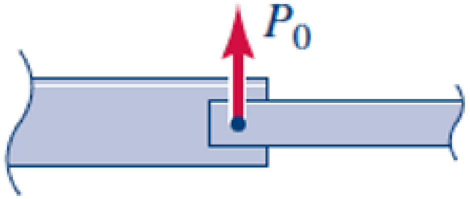


Example 3: Lecture Book, Example 11.4

Determine the deflection curve. The beam has constant EI.

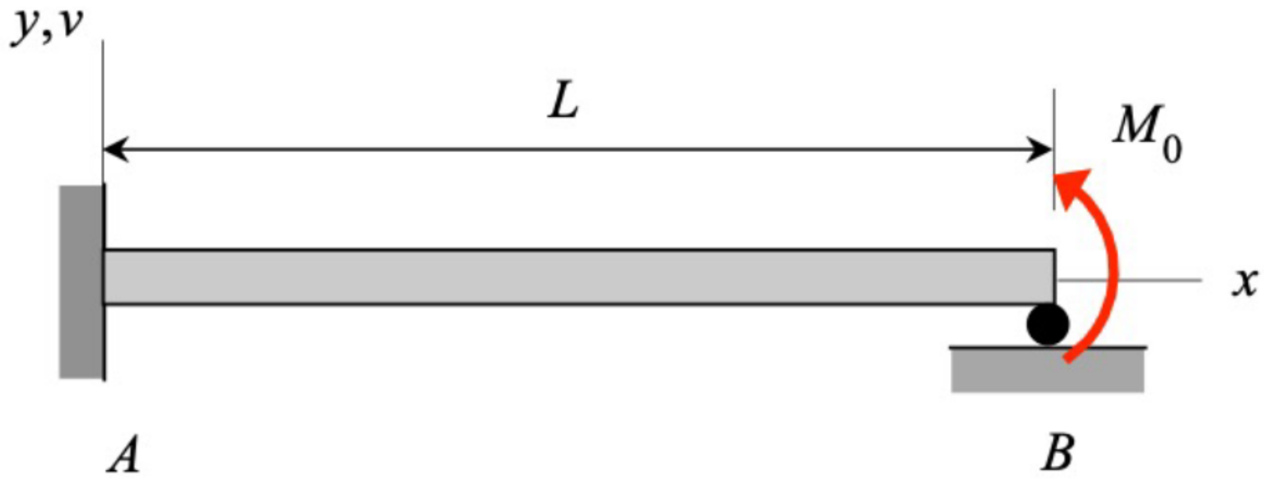


Continuity conditions

CC type		Geometric CCs (2nd *and* 4th order method)	Natural CCs (4th order method)
roller		$v_2 = v_1 = 0$ $v'_2 = v'_1$	$M_1 = M_2$
discontinuity in load function		$v_2 = v_1$ $v'_2 = v'_1$	$V_2 = V_1$ $M_2 = M_1$
point force		$v_2 = v_1$ $v'_2 = v'_1$	$V_2 = V_1 + P_0$ $M_2 = M_1$
point moment		$v_2 = v_1$ $v'_2 = v'_1$	$V_2 = V_1$ $M_2 = M_1 - M_0$
pin with force		$v_2 = v_1$	$V_2 = V_1 + P_0$ $M_2 = M_1 = 0$

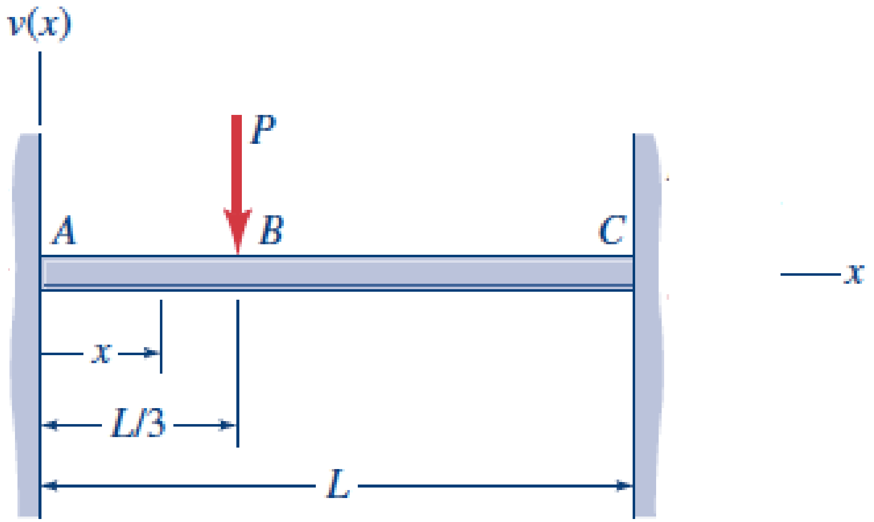
Example 4: Lecture Book, Example 11.10

Determine the reactions at A and B and the deflection curve. The beam has constant EI.



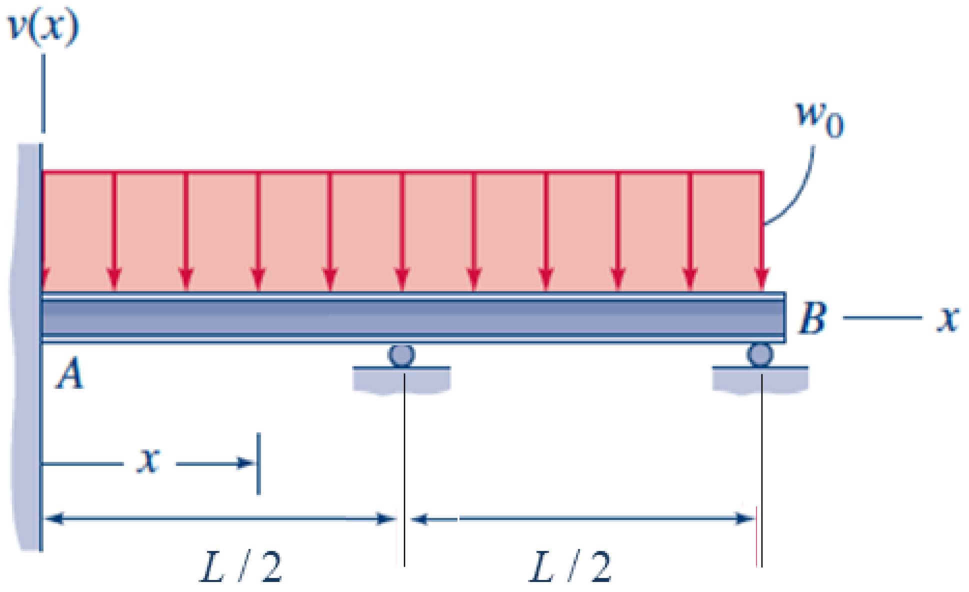
Example 5

Determine the deflection curve. The beam has constant EI.



Example 6

Determine the deflection curve. The beam has constant EI.



Procedure: 2^{nd}-order method

1. FBD and equilibrium for the entire beam → equations for reaction forces and moments
2. Divide the beam into segments. Use FBDs and equilibrium to find equations for the moment $M(x)$ in each segment
3. Write down the **moment-curvature equation** for each segment: $EIv''(x) = M(x)$
4. Integrate the moment-curvature equation twice → equations for $v'(x)$ and $v(x)$. Remember to include the constants of integration.
5. Write down the *geometric* BCs and CCs (i.e. BCs and CCs for v' and v)
6. Use the BCs and CCs to solve for the constants of integration
    1. If the problem is *indeterminate*, you need the BCs and CCs to solve for the reaction forces and moments
7. Calculate $v(x)$ and $v'(x)$ at any required points (typically maxima, minima, endpoints)

Recommendation: use the 2^{nd}-order method when you have to break the beam into multiple segments

Procedure: 4th-order method

1. FBD of the entire beam (do not need to enforce equilibrium)
2. Split the beam into segments. Write down the load function $p(x)$ in each segment.
   1. If there are no distributed loads in a segment, $p(x) = 0$
3. Write down the **load-deflection equation** for each segment: $EIv''''(x) = p(x)$
4. Integrate load-deflection equation four times → equations for $V(x)$, $M(x)$, $v'(x)$, & $v(x)$. Remember to include the constants of integration.
5. Write down the *natural and geometric* BCs and CCs (i.e. BCs and CCs for V, M, v', & v)
6. Use the BCs and CCs to solve for the constants of integration
   1. You can also determine any unknown reaction forces and moments if required
7. Calculate $v(x)$ and $v'(x)$ at any required points (typically maxima, minima, endpoints)

If you are confused about signs, remember: $V(x^+) = V(x^-) + P, \quad M(x^+) = M(x^-) - M_0$
(for upward P and CCW M_0)