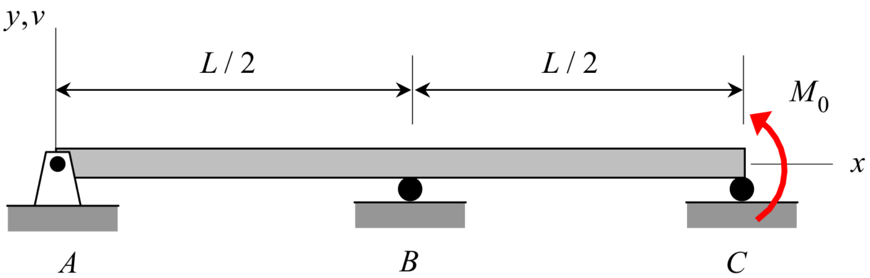
***Example 11.15***

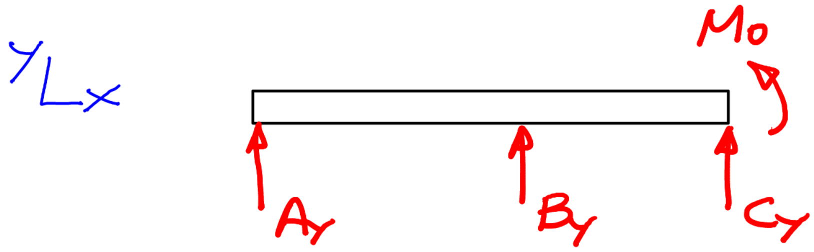
The beam is made up of a material with an elastic modulus E that is constant throughout the beam. The beam has a square cross section of dimensions $h \times h$.

a) Determine the reactions at A, B and C.

b) Determine the bending moment at B.

c) Determine the maximum (magnitude) normal stress at B.




$$\Sigma M_A = B_y\left(\frac{L}{2}\right) + C_y(L) + M_0 = 0$$

$$\hookrightarrow \quad B_y = -2C_y - \frac{2M_0}{L} \qquad (1)$$

$$\Sigma F_y = A_y + B_y + C_y = 0 \qquad (2)$$

Using discontinuity functions:

$$p(x) = A_y\langle x\rangle^{-1} + B_y\langle x - \tfrac{L}{2}\rangle^{-1} + C_y\langle x - L\rangle^{-1} - M_0\langle x - L\rangle^{-2}$$

already included

$$V(x) = \boxed{\cancel{V(0)}} + \int_0^x p(x)\,dx$$

$$= A_y\langle x\rangle^{0} + B_y\langle x - \tfrac{L}{2}\rangle^{0} + C_y\langle x - L\rangle^{0} - M_0\langle x - L\rangle^{-1}$$

(pin joint)

$$M(x) = \cancel{M(0)}^{\,0} + \int_0^x V(x)\,dx$$

$$= A_y\langle x\rangle^{1} + B_y\langle x - \tfrac{L}{2}\rangle^{1} + C_y\langle x - L\rangle^{1} - M_0\langle x - L\rangle^{0}$$

$$\theta(x) = \theta(0) + \frac{1}{EI}\int_0^x M(x)\,dx$$

$$= \overset{?}{\theta_A} + \frac{1}{EI}\left[\frac{A_y}{2}\langle x\rangle^2 + \frac{B_y}{2}\langle x-\tfrac{L}{2}\rangle^2 + \frac{C_y}{2}\langle x-L\rangle^2 - M_0\langle x-L\rangle^1\right]$$

$$v(x) = \cancelto{0}{v(0)} + \int_0^x \theta(x)\,dx$$

$$= \theta_A x + \frac{1}{EI}\left[\frac{A_y}{6}\langle x\rangle^3 + \frac{B_y}{6}\langle x-\tfrac{L}{2}\rangle^3 + \frac{C_y}{6}\langle x-L\rangle^3 - \frac{M_0}{2}\langle x-L\rangle^2\right]$$

<u>BC's</u>

- $$v(0) = \theta_A\cancelto{0}{(0)} + \frac{1}{EI}\left[\frac{A_y}{6}\cancelto{0}{\langle 0\rangle^3} + \frac{B_y}{6}\cancelto{0}{\langle -\tfrac{L}{2}\rangle^3} + \frac{C_y}{6}\cancelto{0}{\langle -L\rangle^3} - \frac{M_0}{2}\cancelto{0}{\langle -L\rangle^3}\right] \overset{\checkmark}{=} 0$$

- $$v(\tfrac{L}{2}) = \theta_A(\tfrac{L}{2}) + \frac{1}{EI}\left[\frac{A_y}{6}(\tfrac{L}{2})^3 + \frac{B_y}{6}\cancelto{0}{\langle 0\rangle^3} + \frac{C_y}{6}\cancelto{0}{\langle -\tfrac{L}{2}\rangle^3} - \frac{M_0}{2}\cancelto{0}{\langle -\tfrac{L}{2}\rangle^3}\right]$$

$$0 = \theta_A \cancel{\frac{L}{2}} + \frac{1}{EI}\,\frac{A_y L^{\cancel{3}\,2}}{\cancel{48}\,24}$$

$$\hookrightarrow \boxed{\theta_A = -\frac{A_y L^2}{24EI}}$$

- $$v(L) = \theta_A L + \frac{1}{EI}\left[\frac{A_y}{6}L^3 + \frac{B_y}{6}(\tfrac{L}{2})^3 + \frac{C_y}{6}\cancelto{0}{\langle 0\rangle^3} - \frac{M_0}{2}\cancelto{0}{\langle 0\rangle^3}\right]$$

$$0 = \left(-\frac{A_y \cancel{L^2}}{24\cancel{EI}}\right)\cancel{L} + \cancel{\frac{1}{EI}}\left[\frac{A_y \cancel{L^3}}{6} + \frac{B_y \cancel{L^3}}{48}\right]$$

$$= \frac{A_y}{8} + \frac{B_y}{48} \Rightarrow A_y = -\frac{1}{6}B_y \qquad (3)$$

<u>Solve</u>

(2) & (3): $-\frac{B_y}{6} + B_y + C_y = 0 \Rightarrow B_y = -\frac{6}{5}C_y \qquad (4)$

(1) & (4): $-\frac{6}{5}C_y = -2C_y - \frac{2M_0}{L}$

$$\hookrightarrow C_y = -\frac{5M_0}{2L}$$

$$\hookrightarrow B_y = -\frac{6}{5}C_y = \frac{3M_0}{L}$$

$\hookrightarrow \quad A_y = -\frac{1}{6} B_y = -\frac{M_0}{2L}$

$\hookrightarrow \quad \theta_A = -\frac{A_y L^2}{2EI} = \frac{M_0 L}{4EI}$

$\therefore$

$$\boxed{\begin{aligned} v(x) &= \frac{M_0 L}{4EI} x + \frac{1}{EI}\Big[-\frac{M_0}{12L}\langle x\rangle^3 + \frac{M_0}{2L}\langle x - \tfrac{L}{2}\rangle^3 \\ &\qquad - \frac{5M_0}{12L}\langle x - L\rangle^3 - \frac{M_0}{2}\langle x - L\rangle^2\Big] \end{aligned}}$$

<u>or</u>:

<u>$0 < x < \frac{L}{2}$</u>:

$$v(x) = \frac{M_0 L}{4EI} x - \frac{M_0}{12EIL} x^3 = \frac{M_0 L^2}{4EI}\left[\frac{x}{L} - \frac{1}{3}\left(\frac{x}{L}\right)^3\right]$$

<u>$\frac{L}{2} < x < L$</u>:

$$v(x) = \frac{M_0 L}{4EI} x + \frac{1}{EI}\left[-\frac{M_0}{12L} x^3 + \frac{M_0}{2L}\left(x - \frac{L}{2}\right)^3\right]$$

$$= \frac{M_0 L^2}{4EI}\left[\frac{x}{L} - \frac{1}{3}\left(\frac{x}{L}\right)^3 + 2\left(\frac{x}{L} - \frac{1}{2}\right)^3\right]$$

(b) $M\left(\frac{L}{2}\right) = A_y \langle \frac{L}{2} \rangle^1 + B_y \langle \frac{L}{2} \rangle^1$

$$= \left(-\frac{M_0}{2L}\right)\left(\frac{L}{2}\right) + \left(\frac{3M_0}{L}\right)\left(\frac{L}{2}\right) = \frac{5}{4} M_0$$

(c) $|\sigma| = \frac{My}{I} = \frac{5M_0}{4}\left(\frac{h/2}{h^4/12}\right) = \frac{15}{2}\frac{M_0}{h^3}$