Example 16.10

An elastic rod BC of uniform cross section is bent into the form of a three quarter (270°) circle such that its mean radius is R. The rod is fixed to a wall at B and is pinned to a roller at C. The rod is composed of a material of Young's modulus E, and the second area moment of the cross section is I. A downward load P is applied at end C of the rod as shown in Fig 9.3. Assuming that elastic strain energies due to shear and axial loads are negligible as compared to bending strain energy:

a) Set up the integral to calculate the total bending strain energy in the rod BC as a function of load P, the unknown reaction F_C at C, and the angle θ.

b) Use Castigliano's theorem to determine the reaction force F_C.

c) Use Castigliano's theorem to determine the downward deflection of end C.

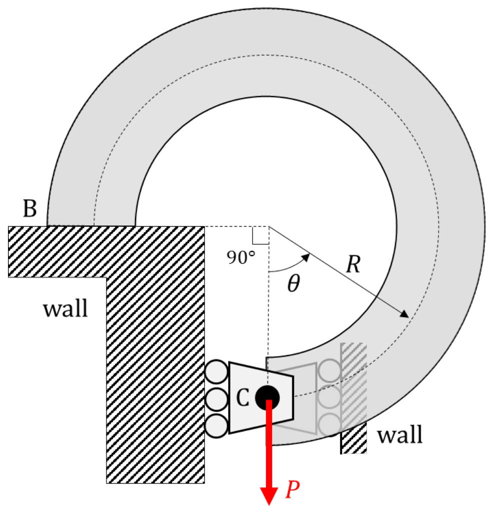


Fig 9.4

Solution

From FBD of segment CD,

$$\Sigma M_D = M_\theta + PR\sin\theta + F_C R(1-\cos\theta) = 0$$

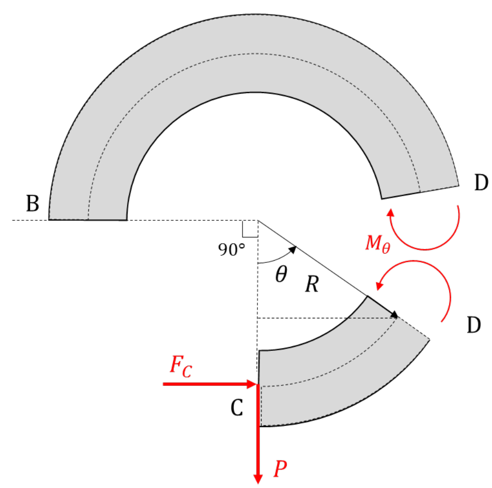
$$\Rightarrow M_\theta = -PR\sin\theta - F_C R(1-\cos\theta)$$

Therefore:

$$U(\theta) = \int_0^{L_{CD}} \frac{M_\theta^2}{2EI}\, d(L_{CD})$$

where, $d(L_{CD}) = d(R\theta) = R$

$$\Rightarrow U(\theta) = \int_0^{\theta} \frac{M_\theta^2 R}{2EI}\, d\theta$$



Total strain energy in BC:

$$U_{BC} = \frac{R}{2EI}\int_0^{\frac{3\pi}{2}} M_\theta^2\, d\theta = \frac{R}{2EI}\int_0^{\frac{3\pi}{2}} R^2[P\sin\theta + F_C(1-\cos\theta)]^2\, d\theta$$

Boundary condition at C: Horizontal deflection at C, $u_C = 0$. Since the horizontal load at C is F_C,

$$\frac{\partial U_{BC}}{\partial F_C} = 0$$

Using the fundamental theorem of calculation (Leibnitz' rule of differentiation within an integral) on (3.3),

$$\frac{\partial U_{BC}}{\partial F_C} = \frac{R}{2EI}\int_0^{\frac{3\pi}{2}} \frac{\partial\{R^2[P\sin\theta + F_C(1-\cos\theta)]^2\}}{\partial F_C} d\theta = 0$$

$$\Rightarrow \frac{R^3}{2EI}\int_0^{\frac{3\pi}{2}} 2(P\sin\theta + F_C(1-\cos\theta))(R(1-\cos\theta))\, d\theta = 0$$

$$\Rightarrow \int_0^{\frac{3\pi}{2}} PR\sin\theta\,(1-\cos\theta) + F_C R(1-\cos\theta)^2\, d\theta = 0$$

$$\Rightarrow \int_0^{\frac{3\pi}{2}} PR\sin\theta\,(1-\cos\theta)\, d\theta + \int_0^{\frac{3\pi}{2}} F_C R(1-\cos\theta)^2\, d\theta = 0$$

$$\Rightarrow \frac{PR}{2} + F_C R\left(2 + \frac{9\pi}{4}\right) = \frac{2P}{4} + F_C\left(\frac{8+9\pi}{4}\right) = 0$$

$$\Rightarrow F_C = -\frac{2P}{8+9\pi} = -0.055P \#(4.4)$$

Downward deflection at end C using Castigliano's theorem,

$$v_C = \frac{\partial U_{BC}}{\partial P} = \frac{R}{2EI}\int_0^{\frac{3\pi}{2}} \frac{\partial\{R^2[P\sin\theta + F_C(1-\cos\theta)]^2\}}{\partial P} d\theta$$

$$\Rightarrow v_C = \frac{R^3}{2EI}\int_0^{\frac{3\pi}{2}} 2\big(P\sin\theta + F_C(1-\cos\theta)\big)\sin\theta)\, d\theta$$

$$\Rightarrow v_C = \frac{PR^3}{EI}\int_0^{\frac{3\pi}{2}} \sin^2\theta\, d\theta + \frac{F_C R^3}{EI}\int_0^{\frac{3\pi}{2}} \sin\theta\,(1-\cos\theta)\, d\theta = \frac{PR^3}{EI}\left(\frac{3\pi}{4}\right) + \frac{F_C R^3}{EI}\left(\frac{1}{2}\right) \#(4.5)$$

Substituting (3.4) into (3.5),

$$\Rightarrow v_C = \frac{PR^3}{4EI}\big(3\pi + 2(-0.055)\big) = 2.329\frac{PR^3}{EI}$$