ME 274: Basic Mechanics II

Lecture 40: Vibrations – Forced response

PURDUE UNIVERSITY | School of Mechanical Engineering

Announcements:

- Fill out the course Eval. and submit proof on Gradescope for bonus points – deadline 5/3
- Submit any questions you have for Friday's class over gradescope

Final Exam Info:

- Tuesday 5/5, 7-9 pm in UC 114
- Inform me of conflicts ASAP
- Review session 5/3 @ 7pm over Zoom – see Exams tab on course blog for the link
- Content: 3 longform problems on material after exam 2, multiple choice covering the entire course

Forced Vibration Response

- Remember that the solution to the EOM $\boldsymbol{M\ddot{x} + C\dot{x} + Kx = F(t)}$ is $\boldsymbol{x(t) = x_c(t) + x_p(t)}$
- Specifically, we are looking at systems that have a harmonic excitation: $\boldsymbol{F(t) = F_0 \sin \omega t}$ OR $\boldsymbol{F(t) = F_0 \cos \omega t}$
- The general form of the particular solution can be assumed to be: $\mathbf{x_p}\boldsymbol{(t) = A\,sin\,\omega t + B\,sin\,\omega t}$

Considering a *sine* excitation:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{M}F_0 sin\omega t$$

Substituting in our assumed solution form:

$$(\text{-}\omega^2 A\sin\omega t - \omega^2 B\cos\omega t) + 2\zeta\omega_n(\omega A\cos\omega t - \omega B\sin\omega t) + \omega_n^2(A\sin\omega t + B\cos\omega t) = \frac{1}{M}F_0\sin\omega t$$

$$\Rightarrow [(\omega^2 + \omega_n^2)A + (2\zeta\omega_n\omega)B]\sin\omega t + [-(2\zeta\omega_n\omega)A + (-\omega^2 + \omega_n^2)B]\cos\omega t = \frac{1}{M}F_0\sin\omega t$$

Balancing coefficients:

$$cosine: \quad -(2\zeta\omega_n\omega)A + (-\omega^2 + \omega_n^2)B = 0$$

$$sine: \quad (\omega^2 + \omega_n^2)A + (2\zeta\omega_n\omega)B = \frac{1}{M}F_0$$

From last lecture: Undamped Force Response

$$x_p(t) = A\sin\omega t = \left[\frac{F_0/K}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]\sin\omega t$$

$$\rightarrow \quad A = \frac{F_0/K}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

Low frequencies ($\frac{\omega}{\omega_n} \ll 1$):

- Quasistatic response → amplitude is $\approx F_0/K$
- System essentially behaves as though there is a constant applied force of magnitude F_0

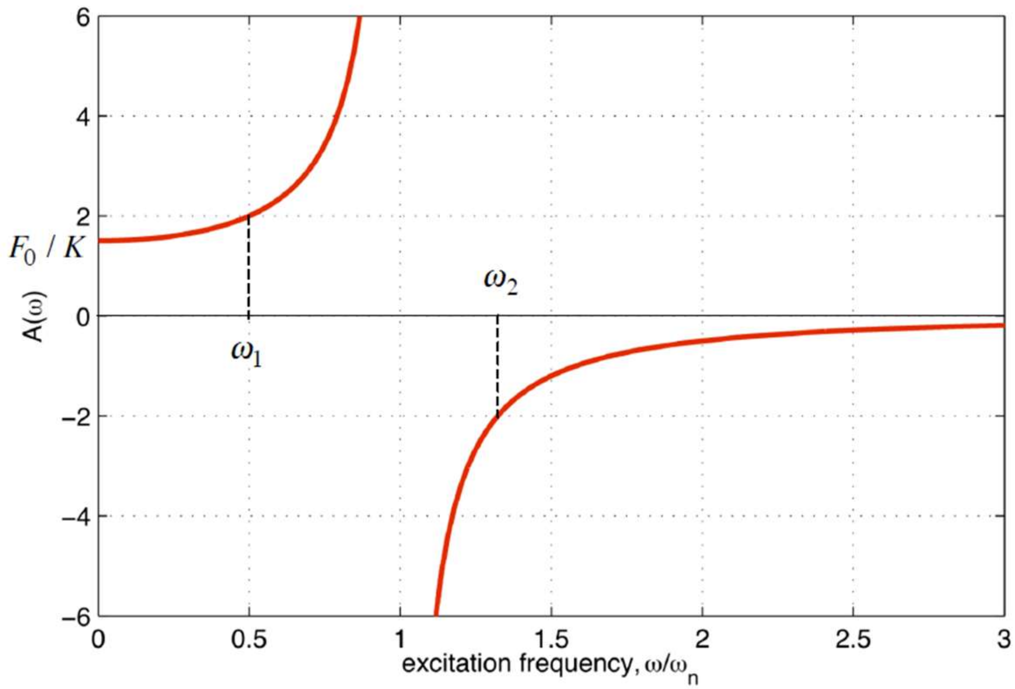
High frequencies ($\frac{\omega}{\omega_n} \gg 1$):

- Amplitude asymptotically approaches 0
- At high frequencies the system cannot respond quickly enough due to inertial effects

Resonance ($\frac{\omega}{\omega_n} = 1$)

- The amplitude of the response is unbounded

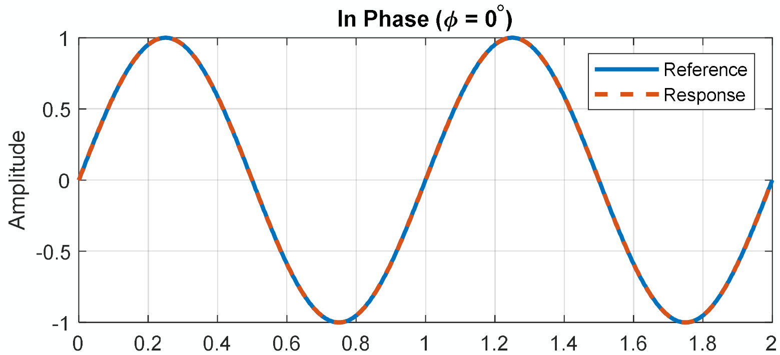
Response amplitude vs. forcing frequency

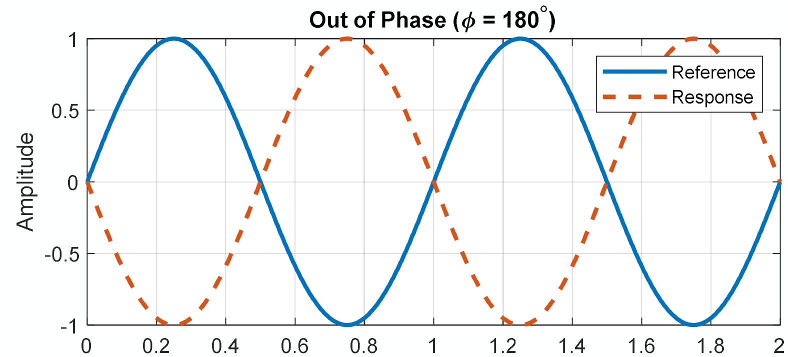


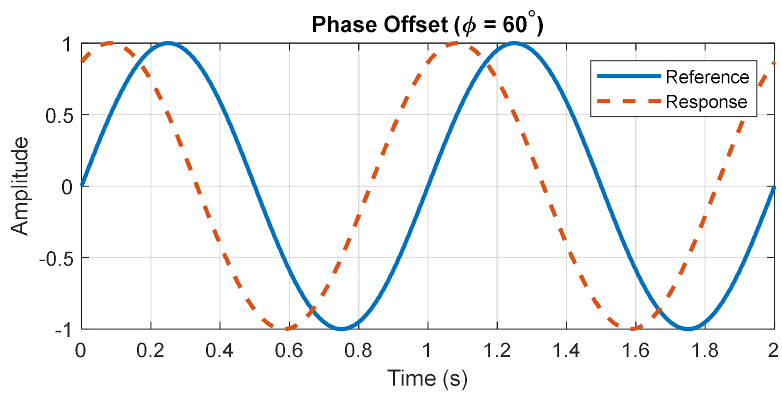
The particular solution $x_p(t) = A\sin\omega t + B\cos\omega t$ can be rewritten:

$$x_p(t) = X\sin(\omega t - \phi)$$

$\phi \rightarrow$ phase
$X \rightarrow$ amplitude



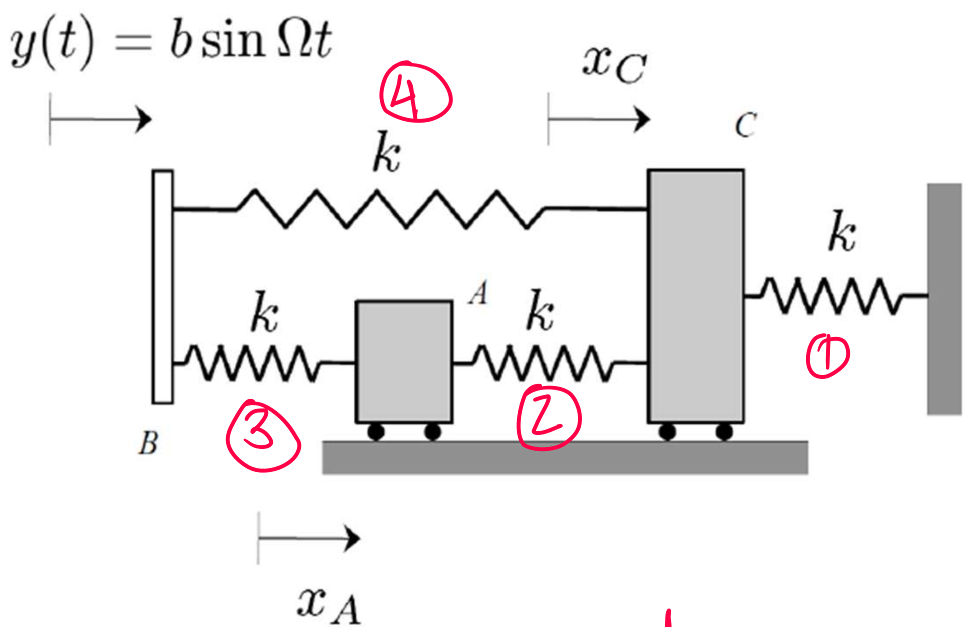


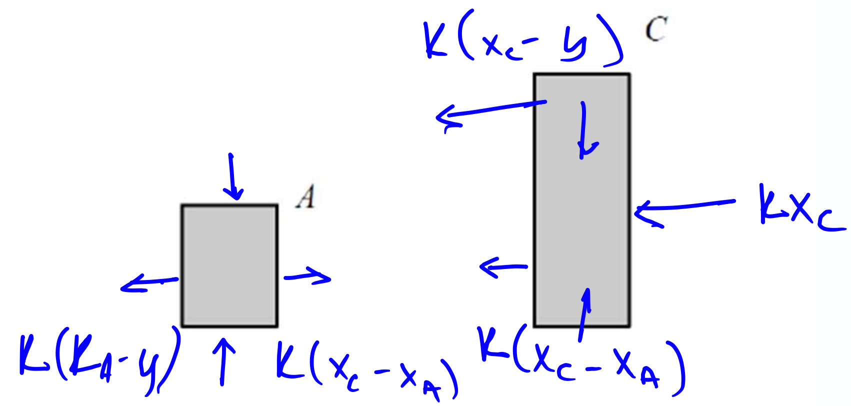


$X = \sqrt{A^2 + B^2}$

Question C6.6

The positions of particles A and C are described by the coordinates x_A and x_C shown below. The base B on the left side is given prescribed motion of $y(t) = b \sin \Omega t$. All springs are unstretched when $x_A = x_C = y = 0$. In the free body diagrams shown below, draw and label [in terms of k, $y(t)$, x_A and x_C] the spring force vectors on particles A and C.





1) spring compressed

2) assume $x_c > x_A$ → spring stretched

3) assume $x_A > y$ → spring stretched

4) assume $x_c > y$ → spring stretched

Question C6.12
The following equation of motion (EOM) has been derived for an undamped single-degree-of-freedom system:

$$2\ddot{x} + 800x = f(t) = 40 \sin \Omega t$$

Let $x_P(t)$ represent the particular solution of this EOM. Is $x_P(t)$ in phase or 180° out of phase with the excitation $f(t)$ when $\Omega = 30$ rad/s? Provide a justification for your answer.

$$\ddot{x} + \underbrace{400}_{\omega_n^2} x = 20 \sin \Omega t$$

$$\omega_n^2 \rightarrow \omega_n = 20 \text{ rad/s}$$

$$x_p(t) = A\cos\Omega t + B\sin\Omega t$$

$$\ddot{x}_p(t) = -A\Omega^2\cos\Omega t - B\Omega^2\sin\Omega t$$

$$(-A\Omega^2\cos\Omega t - B\Omega^2\sin\Omega t) + 400(A\cos\Omega t + B\sin\Omega t) = 20\sin\Omega t$$

$$\cos: \quad -A\Omega^2 + 400A = 0 \rightarrow A = 0$$

$$\sin: \quad -B\Omega^2 + 400B = 20 \rightarrow B = \frac{20}{400-\Omega^2}$$

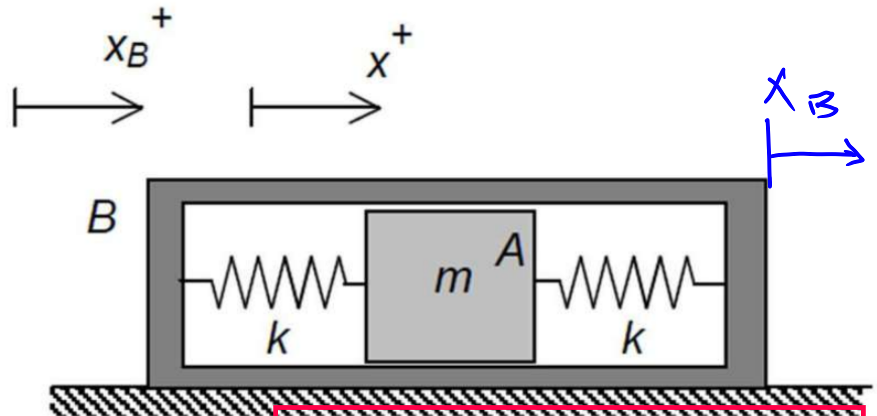
$$B = \frac{20}{400-(30)^2} \rightarrow B < 0 \rightarrow \text{out of phase}$$

make an assumption
-sign doesn't really matter, just be consistent
assume $x > x_B$

Example 6.C.5

Given: Box B is given a prescribed displacement of $x_B(t) = b\sin\omega t$.

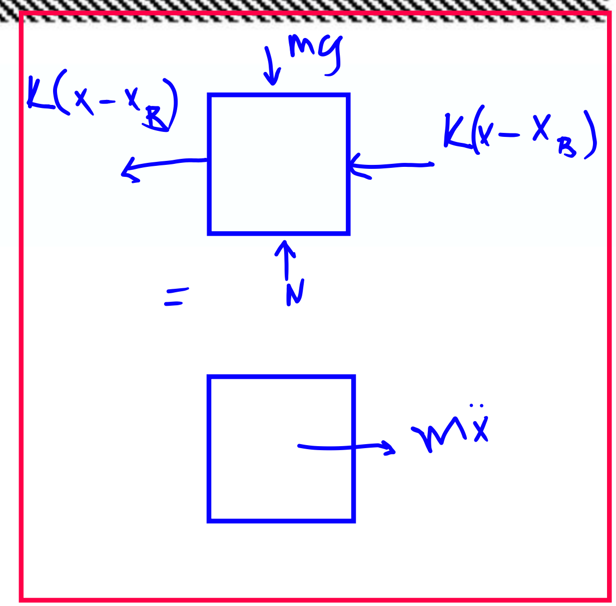
Find: Determine the two values of frequency ω for which the amplitude of forced response of the block A is twice that of the box B.



kinematics

$$\sum F_x = m\ddot{x} = -2k(x - x_B)$$

$$\Rightarrow m\ddot{x} + 2kx = 2kb\sin\omega t \quad \text{EOM}$$



particular soln.

$$x_p(t) = A\cos\omega t + B\sin\omega t$$

$$\ddot{x}_p(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$m(-A\omega^2\cos\omega t - B\omega^2\sin\omega t) + 2k(A\cos\omega t + B\sin\omega t) = 2kb\sin\omega t$$

cos: $(-m\omega^2 + 2k)A = 0 \rightarrow A = 0$

sin: $(-m\omega^2 + 2k)B = 2kb \rightarrow B = \dfrac{2kb}{-m\omega^2 + 2k}$

$$x_p(t) = \frac{2kb}{-m\omega^2 + 2k}\sin\omega t$$ ← amplitude: forced response A $= \left|\dfrac{2kb}{2k - m\omega}\right|$

amplitude forced response $b \Rightarrow b$

$$2b = \left|\frac{2kb}{2k - m\omega}\right| \rightarrow$$

$2b = -\dfrac{2kb}{2k - m\omega^2} \rightarrow 2k - m\omega^2 = k \quad \omega = \sqrt{\frac{k}{m}}$

$2b = \dfrac{2kb}{2k - m\omega^2} \rightarrow 2k - m\omega^2 = -k \quad \boxed{\omega = \sqrt{3k/m}}$

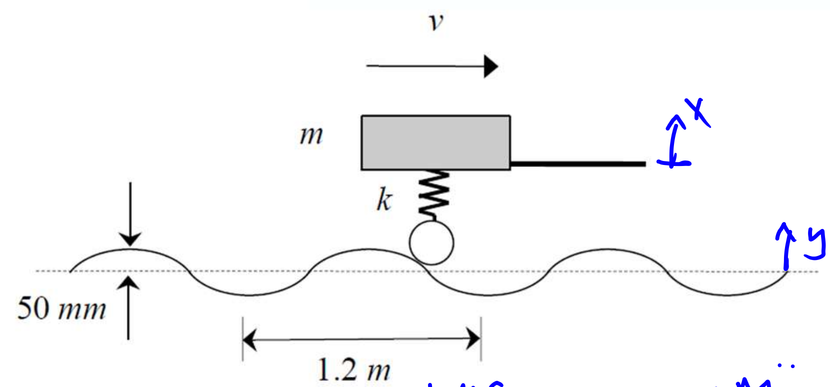
Example 6.C.6

Given: Adding 75 kg of mass to the trailer creates 3 mm of additional sag in the suspension.

Find: Determine:

(a) The vibration amplitude X; and

(b) The critical speed at which the undamped oscillations are the greatest as the trailer is pulled along a roadway with sinusoidal waviness.



$$\Sigma F_x = m\ddot{x} = -mg - k(x-y)$$

EOM $\Rightarrow m\ddot{x} + kx = -mg + ky$

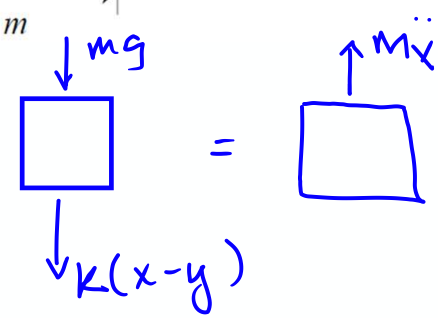
Static $\ddot{x} = 0 \rightarrow kx_{st} = -mg \rightarrow x_{st} = \frac{-mg}{k}$

$(y=0)$

$$x_{st}' = -\frac{(m+\Delta m)g}{k}$$

$$x_{st}' - x_{st} = 3mm = \frac{\Delta m g}{k} \rightarrow \text{find } k$$

FBD:



$z = x - x_{st}$, $\ddot{z} = \ddot{x}$

$$m\ddot{z} + kz = ky = ky_0 \sin\left(2\pi \frac{vt}{L}\right) \longleftarrow \text{find using plot}$$

$$z = A\cos\omega t + B\sin\omega t$$

$$\ddot{z} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$m(-A\omega^2\cos\omega t - B\omega^2\sin\omega t) + k(A\cos\omega t + B\sin\omega t) = K\sin\omega t$$

$\cos$: $\quad -mA\omega^2 + kA = 0 \quad \rightarrow A = 0$

$\sin$: $\quad -mB\omega^2 + kB = ky_0 \quad \longrightarrow \quad B = \dfrac{ky_0}{k - m\omega^2}$

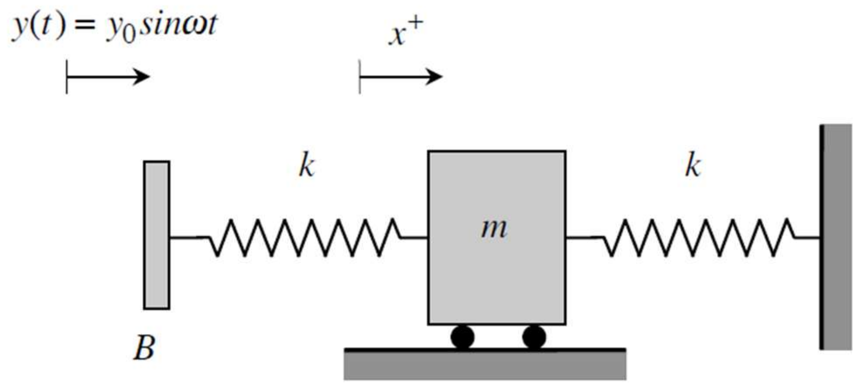
$$z(t) = \frac{ky_0}{k - m\omega^2}\sin\omega t \qquad \omega = \frac{2\pi V}{L}$$

resonance $\rightarrow \omega = \omega_n = \sqrt{\dfrac{k}{m}} \Rightarrow \dfrac{2\pi V}{L} = \sqrt{\dfrac{k}{m}} \longrightarrow \boxed{V_{crit} = \dfrac{L}{2\pi}\sqrt{\dfrac{k}{m}}}$

Example 6.C.7

Given: The system shown below is given a prescribed motion for the base B: $y(t) = y_0 sin\omega t$.

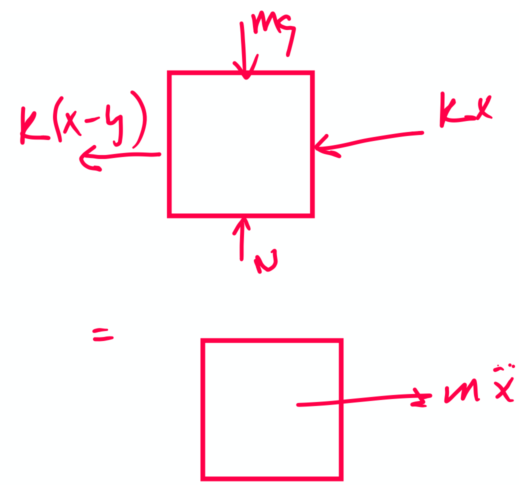
Find: Determine the range of frequencies ω for which the amplitude of the motion for block A does not exceed $1.5y_0$.



Kinetics:

$$\sum F_x = m\ddot{x} = -kx - k(x-y)$$

EOM: $$m\ddot{x} + 2kx = ky_0 \sin\omega t$$



particular soln.

$$x_p(t) = A\cos\omega t + B\sin\omega t$$

$$\ddot{x}_p = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

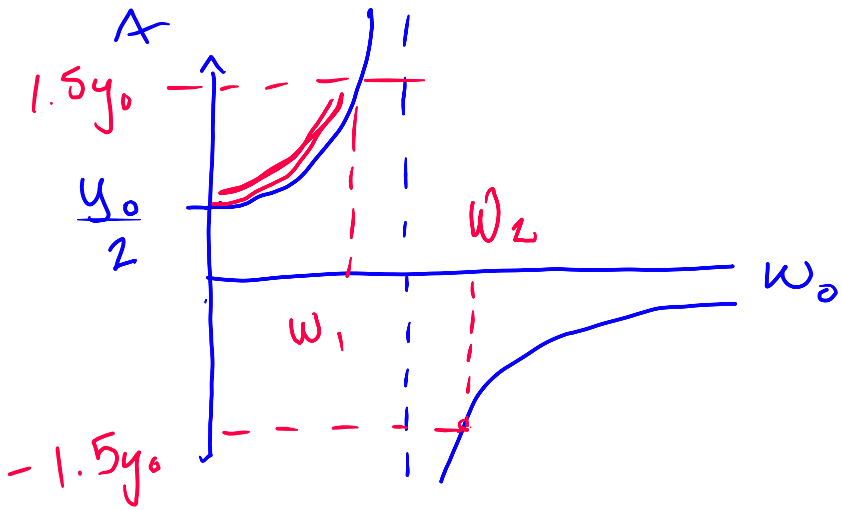
$$-m(A\omega^2\cos\omega t + B\omega^2\sin\omega t) + 2k(A\cos\omega t + B\sin\omega t) = ky_0\sin\omega t$$

cos: $-mA\omega^2 + 2kA = 0 \rightarrow A = 0$

sin: $-mB\omega^2 + 2kB = ky_0 \rightarrow B = \dfrac{ky_0}{(-m\omega^2 + 2k)}$

$$x_p = \frac{ky_0}{(-m\omega^2 + 2k)}\sin\omega t$$

$$|A| = \left|\frac{ky_0}{2k - m\omega^2}\right| < 1.5y_0$$



$$\omega = \sqrt{2k/m} = \omega_n$$

$$A = \frac{k\cancel{y_0}}{2k - m\omega^2} = 1.5\cancel{y_0} \rightarrow \text{solve for } \omega_1$$

$$A = \frac{k\cancel{y_0}}{2k - m\omega^2} = -1.5\cancel{y_0} \rightarrow \text{solve for } \omega_2$$