ME 274: Basic Mechanics II

Lecture 38: Vibrations – Damped Free Response

PURDUE UNIVERSITY | School of Mechanical Engineering

Process for finding the free response vibration of an underdamped system:

EOM for the free response of a vibrating system:

$$\boldsymbol{M\ddot{x} + C\dot{x} + Kx = 0}$$

General solution to the homogeneous equation:

$$x(t) = Ae^{\lambda t}$$

We can therefore rewrite our EOM as:

$$(\lambda^2 M + \lambda C + K)Ae^{\lambda t} = 0$$

$\omega_n = \sqrt{\frac{K}{M}}$

$\zeta = \frac{C}{2\sqrt{KM}}$

Solving for the roots of our characteristic equation:

$$\lambda_1, \lambda_2 = \frac{-C \pm \sqrt{C^2 - 4MK}}{2M} = \left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]\omega_n$$

For an underdamped system, $0 < \zeta < 1$, we will have **two unique, real roots:**

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm i\omega_d, \qquad \omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Our corresponding solution will have the form:

$$x(t) = e^{-\zeta\omega_n t}[C\cos\omega_d t + S\sin\omega_d t]$$

Solve for C, S, using $x(0)$, $\dot{x}(0)$!

Using the standard form of the EOM streamlines this process:

EOM for the free response of a vibrating system:

$$\boldsymbol{M\ddot{x}} + \boldsymbol{C\dot{x}} + \boldsymbol{Kx} = \boldsymbol{f}(\boldsymbol{t})$$

Standard form:

$$\ddot{x} + \frac{C}{M}\dot{x} + \frac{K}{M}x = \frac{1}{M}f(t)$$

Knowing $\omega_n = \sqrt{\frac{K}{M}},\ \zeta = \frac{C}{2\sqrt{KM}}$:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{M}f(t)$$

If we calculate $0.5 < \zeta < 1$, our system is underdamped and the corresponding solution will have the form:

$$x(t) = e^{-\zeta\omega_n t}[C\cos\omega_d t + S\sin\omega_d t]$$

Solve for C, S, using $x(0),\ \dot{x}(0)$!

What happens when we change our system?

For a demo of this concept:
https://www.youtube.com/watch?v=IZPtFDXYQRU

Question C6.5

Consider the free response for a damped, single-DOF system having the following differential equation of motion:

$$m\ddot{x} + c\dot{x} + kx = 0$$

$\omega_n = \sqrt{\frac{k}{m}}$ $\quad \zeta = \frac{c}{2\sqrt{km}}$

This system is known to have a damping ratio of $\zeta = 0.1$ and undamped natural frequency of $\omega_n = 10$ rad/s.

What are the new values for the damping ratio and undamped natural frequency if:

(a) The original value of m is doubled, the original value of k is doubled and the value of c is unchanged? $\omega_n = 10$ $\zeta = 0.05$

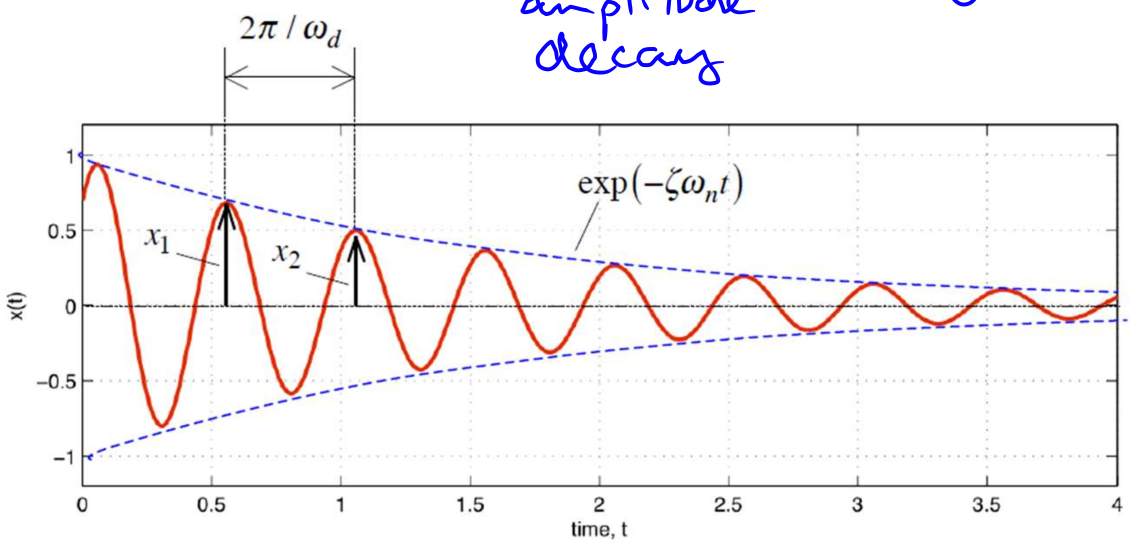
(b) The original value of m is doubled, the original value of k is halved and the value of c is unchanged? $\omega_n = 5$ $\zeta = 0.01$

(c) The original value of k is doubled, the original value of c is doubled and the value of m is unchanged? $\omega_n = 10\sqrt{2}$ $\zeta = 0.1\sqrt{2}$

(d) The original value of m is doubled, the original value of c is doubled and the value of k is unchanged? $\omega_n = 5\sqrt{2}$ $\zeta = 0.1\sqrt{2}$

Response of an underdamped system - $x(t) = \underbrace{e^{-\zeta\omega_n t}}_{\text{amplitude decay}}[C\cos \underline{\omega_d} t + S\sin \underline{\omega_d} t]$

oscillation



CHALLENGE QUESTION: Earlier in the chapter it was pointed out that the coefficients M, C and K in the general form of the EOM

$$M\ddot{x} + C\dot{x} + Kx = f(t)$$

must all be of the same sign. Do you now see why this must be true?

if $\frac{K}{M} < 0 \rightarrow \omega_n$ is imaginary

if $\frac{C}{M} < 0 \rightarrow \zeta < 0 \rightarrow e^{-\zeta\omega_n t}$ increasing amplitude w/ time

Example 6.B.11

Given: $M\ddot{x} + C\dot{x} + Kx = f(t),\ M = 2\text{kg}$

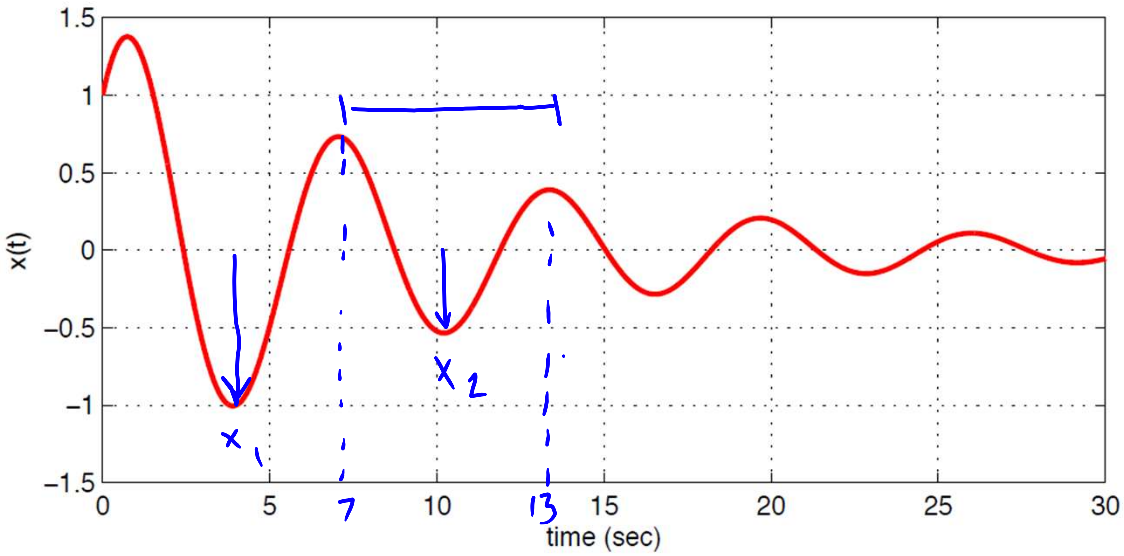
Find: the damping coefficient, C

Hint: use the concept of the logarithmic decrement!

$$\zeta \approx \frac{\delta}{2\pi} = \frac{1}{2\pi}\ln\frac{x_1}{x_2}$$

$$\frac{C}{M} = 2\zeta\omega_n$$

$$\Rightarrow C = 2\zeta\omega_n M$$



$$\zeta \approx \frac{1}{2\pi}\ln\left(\frac{-1}{-0.5}\right) = 0.11$$

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} \rightarrow \omega_n = 2\pi\, T_d\sqrt{1-\zeta^2}$$

$$T_d = 6$$

Example 6.B.10

Given: The addition of damping to an undamped system causes the period to increase by 25 percent.

Find: Find the value of the damping ratio after the addition of damping.

$$T_o = \frac{2\pi}{\omega_n}$$

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.25\, T_o$$

$$\frac{\cancel{2}\pi}{\cancel{\omega_n}\sqrt{1-\zeta^2}} = 1.25\,\frac{\cancel{2\pi}}{\cancel{\omega_n}}$$

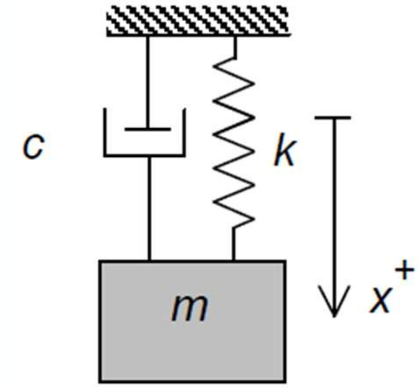
$$\frac{1}{\sqrt{1-\zeta^2}} = \frac{5}{4} \rightarrow 1-\zeta^2 = \frac{16}{25}$$

$$\zeta^2 = \frac{9}{25} \rightarrow \boxed{\zeta = \frac{3}{5}}$$

Example 6.B.12

Given: The system shown below is released from rest under the action of gravity.

Find: Determine the initial overshoot past the static equilibrium state of the system.



Use the following parameters in your analysis: $m = 5$ kg, $c = 6$ N-s/m and $k = 180$ N/m.

EOM: $\downarrow^{+} \Sigma F_x = m\ddot{x}$

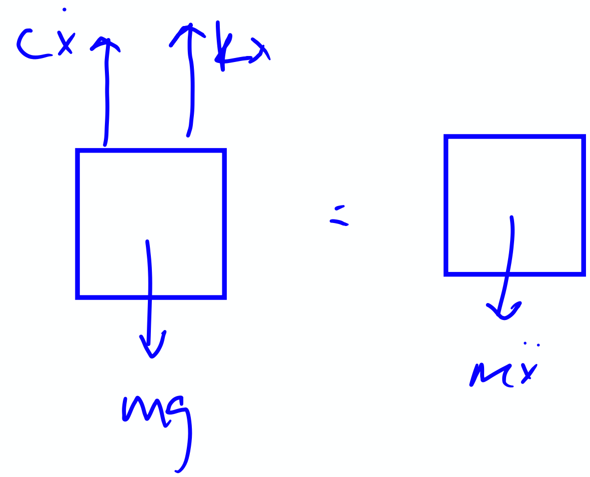
$$-c\dot{x} - kx + mg = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = mg$$

redefine from x_{st}

@ eq. $\dot{x} = \ddot{x} = 0 \rightarrow kx_{st} = mg \quad x_{st} = \frac{mg}{k}$

$$z = x - x_{st}, \quad \dot{z} = \dot{x}, \quad \ddot{z} = \ddot{x}$$



new EOM in terms of z:

$$m\ddot{z} + c\dot{z} + kz = 0$$

$\rightarrow \omega_n = \sqrt{\frac{k}{m}}$

$\rightarrow \zeta = \frac{c}{2\omega_n m}$

$\rightarrow \omega_d = \omega_n\sqrt{1-\zeta^2}$

responce $z(t)$:

$$z(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

$$x(0) = 0 \Rightarrow z = \cancel{x}^{0} - x_{st} \rightarrow z = -x_{st}$$

$$\dot{z}(0) = 0$$

calculate A & B

$$z(0) = -x_{st} = e^{0}(A\cos 0 + B\sin 0) = A \quad \rightarrow \quad A = -x_{st}$$

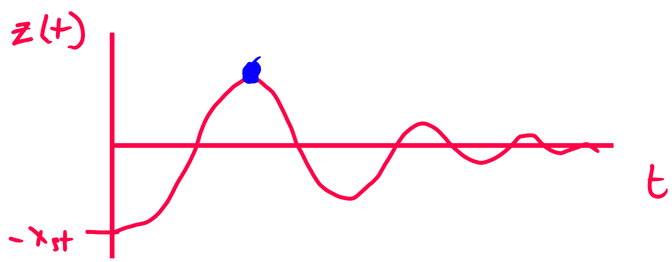
$$\dot{z}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\zeta\omega_n t}(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t)$$

$$\dot{z}(0) = -\zeta\omega_n A + B\omega_d = 0$$

$$\Rightarrow B = \frac{-\zeta\omega_n x_{st}}{\omega_d}$$

$$z(t) = e^{-\zeta\omega_n t}\left(-x_{st}\cos\omega_d t + \frac{-\zeta\omega_n x_{st}}{\omega_d}\sin\omega_d t\right)$$

To find our max overshoot find the max value of $z(t)$ after initial pt.



how?

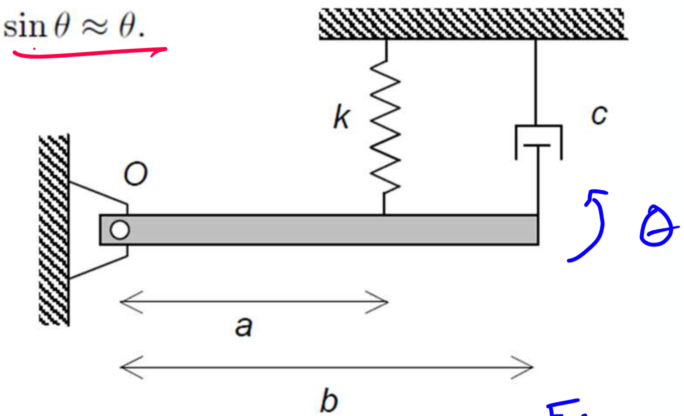
1) derivative $\dot{z}(t) = 0$

2) plot

Example 6.B.14

Given: The system shown below, consisting of a homogenous bar of mass m which is pinned to ground at point O, moves in a horizontal plane. Let θ represent the angular orientation of the bar, measured in a counterclockwise positive sense.

Find: Find the value of c for critical damping. Assume small oscillations, that is, assume $\sin\theta \approx \theta$.



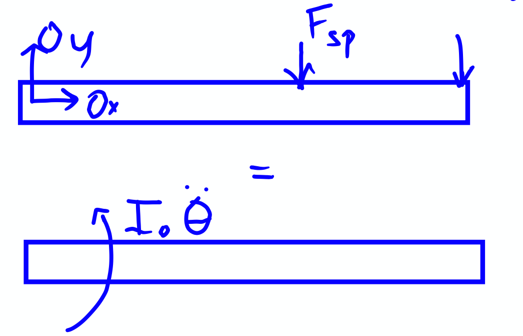
critical damping $\longrightarrow \zeta = 1$

$$\circlearrowleft \Sigma M_o = I_o \ddot{\theta}$$

$$-aF_{sp} - bF_d = I\ddot{\theta}$$

$$-ka^2\theta - cb^2\dot{\theta} = I_o\ddot{\theta} = \left(I_G + m\left(\frac{b}{2}\right)^2\right)\ddot{\theta}$$

$$\Rightarrow \frac{1}{3} m b^2 \ddot{\theta} + c b^2 \dot{\theta} + k a^2 \theta = 0$$

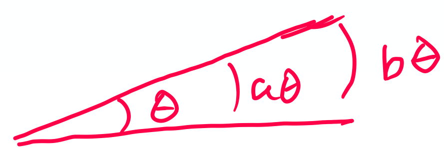


mass normalize

$$\ddot{\theta} + \frac{3c}{m}\dot{\theta} + \frac{3ka^2}{mb^2}\theta = 0$$

$$\omega_n = \sqrt{\frac{3ka^2}{Mb^2}} \qquad 2\zeta\omega_n = \frac{3c}{m}$$

$$2\sqrt{\frac{3ka^2}{mb^2}} = \frac{3c_{cr}}{m} \longrightarrow c_{cr} = \frac{2m}{3}\sqrt{\frac{3ka^2}{mb^2}}$$



arc length approximation