ME 274: Basic Mechanics II

Lecture 37: Vibrations – Damped Free Response

PURDUE UNIVERSITY | School of Mechanical Engineering

Solution to the Homogeneous Form of the EOM

From your studies of linear diff. eq., remember that the solution to the EOM $\boldsymbol{M\ddot{x}} + \boldsymbol{C\dot{x}} + \boldsymbol{Kx} = \boldsymbol{f(t)}$ is

$$\boldsymbol{x(t)} = \boldsymbol{x_c(t)} + \boldsymbol{x_p(t)}$$

$\boldsymbol{x_c(t)}$:

- complimentary (homogeneous) solution
- General form of the solution corresponding to $f(t) = 0$
- Free vibration response

$\boldsymbol{x_p(t)}$:

- Particular solution
- $f(t) \neq 0$

Still only looking at free response

General solution to the homogeneous equation:

$$x(t) = Ae^{\lambda t}$$

We can therefore rewrite our EOM as:

$$\lambda^2 MAe^{\lambda t} + \lambda CAe^{\lambda t} + KAe^{\lambda t} = 0$$

$$(\underbrace{\lambda^2 M + \lambda C + K}_{\text{Characteristic eqn.}})Ae^{\lambda t} = 0$$

Solution to the Homogeneous Form of the EOM

Our Characteristic Equation for the homogeneous form of our EOM is a quadratic in the form:

$$\underbrace{\lambda^2 M + \lambda C + K = 0}$$

gives us all the info we need to tell how our system behaves

To solve our EOM, we need to find the roots of the characteristic eqn.:

$$\lambda_1, \lambda_2 = \frac{-C \pm \sqrt{C^2 - 4MK}}{2M}$$

What do these roots tell us about the behavior of our system?

$$\begin{aligned}
\lambda_1, \lambda_2 &= \frac{-C \pm \sqrt{C^2 - 4MK}}{2M} \\
&= -\frac{C}{2M} \pm \sqrt{\left(\frac{C}{2M}\right)^2 - \frac{K}{M}} \\
&= \left[-\frac{C}{2M}\sqrt{\frac{M}{K}} \pm \sqrt{\left(\frac{C}{2M}\right)^2 \frac{M}{K} - 1}\right]\sqrt{\frac{K}{M}} \\
&= \left[-\frac{C}{2\sqrt{KM}} \pm \sqrt{\left(\frac{C}{2\sqrt{KM}}\right)^2 - 1}\right]\sqrt{\frac{K}{M}} \\
&= \left[\underline{-\zeta} \pm \sqrt{\zeta^2 - 1}\right]\sqrt{\frac{K}{M}}
\end{aligned}$$

$\sqrt{\frac{K}{M}}$ ← ω_n ← natural frequency

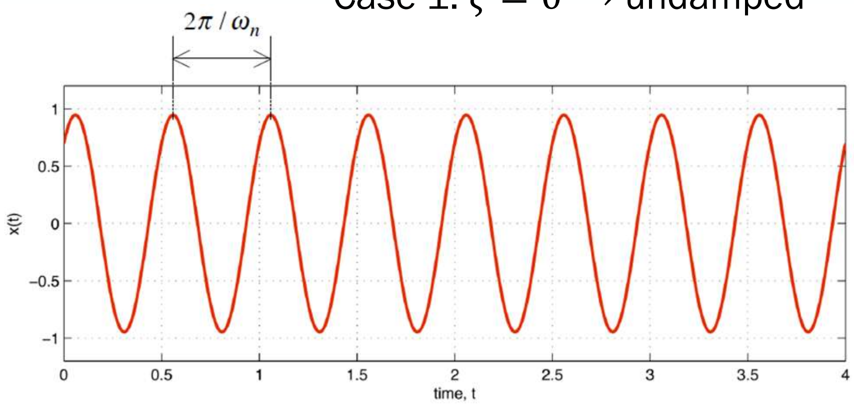
$\zeta = \frac{C}{2\sqrt{KM}}$ ← damping ratio

Discussion: Damping Ratio - ζ

$$\lambda_1, \lambda_2 = \left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]\sqrt{\frac{K}{M}}$$

ζ is a normalized measure of the amount of damping in your system - $\zeta = \frac{actual\ damping}{critical\ damping}$

Case 1: $\zeta = 0$ $\rightarrow$ undamped

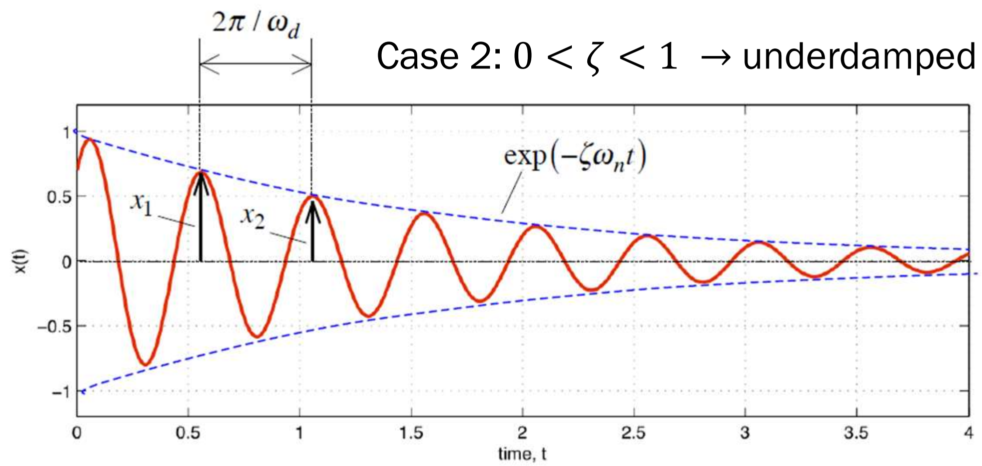


$$\underbrace{\lambda_1, \lambda_2 = \pm i\omega_n}, \quad \omega_n = \sqrt{\frac{K}{M}}$$

2 imaginary roots

system oscillates forever @ natural frequency with no amplitude decay

Case 2: $0 < \zeta < 1$ $\rightarrow$ underdamped



$$\lambda_1, \lambda_2 = \underbrace{-\zeta\omega_n \pm i\omega_d}, \quad \underbrace{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

2 complex roots

new parameter
Wd→
"damped
natural frequency:

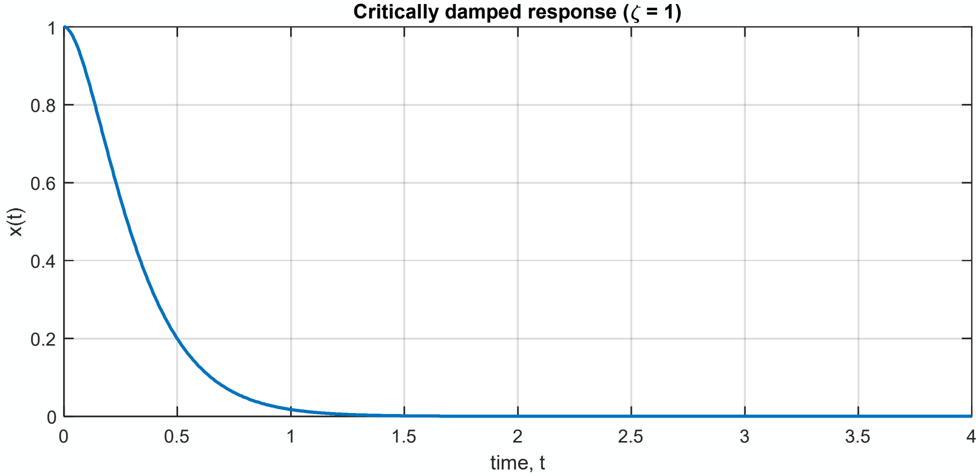
Oscillation + decay, most systems we're looking at

Discussion: Damping Ratio - ζ

ζ is a normalized measure of the amount of damping in your system - $\zeta = \dfrac{actual\ damping}{critical\ damping}$

Case 3: $\zeta = 1 \rightarrow$ critically damped

Critically damped response (ζ = 1)



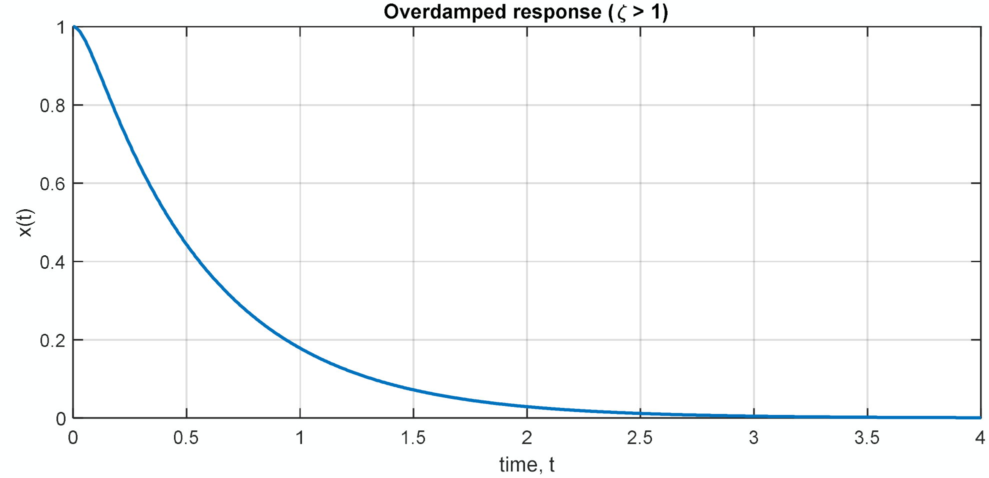
$\lambda_1, \lambda_2 = -\omega_n$

repeated real roots $\rightarrow -\omega_n$

$\rightarrow$ system moves to eq. as quickly as possible w/o oscillating

Case 4:

Overdamped response (ζ > 1)



Two distinct real roots

also does not oscillate, takes longer to reach equilibrium

Discussion: Underdamped systems

The solution to the homogeneous ODE for the underdamped case is:

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

Substituting our roots in: $\lambda_1, \lambda_2 = -\zeta\omega_n \pm i\omega_d$

$$\begin{aligned} x(t) &= A_1 e^{(-\zeta\omega_n - i\omega_d)t} + A_2 e^{(-\zeta\omega_n + i\omega_d)t} \\ &= e^{-\zeta\omega_n t}\left[A_1 e^{-i\omega_d t} + A_2 e^{i\omega_d t}\right] \\ &= e^{-\zeta\omega_n t}\left[A_1(\cos\omega_d t - i\sin\omega_d t) + A_2(\cos\omega_d t + i\sin\omega_d t)\right] \\ &= e^{-\zeta\omega_n t}\left[(A_1 + A_2)\cos\omega_d t + i(-A_1 + A_2)\sin\omega_d t\right] \\ &= e^{-\zeta\omega_n t}\left[C\cos\omega_d t + S\sin\omega_d t\right] \end{aligned}$$

rewrite using Euler's formula

controls decay in amplitude

controls oscillation

Note the difference between ω_n and ω_d:

- $\omega_n = \sqrt{\frac{K}{M}}$ → natural frequency – the frequency the system would oscillate at if there was no damping
- $\omega_d = \omega_n\sqrt{1-\zeta^2}$ → damped natural frequency – the actual frequency of the oscillations

now to find C & S:

$x(0) = e^0[C\cos 0 + S\sin 0] = C$ ← initial position

$v(t) = -\zeta\omega_n e^{-\zeta\omega_n t}[C\cos\omega_d t + S\sin\omega_d t] + e^{-\zeta\omega_n t}[-C\omega_d\sin\omega_d t + S\omega_d\cos\omega_d t]$

$\Rightarrow v(0) = -\zeta\omega_n C + S\omega_d$

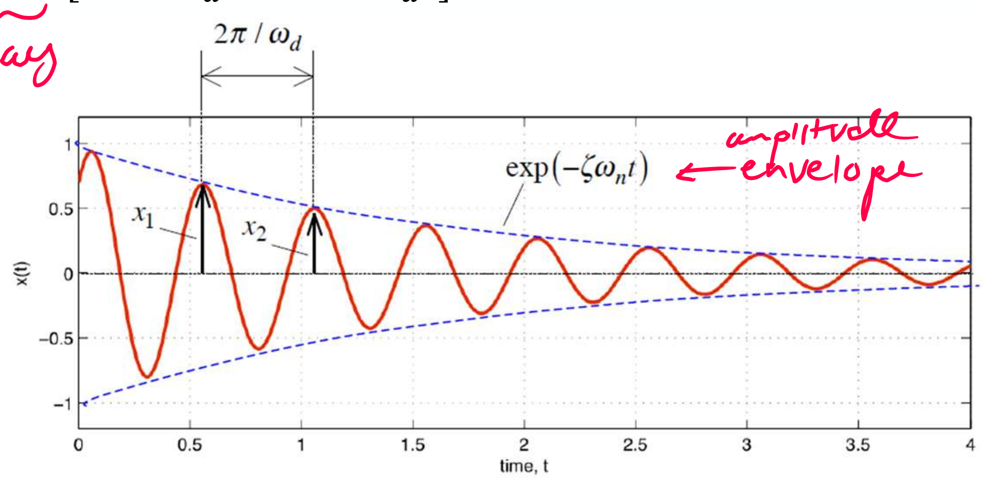
→ solve for S using v_0

Discussion: Underdamped systems

oscillation

Plotting our solution: $x(t) = e^{-\zeta\omega_n t}[C\cos\omega_d t + S\sin\omega_d t]$

decay



amplitude envelope

The natural log of the ratio of amplitudes for successive peaks is called the <u>Logarithmic Decrement:</u>

$$\delta = \ln\frac{x_1}{x_2} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

practical way to estimate damping from measured experimental data

Which can be approximated for small damping ($\zeta \ll 1$) as:

$$\zeta \approx \frac{\delta}{2\pi} = \frac{1}{2\pi}\ln\frac{x_1}{x_2}$$

tells us how much the amplitude shrinks from one cycle to the next

Standard Form for the EOM of a Single-Degree-of-Freedom System

We can now express our EOM in a form that gives us immediate information about the behavior of our system:

$$M\ddot{x} + C\dot{x} + Kx = f(t)$$

$$\ddot{x} + \frac{C}{M}\dot{x} + \frac{K}{M}x = \frac{1}{M}f(t)$$ ← mass normalize

$$\omega_n = \sqrt{K/M} \qquad \zeta = C/2\sqrt{KM}$$

$$\begin{aligned}\frac{C}{M} &= \frac{2\zeta\sqrt{KM}}{M} \\ &= 2\zeta\sqrt{\frac{K}{M}} \\ &= 2\zeta\omega_n\end{aligned}$$

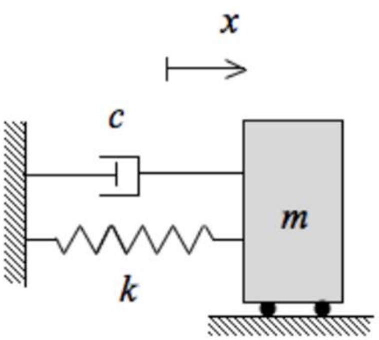
$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + {\omega_n}^2 x = \frac{1}{M}f(t)}$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m} = 0$$

$$\omega_n^2 = 100 \qquad 2\zeta\omega_n = 10$$

$$\ddot{\theta} + \frac{2c}{5m}\dot{\theta} + \frac{2k}{5m}\theta = 0$$

$$\omega^2 = \frac{2(2500)}{5(10)} = 100 \qquad 2\zeta\omega_n = \frac{2(250)}{5(10)} = 10$$



$\omega_n = 10$ rad/s

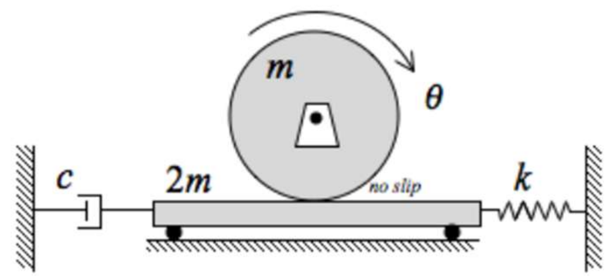
$\zeta = 0.5$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$m = 10kg$

$c = 100kg/sec$

$k = 1000N/m$

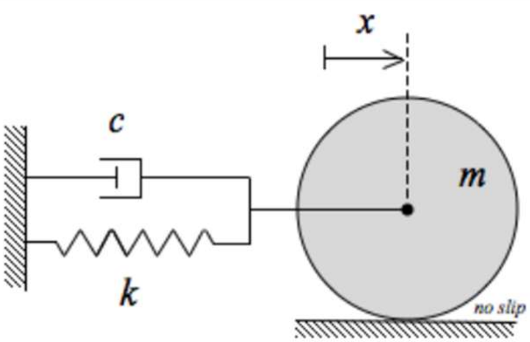


$$\frac{5m}{2}\ddot{\theta} + c\dot{\theta} + k\theta = 0$$

$m = 10kg$

$c = 250kg/sec$

$k = 2500N/m$

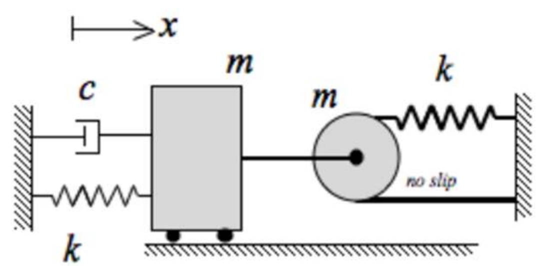


$$\frac{3m}{2}\ddot{x} + c\dot{x} + kx = 0$$

$m = 10kg$

$c = 150kg/sec$

$k = 1500N/m$



$$\frac{5m}{2}\ddot{x} + c\dot{x} + 5kx = 0$$

$m = 10kg$

$c = 250kg/sec$

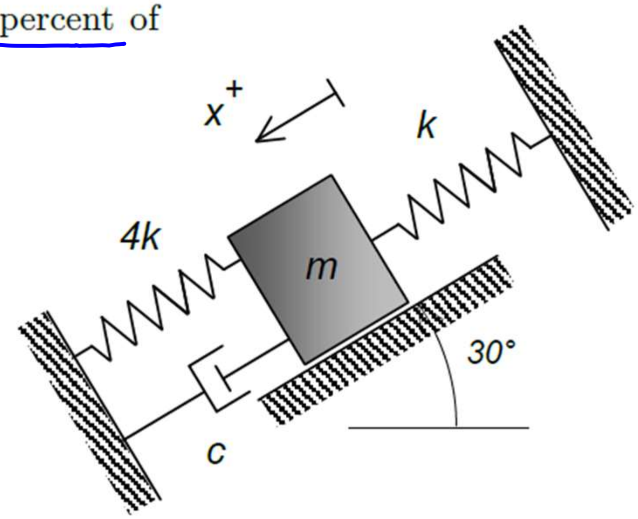
$k = 500N/m$

→ all have the same ω_n, ζ → will have the same $x(t)$ with same IC

Example 6.B.8

Given: The system shown below consists of a block of mass m, which rests upon a smooth inclined surface, that is connected to ground by two springs of stiffness $4k$ and k, respectively, and a dashpot with damping constant c.

Find: Determine the value of the damping constant c, such that the system has 50 percent of critical damping.

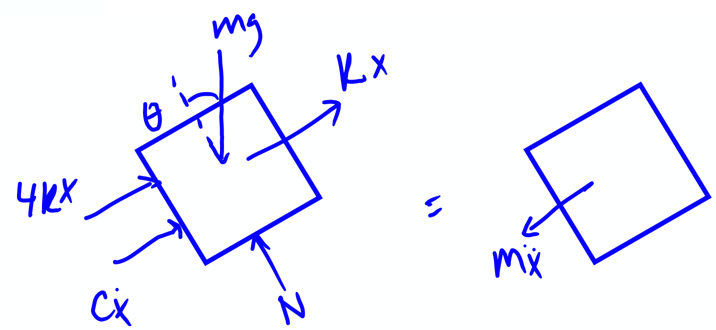


$$\Sigma F_x = m\ddot{x}$$

$$-kx - 4kx - c\dot{x} + mg\sin\theta = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + 5kx = mg\sin\theta$$ ← mass normalize

$$\ddot{x} + \frac{c}{m} + \frac{5k}{m} = g\sin\theta$$



$$2\xi\omega_n = \frac{c}{m} \qquad \omega_n^2 = \frac{5k}{m}$$

50% of crit. damping → $\xi = 0.5$

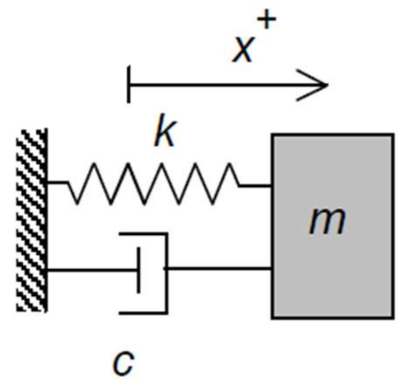
$$\omega_n = \frac{c}{m} \rightarrow \frac{c}{m} = \sqrt{\frac{5k}{m}}$$

$$c = \sqrt{5km}$$ ← note result is the same regardless of angle of incline

Example 6.B.9

Given: The system shown below. assume horz. plane

Find: Find the free response of the system corresponding to $x(0) = x_0$ and $\dot{x}(0) = 0$.

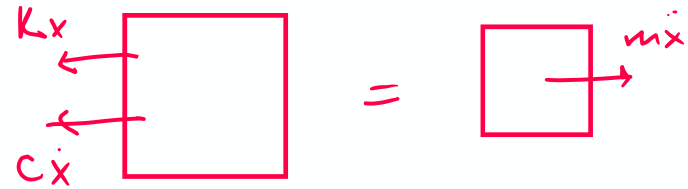


$$\sum F_x = m\ddot{x}$$

$$-kx - c\dot{x} = m\ddot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m} = 0$$



$$\omega_n^2 = \frac{k}{m} \Rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

$$2\zeta\omega_n = \frac{c}{m} \Rightarrow \zeta = \frac{c}{2\sqrt{km}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m}}\sqrt{1-\frac{c^2}{4km}}$$

General Solution:

$$x(t) = e^{-\zeta\omega_n t}\left[C\cos\omega_d t + S\sin\omega_d t\right]$$

$$x(0) = x_0 = C$$

$$v(0) = \dot{x}(0) = 0 \Rightarrow 0 = -\zeta\omega_n C + \omega_d S$$

$$\Rightarrow S = \frac{\zeta\omega_n C}{\omega_d} = \frac{\zeta\omega_n x_0}{\omega_d}$$

$$x(t) = e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \frac{\zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right]$$