ME 274: Basic Mechanics II

Lecture ~~35~~ 36: Vibrations – Free Response (Undamped)

School of Mechanical Engineering

Free vibration response

Last lecture we learned the general form of an EOM for a single-degree of freedom vibrating system is:

$$\rightarrow M\ddot{x} + C\dot{x} + Kx = f(t) \quad \text{non-homogeneous}$$

Where:

- $M\ddot{x} \longrightarrow$ inertia (mass)
- $C\dot{x} \longrightarrow$ damping (dashpots)
- $Kx \longrightarrow$ stiffness (springs)
- $f(t) \longrightarrow$ external forcing

Today we will look at a special form where there is no externally applied forcing:

$$M\ddot{x} + C\dot{x} + Kx = 0 \leftarrow \text{homogeneous}$$

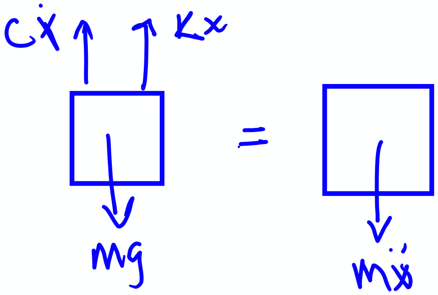
This is called the free response of the system.

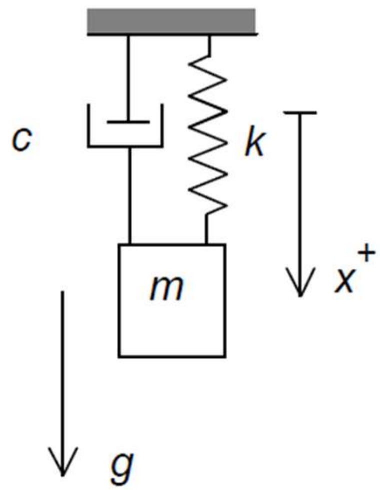
In the free response, the motion of the system is defined only by the initial conditions

$x(t=0) \quad v(t=0)$

$\Sigma F = 0 \qquad v, a = 0$

Static Deformation and the EOM for Vibration





The EOM for the shown equation in terms of x:

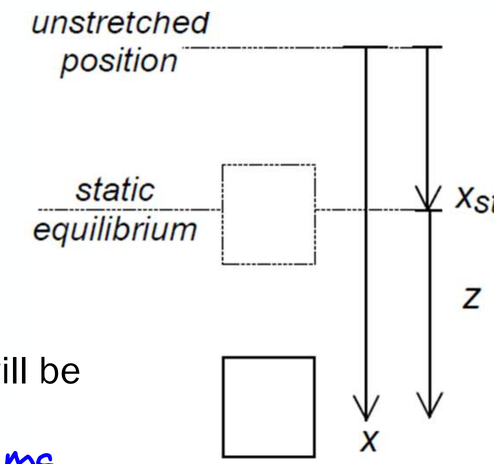
$$m\ddot{x} + c\dot{x} + kx = mg$$

This is a diff. eq. with a constant, inhomogeneous term mg.
We want to re-write this equation to be in free-response form.

We can find the static equilibrium position, x_{st}, of the block by setting $\dot{x} = \ddot{x} = 0$

$$\cancel{m\ddot{x}} + \cancel{c\dot{x}} + kx_{st} = mg$$

$$x_{st} = \frac{mg}{k}$$



We can now introduce a new coordinate, z, noting that x_{st} will be *constant*

$$z = x - x_{st} = x - \frac{mg}{k}$$

$$\dot{z} = \dot{x} \qquad \ddot{z} = \ddot{x}$$

$$m\ddot{z} + c\dot{z} + k(z + x_{st}) = mg$$

$$m\ddot{z} + c\dot{z} + kz + \cancel{k\left(\frac{mg}{k}\right)} = \cancel{mg}$$

$$m\ddot{z} + c\dot{z} + kz = 0$$

Solution to the Homogeneous Form of the EOM

From your studies of linear diff. eq., remember that the solution to the EOM $\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{t})$ is

$$\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x}_c(\boldsymbol{t}) + \boldsymbol{x}_p(\boldsymbol{t})$$

$\boldsymbol{x}_c(\boldsymbol{t})$:

- complimentary (homogeneous) solution
- General form of the solution corresponding to $f(t) = 0$
- Free vibration response

$\boldsymbol{x}_p(\boldsymbol{t})$:

- Particular solution
- $f(t) \neq 0$

General solution to the homogeneous equation:

$$x(t) = Ae^{\lambda t} \rightarrow A, \lambda \quad \text{solve w/ IC}$$

$\dot{x} = A\lambda e^{\lambda t}$, $\ddot{x} = A\lambda^2 e^{\lambda t}$

We can therefore rewrite our EOM as:

$$\lambda^2 MAe^{\lambda t} + \lambda CAe^{\lambda t} + KAe^{\lambda t} = 0$$

$$\rightarrow \underbrace{(\lambda^2 M + \lambda C + K)}_{\text{characteristic eqn.} = 0} \underline{Ae^{\lambda t}} = 0$$

$\hookrightarrow \neq 0$ trivial

$\rightarrow$ solve for λ

Special Case: Free, undamped response

If our system contains no damping ($C = 0$), then our EOM becomes:

$$M\ddot{x} + Kx = 0$$

$x(t) = Ae^{\lambda t}$

This makes our characteristic equation:

$$\lambda^2 M + K = 0 \qquad \Rightarrow \lambda = \pm\sqrt{-\frac{K}{M}} = \pm i\sqrt{\frac{K}{M}}$$

Our general solution to the EOM is therefore:

$$x(t) = A_1 e^{+i\sqrt{\frac{K}{M}}t} + A_2 e^{-i\sqrt{\frac{K}{M}}t}$$

Re-writing using Eulers equation, $e^{i\theta} = \cos\theta + isin\theta$:

$$x(t) = Ccos\sqrt{\frac{K}{M}}t + Ssin\sqrt{\frac{K}{M}}t$$

Where C and S are constants that should be solved for using initial conditions:

$$x(0) = C = initial\ position$$

$$\dot{x}(0) = S\sqrt{\frac{K}{M}} = v(0) = initial\ velocity$$

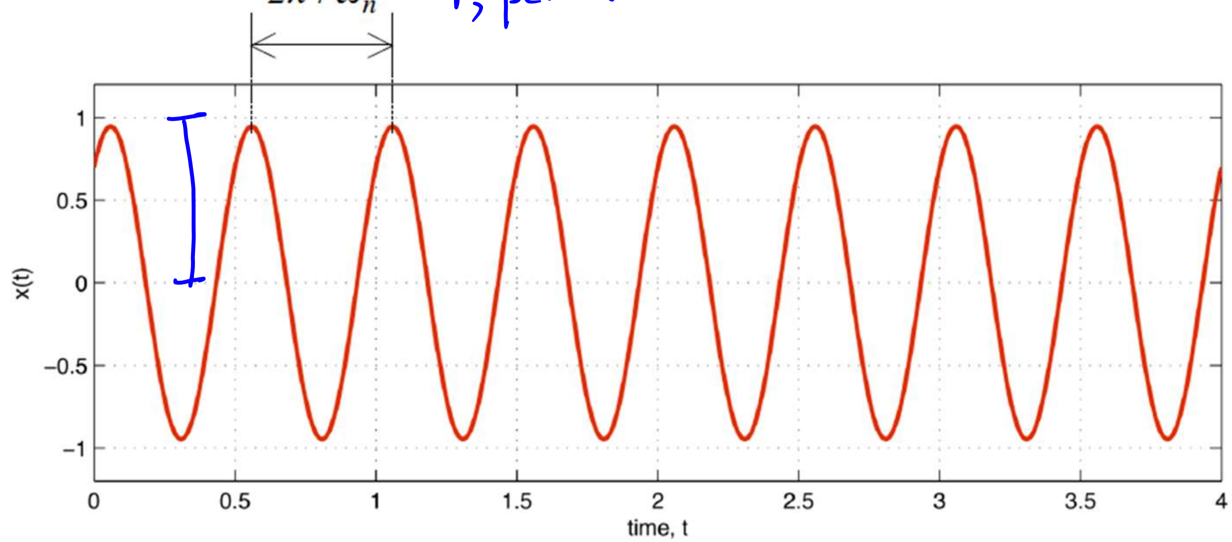
Discussion: Free vibration of undamped systems

- $\zeta \rightarrow$ damping ratio. For systems with no damping, $\zeta = 0$
- For undamped systems the natural frequency (rate at which the system oscillates) is

$$\omega_n = \sqrt{\frac{K}{M}}$$

- The system response will be harmonic (it will not decay over time)

$2\pi / \omega_n$ = T, period

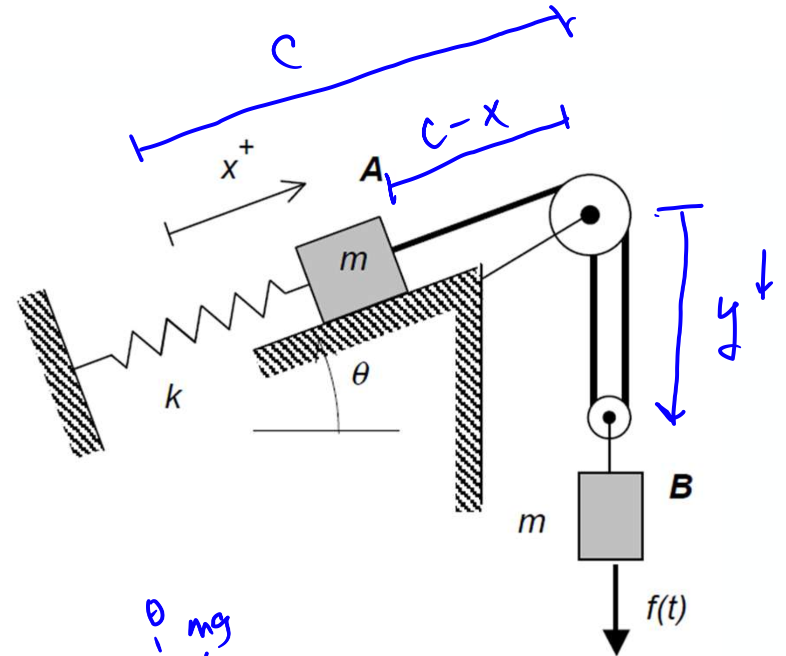


Example 6.B.1

Find: For this problem:

(a) Determine the EOM of this system in terms of the coordinate x; and

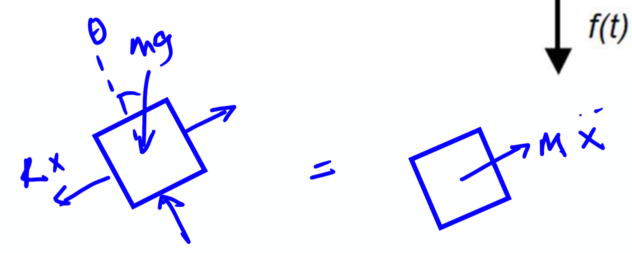
(b) Determine the natural frequency of the system.



Kinetics

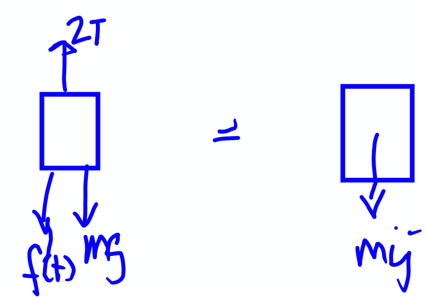
Block A: $\Sigma F_x = m\ddot{x}$

$$T - kx - mg\sin\theta = m\ddot{x} \quad (1)$$



Block B: $\Sigma F_y = m\ddot{y}$

$$-2T + mg + f(t) = m\ddot{y} \quad (2)$$



$$T = m\ddot{x} + kx - mg\sin\theta$$

$$-2m\ddot{x} - 2xk - 2mg\sin\theta + mg + f(t) = m\ddot{y}$$

Kinematics → cable

$$L = c - x + 2y \qquad 0 = -\ddot{x} + 2\ddot{y} \rightarrow \ddot{y} = \frac{\ddot{x}}{2}$$

$$-2m\ddot{x} - 2xk - 2mg\sin\theta + mg + f(t) = m\frac{\ddot{x}}{2}$$

$$\boxed{\underbrace{\tfrac{5}{2}m}_{M}\ddot{x} + \underbrace{2k}_{K}x = f(t) + mg - 2mg\sin\theta} \quad \text{EOM}$$

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{2k}{5/2\,m}} = \sqrt{\frac{4k}{5m}}$$

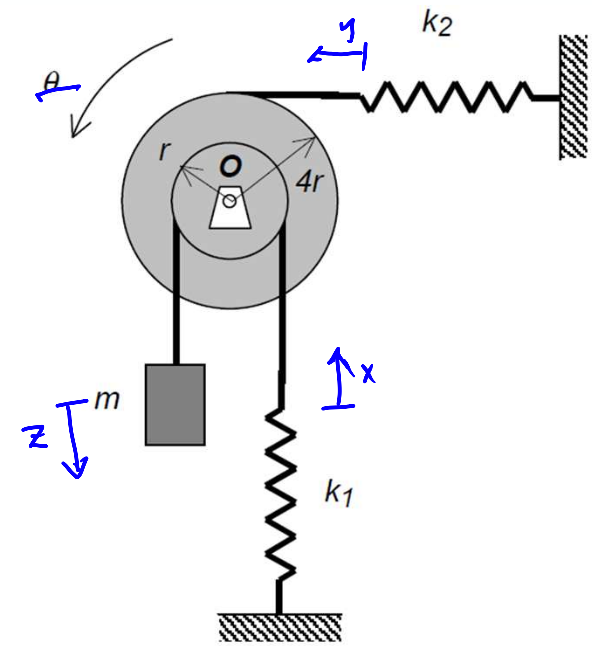
Example 6.B.5

Given: The pulley has a mass moment of inertia of I_O.

Find:

(a) Determine the EOM of this system in terms of the coordinate θ; and

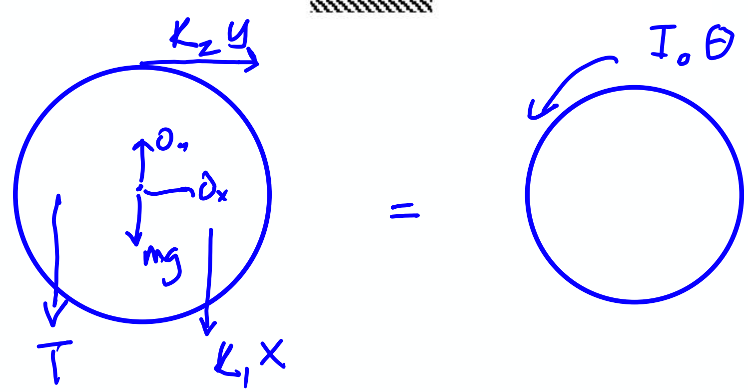
(b) Determine the natural frequency of the system.



$$\circlearrowleft\!+\ \Sigma M_o = I_o\ddot{\theta}$$

$$Tr - k_1 x r - k_2 y(4r) = I_o\ddot{\theta} \quad (1)$$

$$+\downarrow\ \Sigma F_z = m\ddot{z} \rightarrow mg - T = m\ddot{z} \quad (2)$$



kinematics

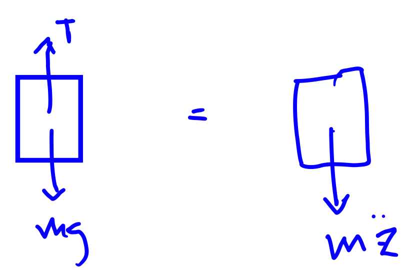
$z = r\theta \qquad \ddot{z} = r\ddot{\theta}$

$x = r\theta \qquad y = 4r\theta$

(2) $T = mg - mr\ddot{\theta}$

(1) $mgr - mr^2\theta - k_1 r^2\theta - k_2(4r)^2\theta = I_o\ddot{\theta}$

EOM $\rightarrow$ $$\boxed{\underbrace{(I_o + mr^2)}_{M}\ddot{\theta} + \underbrace{(k_1 r^2 + 16k_2 r^2)}_{K}\theta = mgr}$$



$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{k_1 r^2 + 16k_2 r^2}{I_o + mr^2}}$$