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ME 274 Lecture 37

Vibrations: Free Response – Part 2

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04/20/26

Housekeeping/Announcements

***Reminder for Henny to wear a mic during the lecture.

1. **HW 35 (6.C and 6.D) due Today!!**

2. Office hours are on ME2008. Located on second floor of renovated side of ME.

Chapter 6: Vibrations

$$\boxed{\begin{aligned} \sum \vec{F} &= m\vec{a}_G \\ \sum \vec{M}_A &= I_A\vec{\alpha} \end{aligned}}$$

$$\boxed{M\ddot{x} + C\dot{x} + Kx = f(t)}$$

Goals for Chapter 6:

- *We are used* to solving acceleration/velocity/position for a given **instant in time** (using Newton-Euler Equations, upper left).

- In this chapter, we will study the response of systems whose motion is *oscillatory in nature* (~vibrations).

- We will focus on understanding the change in position as a **general function of time** (ie. **Deriving the EOMs of a system**, upper right).

Standard Form of EOM

A) General Form EOM:

1. Particles/rigid bodies
2. Dashpots (a.k.a. Damper)
3. Springs
4. External forcing

$$\boxed{M\ddot{x}}+\boxed{C\dot{x}}+\boxed{Kx}=\boxed{f(t)}$$

B) Standard Form EOM:

- If you divide general form by M you get standard form (bottom left).
- Standard form gives us the EOM in terms of damping ratio, ζ (zeta), and undamped natural frequency, ω_n (bottom right)
- Why? Damping ratio and undamped natural frequency are values with physical interpretations:
  - Damping ratio -> Ratio for which an objects oscillations will decay over time
  - Undamped natural frequency -> a systems preferred state of motion based on its weight and stiffness

$$\ddot{x}+\frac{C}{M}\dot{x}+\frac{K}{M}x=\frac{1}{M}f(t)$$

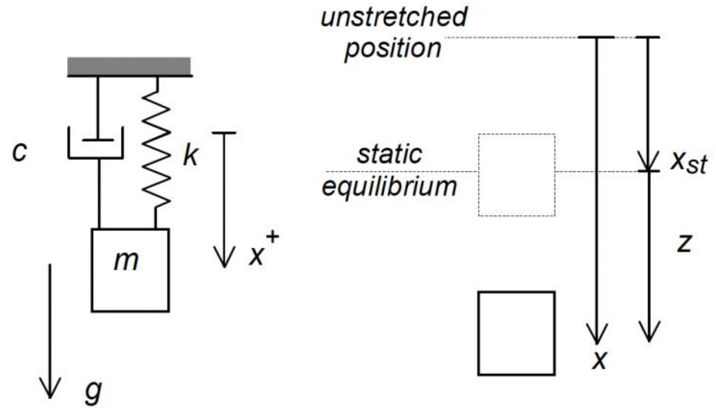
$$\ddot{x}+2\zeta\omega_n\dot{x}+{\omega_n}^2x=\frac{1}{M}f(t)$$

[p. 392]

Static Deformation

1. The EOM for the system shown is given as:

$$m\ddot{x} + c\dot{x} + kx = mg$$



2. The **static displacement** can be found by setting $\boxed{\dot{x} = \ddot{x} = 0}$, which gives:

$$kx_{st} = mg \longrightarrow x_{st} = \frac{mg}{k}$$

3. Now instead of measuring the position form the unstretched position, we measure its position from the static deformation state. **We define this coordinate as z:**

$$\boxed{z = x - x_{st}}$$

4. This gives a **homogenous equation (eqn equal to 0):**

$$m\ddot{z} + c\dot{z} + kz = 0$$

[p. 385-386]

Types of Responses

$$\left(\lambda^2 M + \lambda C + K\right) A e^{\lambda t} = 0$$

Solving for the characteristic equation gives us two roots:

$$\lambda_1, \lambda_2 = \frac{-C \pm \sqrt{C^2 - 4MK}}{2M} = \left[-\zeta \pm \sqrt{\zeta^2 - 1}\right] \sqrt{\frac{K}{M}} \quad \text{where } \zeta = C/2\sqrt{KM}$$

The value of the parameter zeta (damping ratio) tells us the type of response of the system:

1. Undamped $\zeta = 0$
- We have two imaginary roots

2. Underdamped $0 < \zeta < 1$
- We have two complex roots

3. Critically damped $\zeta = 1$
- We have two repeated, real roots

4. Overdamped $\zeta > 1$
- We have two distinct, real roots

[p. 388]

The Underdamped Response

1. In this course we will focus **primarily on the Underdamped response**, with undamped being a special case.
2. Substituting the underdamped roots of the characteristic equation into the total complementary solution (first equation below) gives the second equation:

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

$$\boxed{\begin{gathered} = e^{-\zeta\omega_n t}\left[C\cos\omega_d t + S\sin\omega_d t\right] \\ \text{where } \omega_d = \omega_n\sqrt{1-\zeta^2} \end{gathered}}$$

[p. 390]

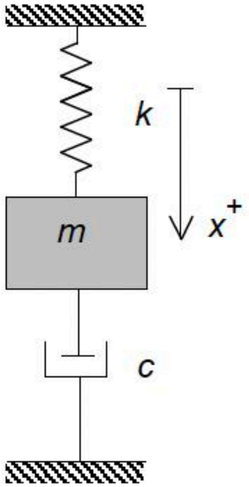
Example 6.B.6

Given: The system shown below.

Find:

(a) The static deformation of the system; and
(b) The undamped natural frequency and damping ratio of the system.

Use the following parameters in your analysis: $m = 0.25$ m, $k = 100$ N/m and $c = 2$ kg/s.



[pg. 400]

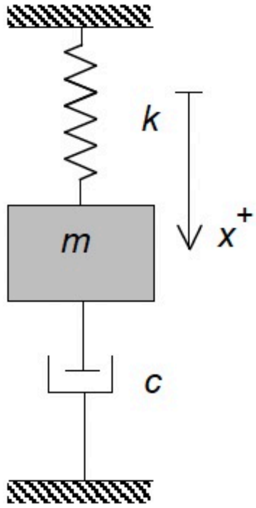
Example 6.B.6

p. 400

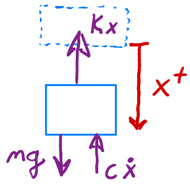
Given: The system shown below.

Find:
(a) The static deformation of the system; and
(b) The undamped natural frequency and damping ratio of the system.

Use the following parameters in your analysis: $m = 0.25$ m, $k = 100$ N/m and $c = 2$ kg/s.



① FBD $x = 0$ when spring unstretched



② Newton

$$\Sigma F_x = mg - c\dot{x} - kx = m\ddot{x}$$

③ EOM

$$m\ddot{x} + c\dot{x} + kx = mg$$

④ x_{st} = def. when system is in static equilibrium ($\dot{x} = \ddot{x} = 0$)

$$m\cancel{\ddot{x}} + \cancel{c\dot{x}} + kx = mg$$

$$\hookrightarrow \quad x_{st} = mg/k$$

⑤ Standard form. $\div$ by m.

$$\ddot{x} + \boxed{\frac{c}{m}}\dot{x} + \boxed{\frac{k}{m}}x = g$$

⑥ Find ω_n

$$\omega_n = \sqrt{\frac{k}{m}}$$

⑦ Find Damping Ratio, $\zeta = \dfrac{c}{2\sqrt{km}}$

$$2\zeta\omega_n = \frac{c}{m}$$

$$\zeta = \frac{c}{2m\omega_n}$$

$$= \frac{c}{2m\sqrt{\frac{k}{m}}}$$

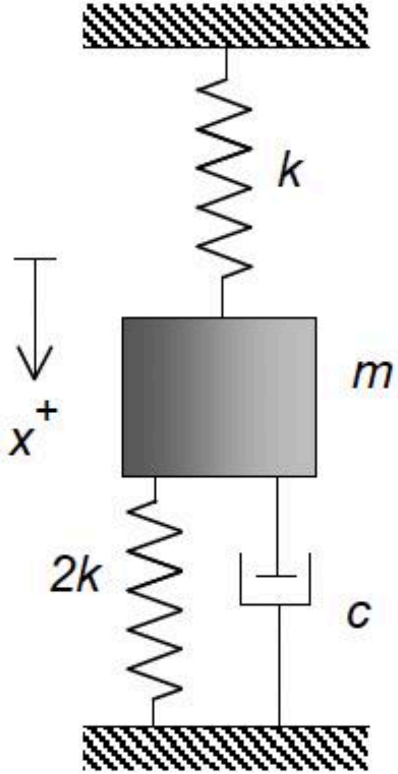
$$= \frac{c}{2\sqrt{km}}$$

Example 6.B.7

Given: The system shown below.

Find: Determine the value of the damping constant c, such that the system has 50 percent of critical damping.

Use the following parameters in your analysis: $k = 3000$ N/m and $m = 10$ kg.



[pg. 401]

Example 6.B.7

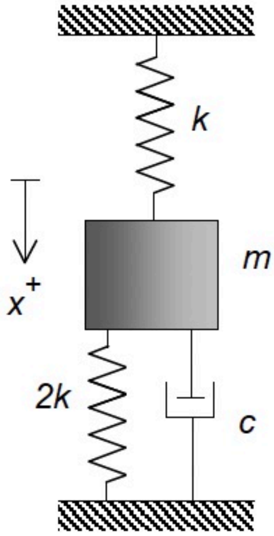
p. 401

Given: The system shown below.

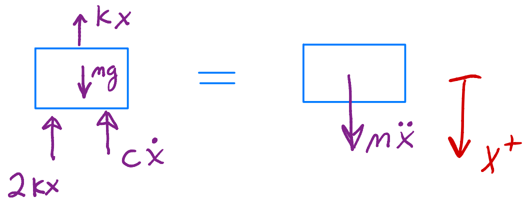
Find: Determine the value of the damping constant c, such that the system has 50 percent of critical damping.

c ?

Use the following parameters in your analysis: $k = 3000$ N/m and $m = 10$ kg.



(1) FBD



(2) Kinetics N/E

$+\downarrow \Sigma F_x = m\ddot{x}$

$-kx - 2kx + mg - c\dot{x} = m\ddot{x}$

$\Rightarrow m\ddot{x} + c\dot{x} + 3kx = mg$ (EOM)

(3) Put in standard form, ÷ by m.

$\Rightarrow \ddot{x} + \frac{c}{m}\dot{x} + \frac{3k}{m}x = g$

(4) ζ & ω_n eqns

$2\zeta\omega_n = \frac{c}{m}$ (1)

$\omega_n^2 = \frac{3k}{m}$ (2)

(5) @ $\zeta \to 0.5$

(1) $\Rightarrow \omega_n = \frac{c}{m}$; Plug in $\zeta = 0.5$ on (1)

$\Rightarrow \frac{c}{m} = \sqrt{\frac{3k}{m}}$; use (2)

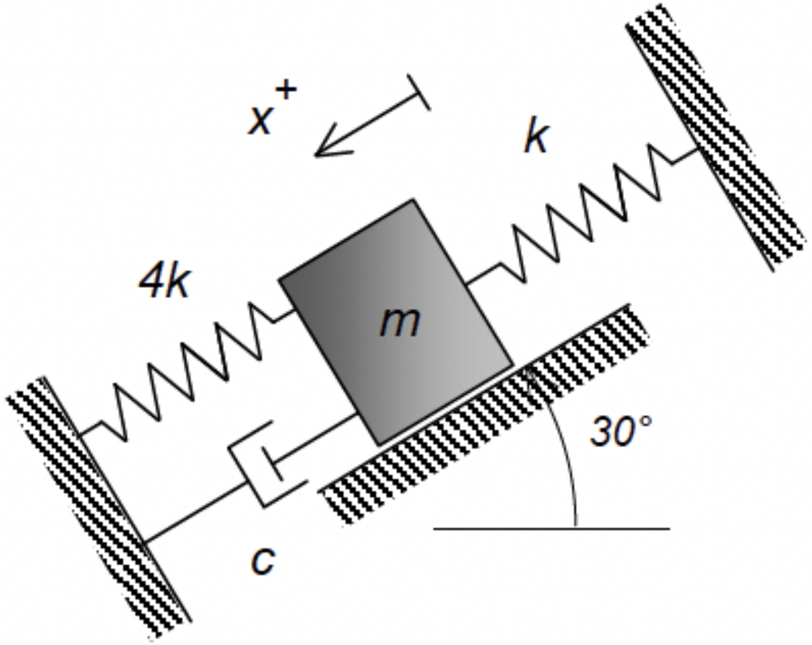
$\Rightarrow c = m\sqrt{\frac{3k}{m}}$

Example 6.B.8

Given: The system shown below consists of a block of mass m, which rests upon a smooth inclined surface, that is connected to ground by two springs of stiffness $4k$ and k, respectively, and a dashpot with damping constant c.

Find: Determine the value of the damping constant c, such that the system has 50 percent of critical damping.

Use the following parameters in your analysis: $k = 2000$ N/m and $m = 10$ kg.



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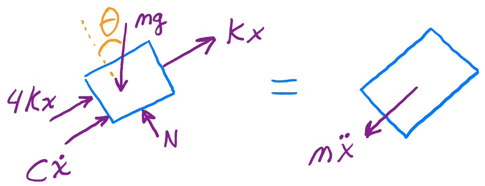
Example 6.B.8 p. 402

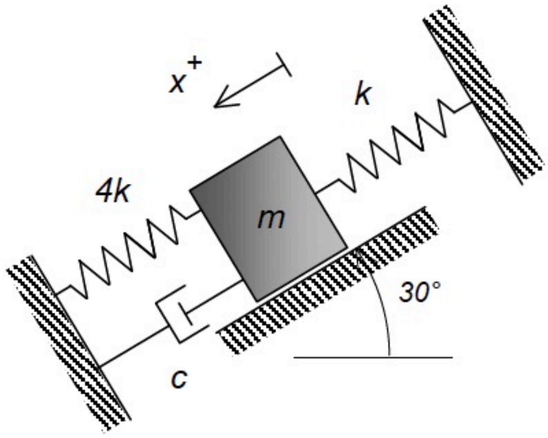
Given: The system shown below consists of a block of mass m, which rests upon a smooth inclined surface, that is connected to ground by two springs of stiffness $4k$ and k, respectively, and a dashpot with damping constant c.

Find: Determine the value of the damping constant c, such that the system has 50 percent of critical damping. c?

Use the following parameters in your analysis: $k = 2000$ N/m and $m = 10$ kg.

1 FBD





2 Kinetics N/E

$\overset{+}{\swarrow} \Sigma F_x = m\ddot{x}$

$-kx - 4kx - c\dot{x} + mg\sin\theta = m\ddot{x}$

$\Rightarrow m\ddot{x} + c\dot{x} + 5kx = mg\sin\theta$ (EOM)

3 Standard form. Find ξ & ω_n

$\Rightarrow \ddot{x} + \frac{c}{m}\dot{x} + \frac{5k}{m}x = g\sin\theta$

$2\xi\omega_n = \frac{c}{m}$

$\omega_n^2 = \frac{5k}{m}$

4 c @ $\xi \to 0.5$

$\omega_n = \frac{c}{m}$

$\Rightarrow \frac{c}{m} = \sqrt{\frac{5k}{m}}$

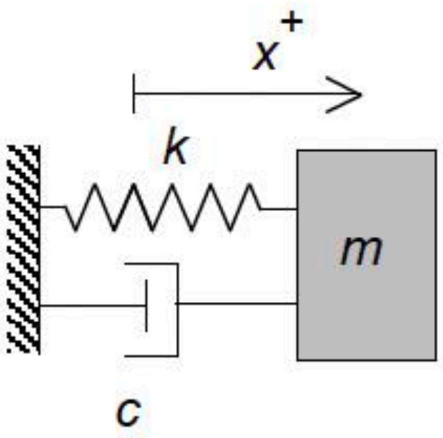
$\Rightarrow c = m\sqrt{\frac{5k}{m}}$

Example 6.B.9

Given: The system shown below.

Find: Find the free response of the system corresponding to $x(0) = x_0$ and $\dot{x}(0) = 0$.

Use the following parameters in your analysis: $x_0 = 6$ mm, $c = 32$ kg/s, $k = 1600$ N/m and $m = 1$ kg.



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Example 6.B.9 p. 403 Classic Spring-Mass-Damper problem

Given: The system shown below.

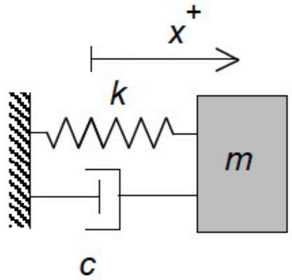
Find: Find the free response of the system corresponding to $x(0) = x_0$ and $\dot{x}(0) = 0$.

$x(t) = ?$

Use the following parameters in your analysis: $x_0 = 6$ mm, $c = 32$ kg/s, $k = 1600$ N/m and $m = 1$ kg.

① FBD

ppl in this classical problem usually ignore mg.



② Kinetics N/E

$\xrightarrow{+} \Sigma F_x = m\ddot{x}$

$\Rightarrow -kx - c\dot{x} = m\ddot{x}$

$\Rightarrow m\ddot{x} + c\dot{x} + kx = 0$ (EOM)

③ Standard form. Find ξ & ω_n

$\Rightarrow \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$

$2\xi\omega_n = \frac{c}{m}$; $\omega_n^2 = \frac{k}{m}$

$\xi = \frac{c}{2\omega_n m}$; is ξ : $\begin{cases} \text{undamped} & (\xi = 0) \\ \text{underdamped} & (\xi < 1) \\ \text{critically damped} & (\xi = 1) \\ \text{overdamped} & (\xi > 1) \end{cases}$

$= 0.4$

④ Underdamped. Apply initial conditions

$x(t) = e^{-\omega_n \xi t}(A\cos\omega_d t + B\sin\omega_d t)$

$\omega_d = \omega_n\sqrt{1-\xi^2}$ ⟸ damped natural frequency. for underdamped

⑤ $x(0)$

$x(0) = x_0$

$\Rightarrow x(0) = \overset{1}{\cancel{e^{-\omega_n \xi 0}}}(A\overset{1}{\cancel{\cos\omega_d 0}} + B\overset{0}{\cancel{\sin\omega_d 0}})$

$\Rightarrow x_0 = A$

⑥ $\dot{x}(0)$

$\dot{x}(t) = -\xi\omega_n \overset{1}{\cancel{e^{-\xi\omega_n t}}}(A\overset{1}{\cancel{\cos\omega_d t}} + \overset{0}{\cancel{B\sin\omega_d t}}) + \overset{1}{\cancel{e^{-\xi\omega_n t}}}(\overset{0}{\cancel{-\omega_d A\sin\omega_d t}} + B\omega_d \overset{1}{\cancel{\cos\omega_d t}})$; product rule

$\dot{x}(0) = 0$

$= -\xi\omega_n A + B\omega_d$

$\Rightarrow -\xi\omega_n x_0 + B\omega_d = 0$

$\Rightarrow B = \frac{\xi\omega_n x_0}{\omega_d}$

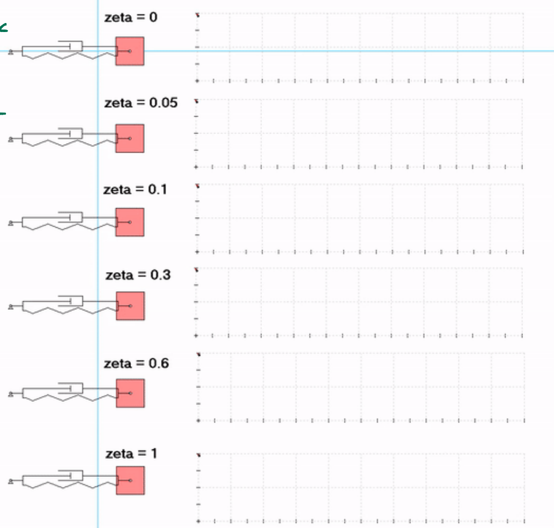
⑦ Free Response of the system

$x(t) = e^{-\xi\omega_n t}\left(x_0\cos\omega_d t + \frac{\xi\omega_n x_0}{\omega_d}\sin\omega_d t\right)$

where: $\omega_d = \omega_n\sqrt{1-\xi^2}$

harmonic response

exponentially decaying harmonic response



Summary: Vibrations - damped free response

EOM: For free response of damped system:

$$m\ddot{x} + c\dot{x} + kx = 0$$

STANDARD FORM OF EOM: Divide EOM by "m" to get:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

where

$$2\zeta\omega_n = c/m \quad \Rightarrow \quad \zeta = c/2\sqrt{km} = damping\ ratio$$

$$\omega_n = \sqrt{k/m} = natural\ \ frequency$$

SOLUTION OF EOM: For $0 \le \zeta < 1\left(UNDERdamped\right)$

$$x(t) = e^{-\zeta\omega_n t}\left(C\cos\omega_d t + S\sin\omega_d t\right) \quad ; \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

HOW TO FIND THE RESPONSE COEFFICIENTS, C AND S?

Enforce the initial conditions on the solution:

$$x(0) = x_0 = e^0\left(C\cos 0 + S\sin 0\right) \ \Rightarrow \ C = x_0$$

$$\dot{x}(0) = \dot{x}_0 = -\zeta\omega_n e^0\left(C\cos 0 + S\sin 0\right) + \omega_d e^0\left(-C\sin 0 + S\cos 0\right) \ \Rightarrow \ S = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}$$

Lec 37 Short Feedback Form: