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ME 274 Lecture 36

Vibrations: Free Response – Part 1

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04/17/26

Housekeeping/Announcements

***Reminder for Henny to wear a mic during the lecture.

1. **HW 35 (6.A and 6.B) due Today!!**

2. Office hours are on ME2008. Located on second floor of renovated side of ME.

Chapter 6: Vibrations – Free Response

$$\sum \vec{F} = m\vec{a}_G$$

$$\sum \vec{M}_A = I_A\vec{\alpha}$$

$$M\ddot{x} + C\dot{x} + Kx = f(t)$$

Goals for Chapter 6:

- *We are used* to solving acceleration/velocity/position for a given **instant in time** (using Newton-Euler Equations, upper left).

- In this chapter, we will study the response of systems whose motion is *oscillatory in nature* (~vibrations).

- We will focus on understanding the change in position as a **general function of time** (ie. **Deriving the EOMs of a system**, upper right).

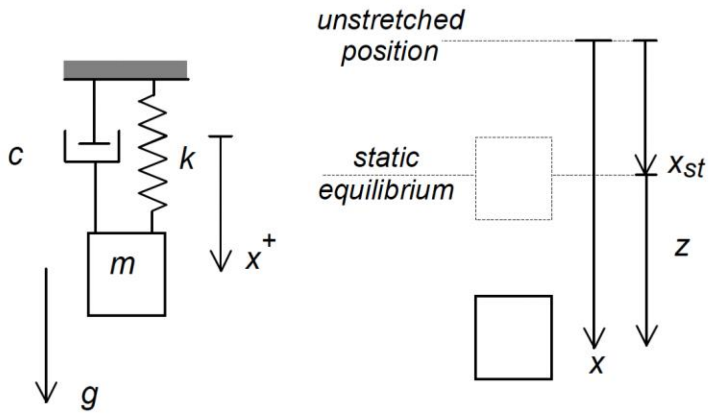
Equations of Motion (EOMs)

The equations of motion for the vibrational systems studied in this course are typically composed of the following components:

1. Particles/rigid bodies
2. Dashpots (a.k.a. Damper)
3. Springs
4. External forcing

$$\boxed{\boxed{M\ddot{x}}+\boxed{C\dot{x}}+\boxed{Kx}=\boxed{f(t)}}$$

Static Deformation and the EOM for Vibration



1. Last lecture, we saw that the EOM for the system shown is given as:

$$m\ddot{x} + c\dot{x} + kx = mg$$

2. Throughout this chapter, we will be evaluating and finding EOMs for different systems. **The above equation shown an inhomogeneous equation with a constant inhomogeneous term of mg.**

3. This static displacement can be found by setting $\dot{x} = \ddot{x} = 0$, which gives:

$$kx_{st} = mg \longrightarrow x_{st} = \frac{mg}{k}$$

4. Now instead of measuring the position form the unstretched position, we measure its position from the static deformation state. **We define this coordinate as z:** $z = x - x_{st}$

5. Introducing this into the original EOM above gives a **homogenous equation (eqn equal to 0):**

$$m\ddot{z} + c\dot{z} + kz = 0$$

Solution of the Homogeneous Form of the EOM

$$M\ddot{x} + C\dot{x} + Kx = f(t)$$

1. The EOM shown above is, in general, inhomogeneous. From linear differential equations you can recall that the solution of this EOM can be written as:

$$x(t) = x_c(t) + x_p(t)$$

2. Where:
   - x_c(t) is the complementary (or homogeneous) solution
   - x_p(t) is the particular solution

3. Note: Lectures 36-39 will focus on the complementary solution, then we will look at the particular solution (where f(t) is not equal to zero)

4. The general form of the complementary solution can be written as: $x(t) = Ae^{\lambda t}$

5. Where A and lamda are to be determined parameters. Substituting this general form of the complementary solution into the homogeneous equation of motion gives:

$$\lambda^2 MAe^{\lambda t} + \lambda CAe^{\lambda t} + KAe^{\lambda t} = 0$$
$$\left(\lambda^2 M + \lambda C + K\right) Ae^{\lambda t} = 0$$

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The characteristic equation and undamped response

$$\left(\lambda^2 M + \lambda C + K\right) A e^{\lambda t} = 0$$

6. In order for above to be true one of the following terms must be zero:
   - $e^{\lambda t} = 0$: This is not possible for any value of (lambda)*(t)
   - $A = 0$: This would mean the system does not move.
   - $\lambda^2 M + \lambda C + K = 0$: **This is the only possible term that corresponds to the motion of the system**

7. This final, third condition above is known as the ***characteristic equation*** of the problem.
- Solving the ***characteristic equation*** gives us two roots:

$$\lambda_1, \lambda_2 = \frac{-C \pm \sqrt{C^2 - 4MK}}{2M} = \left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]\sqrt{\frac{K}{M}} \quad \text{where } \zeta = C/2\sqrt{KM}$$

8. For today, we will evaluate the specific case where the damping ratio is equal to 0.
- Meaning our system is **undamped.**
- **Notice the natural frequency, w_n is equal to sqrt(K/M)**

Case 1: $\zeta = 0$ ("undamped")
Here, we have two IMAGINARY roots of the characteristic equation:

$$\lambda_1, \lambda_2 = \pm i\omega_n$$

where $i = \sqrt{-1}$ = imaginary number, and $\omega_n = \sqrt{K/M}$

[p. 388]

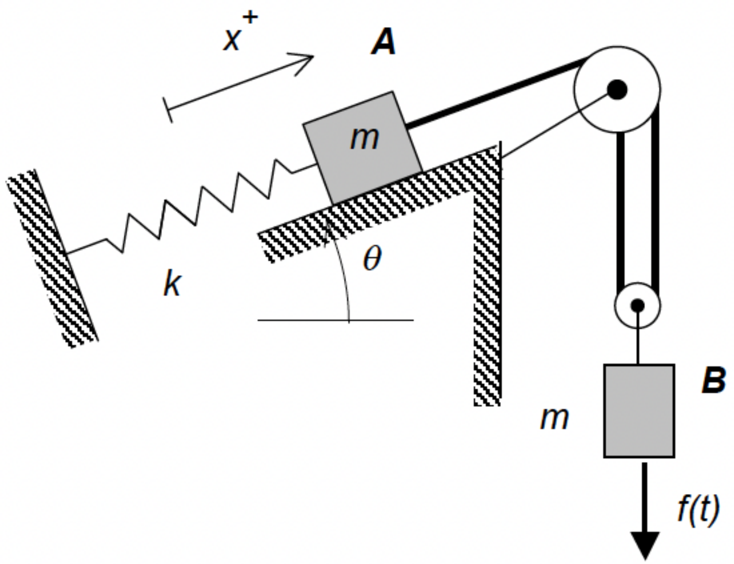
Example 6.B.1

Given: The system shown below. Assume the inclined surface is smooth.

Find: For this problem:

(a) Determine the EOM of this system in terms of the coordinate x; and

(b) Determine the natural frequency of the system.



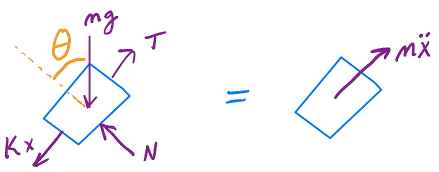
[pg. 395]

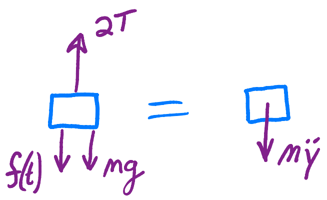
Example 6.B.1 p. 395

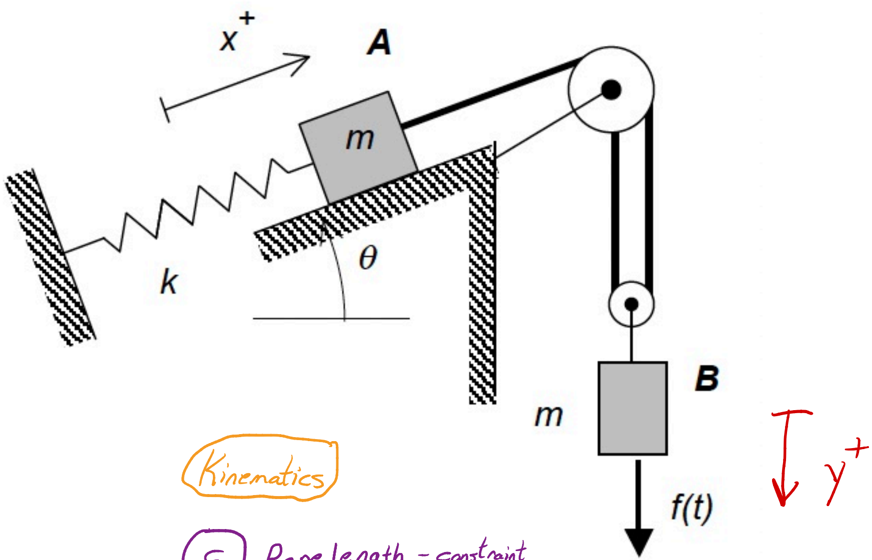
Given: The system shown below. Assume the inclined surface is smooth.

Find: For this problem:
(a) Determine the EOM of this system in terms of the coordinate x; and EOM?
(b) Determine the natural frequency of the system. ω_n?

① FBD







Kinetics

② N/E

$+\nearrow \Sigma F_x = m\ddot{x}$

$\Rightarrow T - kx - mg\sin\theta = m\ddot{x}$ (1)

$+\downarrow \Sigma F_y = m\ddot{y}$

$\Rightarrow -2T + f(t) + mg = m\ddot{y}$ (2)

③ Plug in (1) onto (2)

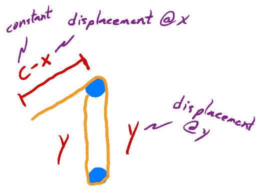
(1) $\Rightarrow T = m\ddot{x} + kx + mg\sin\theta$

(2) $\Rightarrow -2m\ddot{x} - 2kx - 2mg\sin\theta + f(t) + mg = m\ddot{y}$

④ x & y are dependent variables

Kinematics

⑤ Rope length - constraint



$C_1 - x + 2y = C_2$

$\Rightarrow -x + 2y = C$

$\Rightarrow -\ddot{x} + 2\ddot{y} = 0$

$\Rightarrow \ddot{y} = \frac{\ddot{x}}{2}$ (3)

EOM

⑥ Plug (3) back into step ③

$-2m\ddot{x} - 2kx - 2mg\sin\theta + f(t) + mg = \frac{1}{2}m\ddot{x}$

$\Rightarrow \underbrace{\frac{5}{2}m}_{M}\ddot{x} + \underbrace{2k}_{K}x = f(t) + mg - 2mg\sin\theta$ (EOM)

⑦ Natural Freq. Find M & K in EOM

$\omega_n = \sqrt{\frac{K}{M}}$

$= \sqrt{\frac{2k}{\frac{5}{2}m}}$

$= \sqrt{\frac{4k}{5m}}$

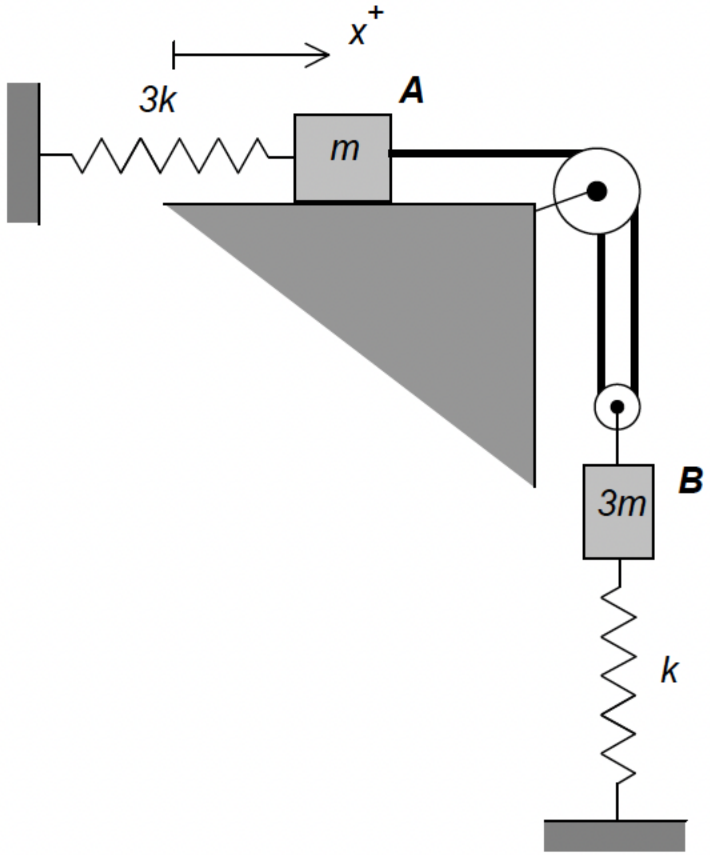
Example 6.B.2

Given: Blocks A and B (having masses of m and $3m$, respectively) are connected by a cable-pulley system as shown below. A spring of stiffness $3k$ is attached between A and ground. A second spring of stiffness k is attached between block B and ground. Let x describe the position of A, where the springs are unstretched when $x = 0$, and let the horizontal surface be smooth.

Find:

(a) Determine the EOM of this system in terms of the coordinate x; and

(b) Determine the natural frequency of the system.



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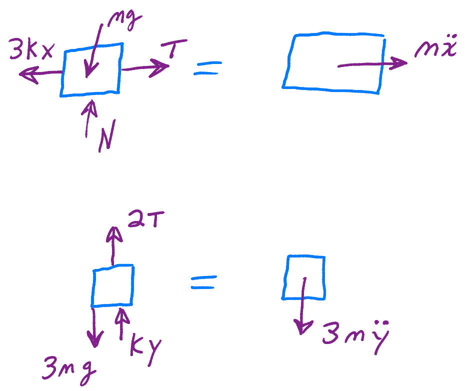
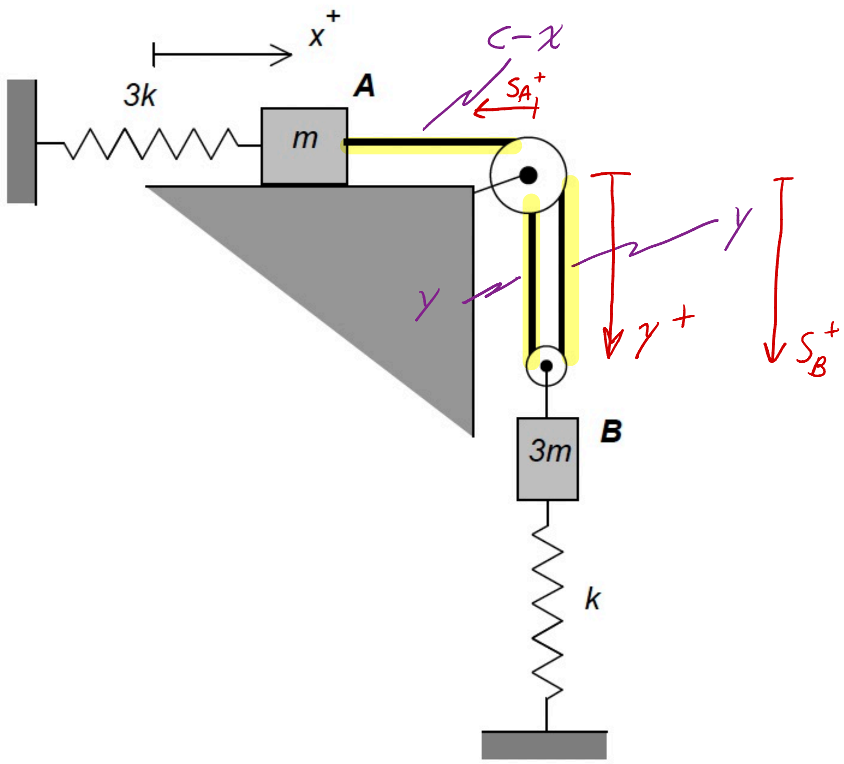
Example 6.B.2 p.396

Given: Blocks A and B (having masses of m and $3m$, respectively) are connected by a cable-pulley system as shown below. A spring of stiffness $3k$ is attached between A and ground. A second spring of stiffness k is attached between block B and ground. Let x describe the position of A, where the springs are unstretched when $x = 0$, and let the horizontal surface be smooth.

Find:
(a) Determine the EOM of this system in terms of the coordinate x; and EOM ?
(b) Determine the natural frequency of the system. ω_n ?

① FBD





Kinetics

② N/E

Block A: $\xrightarrow{+} \Sigma F_x = m\ddot{x}$

$\Rightarrow T - 3kx = m\ddot{x}$ (1)

Block B: $+\downarrow \Sigma F_y = 3m\ddot{y}$

$\Rightarrow 3mg - 2T - ky = 3m\ddot{y}$ (2)

③ 4 unkns/dep vars. Need eqn in 1

④ Kinematics Rope Length - constraint

$C + s_A + 2s_B = L$

$\Rightarrow \ddot{s}_A + 2\ddot{s}_B = 0$

$\ddot{s}_A = -\ddot{x}$; $\ddot{s}_B = +\ddot{y}$

$-\ddot{x} + 2\ddot{y} = 0$

$\Rightarrow \ddot{x} = 2\ddot{y}$ (3)

⑤ $\ddot{x}$ & $\ddot{y}$ are about unstretched length

$\ddot{x} = 2\ddot{y}$

$\Rightarrow \Delta x = 2\Delta y$; about unstretched

$\Rightarrow x = 2y$ (4)

⑥ EOM Plug in (1) into (2)

(1) $\Rightarrow T = m\ddot{x} + 3kx$

(2) $\Rightarrow 3mg - 2m\ddot{x} - 6kx - k\frac{x}{2} = 3m\frac{\ddot{x}}{2}$

$\frac{7}{2}m\ddot{x} + \frac{13}{2}kx = 3mg$ (EOM)

⑦ Find ω_n. Get K & M from EOM

$\Rightarrow \ddot{x} + \frac{13}{7}\frac{k}{m}x = \frac{6mg}{7m}$; ÷ by $\frac{7}{2}m$

$= \frac{6g}{7}$

$\omega_n = \sqrt{\frac{K}{M}}$

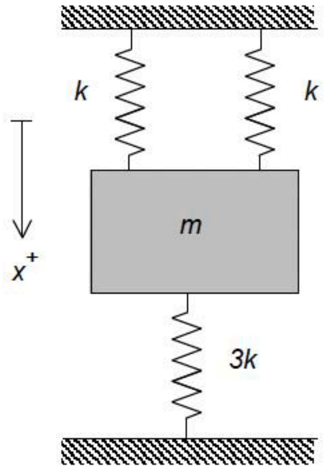
$\Rightarrow \omega_n^2 = \frac{13k}{7m}$

$\Rightarrow \omega_n = \sqrt{\frac{13k}{7m}}$

Example 6.B.3

Given: The block has a downward speed of v_0 as it passes through its equilibrium position.

Find: Determine the maximum acceleration of the block over one cycle of oscillation.



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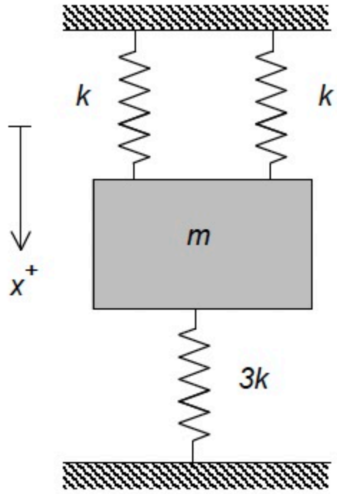
Example 6.B.3 p. 397

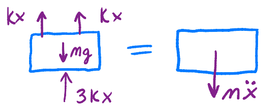
Given: The block has a downward speed of v_0 as it passes through its equilibrium position.

Find: Determine the maximum acceleration of the block over one cycle of oscillation.

$|\ddot{z}|_{max}$?



① FBD



② Kinetics N/E

$\downarrow^{+} \Sigma F_x = m\ddot{x}$

$\Rightarrow -kx - kx - mg - 3kx = m\ddot{x}$

$\Rightarrow m\ddot{x} + 5kx = mg$ (EOM)

③ Simple harmonic oscillator w/ a static offset.
get rid of

$\dot{x} = \ddot{x} = 0$

$\Rightarrow 5k x_{st} = mg$

$\Rightarrow x_{st} = \frac{mg}{5k}$

④ Introduce coordinate z

$z = x - x_{st}$

$= x - \frac{mg}{5k}$; $\dot{z} = \dot{x}$; $\ddot{z} = \ddot{x}$

⑤ Plug results from ④ into EOM

$m\ddot{z} + 5kz = 0$

$z = A\cos\omega_n t + B\sin\omega_n t$; $\omega_n = \sqrt{\frac{5k}{m}}$

⑥ Initial Conditions

$z(0) = 0$

$= A$

$\Rightarrow A = 0$

⑦ Use z

$z = B\sin\omega_n t$

$\dot{z} = B\omega_n \cos\omega_n t$

$\dot{z}(0) = v_0$

$= B\omega_n$

$\Rightarrow B = \frac{v_0}{\omega_n}$

⑧

$\dot{z} = \frac{v_0}{\omega_n}\omega_n \cos\omega_n t$

$= v_0 \cos\omega_n t$

⑨ Take one more derivative for accel

$\ddot{z} = -v_0\omega_n \sin\omega_n t$

⑩ max accel value $[-1, 1]$ ∴

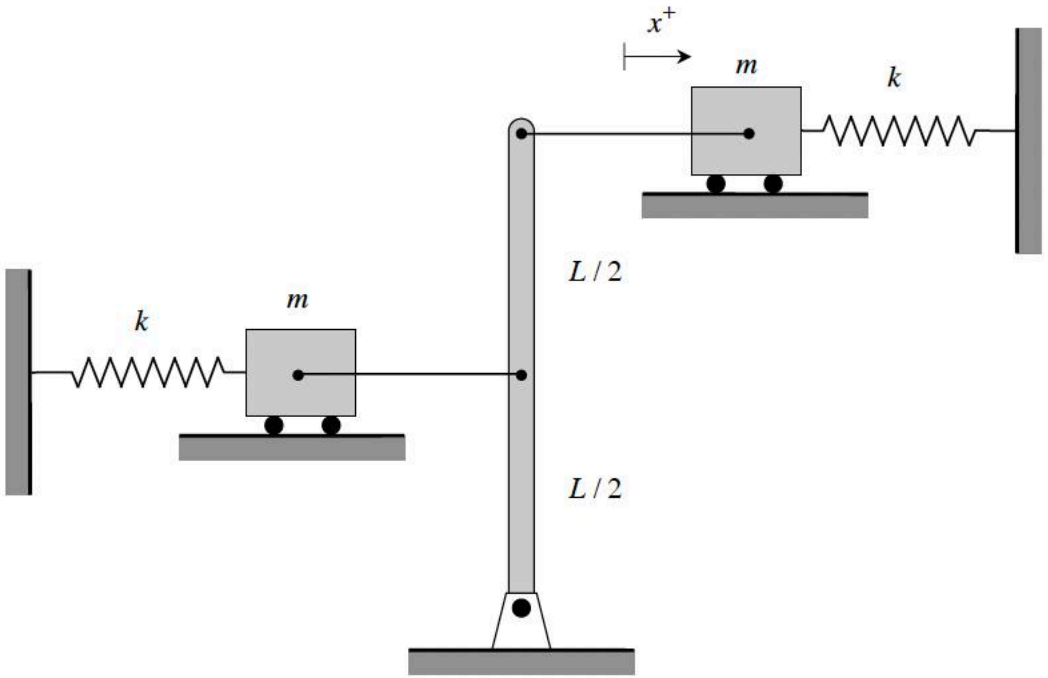
$|\ddot{z}|_{max} = v_0\omega_n$

$= v_0\sqrt{\frac{5k}{m}}$

Example 6.B.4

Given: The system shown below consists of two masses, each of mass m, which are connected by a massless bar with the bar able to rotate about a pin joint at its lower end. Each mass is connected to ground by a spring of stiffness k, and moves along a smooth surface. The coordinate x is measured from the position of the particle when the bar is vertical. The springs are unstretched when $x = 0$. Assume small motions of the system; i.e., restrict the motion of the bar to small angles of rotation.

Find:
(a) Determine the equation of motion of this system in terms of the coordinate x; and
(b) Determine the natural frequency of the system.



[pg. 398]

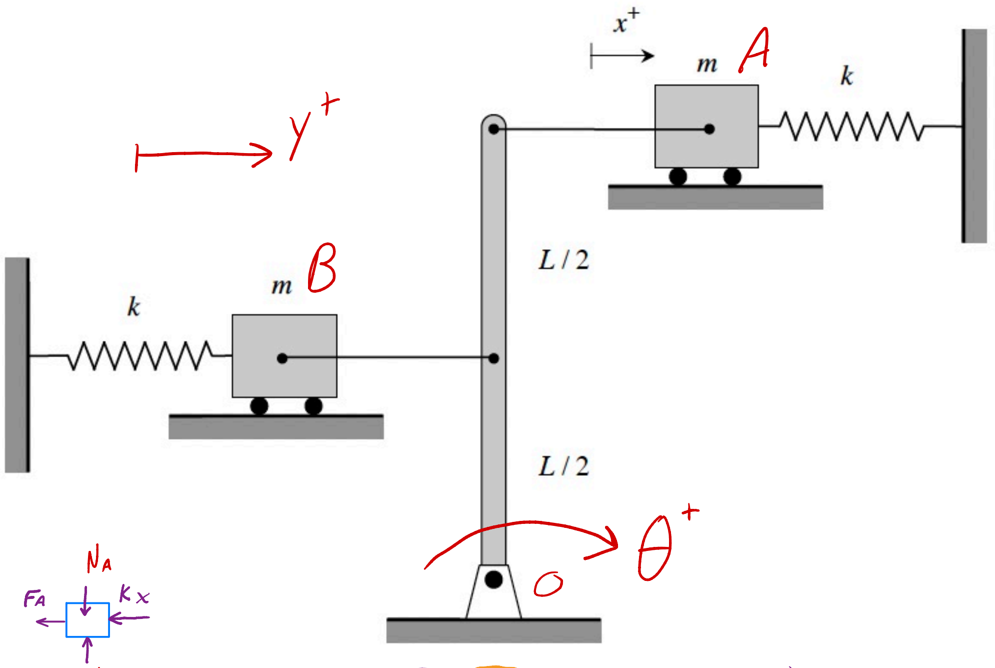
Example 6.B.4 p.398

Given: The system shown below consists of two masses, each of mass m, which are connected by a massless bar with the bar able to rotate about a pin joint at its lower end. Each mass is connected to ground by a spring of stiffness k, and moves along a smooth surface. The coordinate x is measured from the position of the particle when the bar is vertical. The springs are unstretched when $x = 0$. Assume small motions of the system; i.e., restrict the motion of the bar to small angles of rotation.

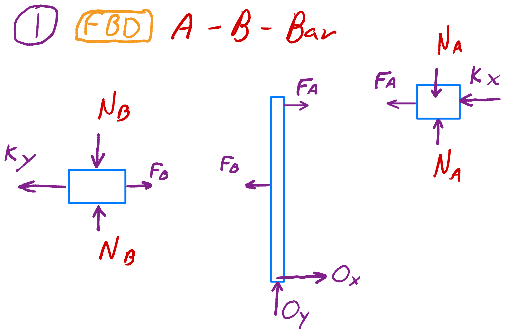
Find:

(a) Determine the equation of motion of this system in terms of the coordinate x; and EOM?

(b) Determine the natural frequency of the system. ω_n ?



① FBD A - B - Bar



② Kinetics Newton-Euler

A: $\overset{+}{\rightarrow}\Sigma F_x = -kx - F_A = m\ddot{x}$ (1)

B: $\overset{+}{\rightarrow}\Sigma F_y = -ky + F_B = m\ddot{y}$ (2)

bar: $\overset{+}{\circlearrowright}\Sigma M_O = +F_A L - F_B \frac{L}{2} = I_O \ddot{\theta}$ (3)

w/ $I_O = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2$; P.A.T.

$= \frac{1}{3}mL^2$

③ Put in standard form

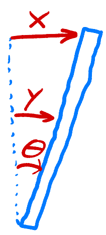
(1) $\Rightarrow F_A = -kx - m\ddot{x}$

(2) $\Rightarrow F_B = ky + m\ddot{y}$

(3) $\Rightarrow (-kx - m\ddot{x})L - (ky + m\ddot{y})\frac{L}{2} = \frac{1}{3}mL^2\ddot{\theta}$

④ 4 unkns/dep vars. Need everything in x

⑤ Kinematics Assume θ is small



$x = L\sin\theta$ ($\sin\theta \approx \theta$)

$\approx L\theta$

$y = \frac{L}{2}\sin\theta$ ($\sin\theta \approx \theta$)

$= \frac{L}{2}\theta$

$\ddot{x} = L\ddot{\theta}$

$\Rightarrow \ddot{\theta} = \frac{\ddot{x}}{L}$

$\ddot{y} = \frac{L}{2}\ddot{\theta}$

$\Rightarrow \ddot{y} = \frac{1}{2}\ddot{x}$

⑥ EOM

$(-kx - m\ddot{x})L - \left(k\frac{x}{2} + m\frac{\ddot{x}}{2}\right)\frac{L}{2} = \left(\frac{1}{3}mL^2\right)\frac{\ddot{x}}{L}$; Plug-in Kinematics on (3)

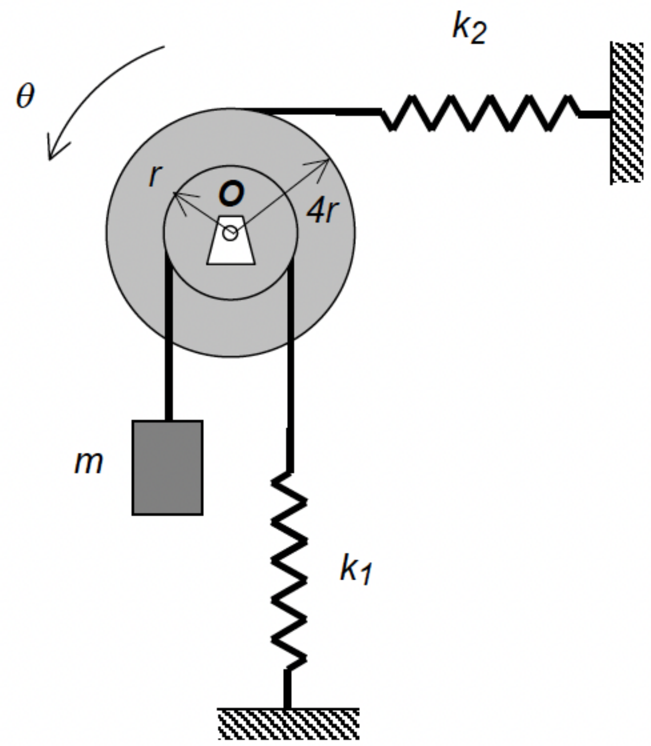
$\left[\frac{1}{3}m + \underbrace{m + \frac{1}{2}m}_{3/2}\right]\ddot{x} + \left[k + \frac{k}{2}\right]x = 0$

$\boxed{\left(\frac{1}{3} + \frac{3}{2}\right)m\ddot{x} + \frac{3}{2}kx = 0 \quad \text{EOM for small } \theta}$

Example 6.B.5

Given: The pulley has a mass moment of inertia of I_O.

Find:
(a) Determine the EOM of this system in terms of the coordinate θ; and
(b) Determine the natural frequency of the system.

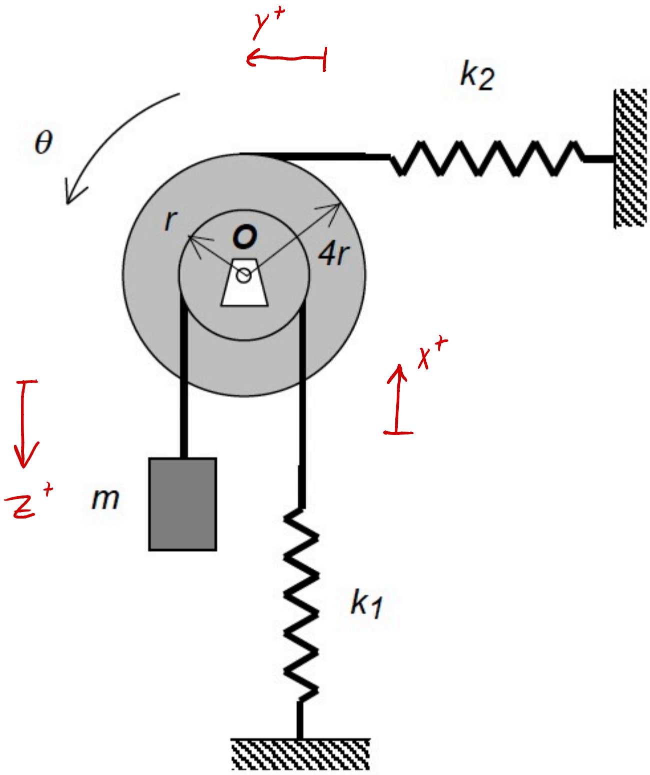


[pg. 399]

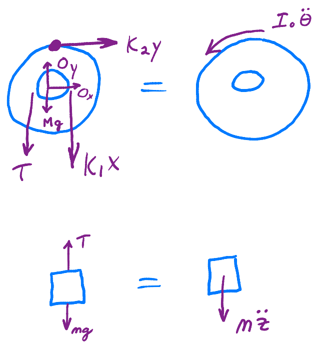
Example 6.B.5 p.399

Given: The pulley has a mass moment of inertia of I_O.

Find:
(a) Determine the EOM of this system in terms of the coordinate θ; and EOM?
(b) Determine the natural frequency of the system. ω_n?



① FBD



② Kinetics

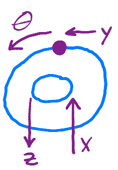
$\circlearrowleft + \sum M_O = I_o\ddot{\theta}$

$Tr - K_1 x r - K_2 y(4r) = I_o\ddot{\theta} \quad (1)$

$+\downarrow \sum F_z = m\ddot{z}$

$mg - T = m\ddot{z} \quad (2)$

③ Kinematics



$z = r\theta \Rightarrow \ddot{z} = r\ddot{\theta} \quad (3)$

$x = r\theta \quad (4)$

$y = 4r\theta \quad (5)$

④ EOM

$(2) \Rightarrow T = mg - m\ddot{z} = mg - mr\ddot{\theta}$

$(1) \Rightarrow mgr - mr^2\ddot{\theta} - k_1 r^2\theta - k_2(4r)^2\theta = I_o\ddot{\theta}$

$\Rightarrow \underbrace{(I_o + mr^2)}_{M}\ddot{\theta} + \underbrace{(k_1 r^2 + 16 k_2 r^2)}_{K}\theta = mgr$ (EOM)

⑤ Find ω_n. Get K & M from EOM

$\omega_n = \sqrt{\frac{K}{M}}$

$= \sqrt{\frac{k_1 r^2 + 16 k_2 r^2}{I_o + mr^2}}$

Summary: Vibrations - Free Undamped Response

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EOM: For free response of undamped system:

$$m\ddot{x} + kx = 0$$

STANDARD FORM OF EOM: Divide EOM by "m" to get:

$$\ddot{x} + \omega_n^2 x = 0$$

where $\omega_n = \sqrt{\dfrac{k}{m}} = natural\ \ frequency$

SOLUTION OF EOM:

$$x(t) = C\cos\omega_n t + S\sin\omega_n t$$

HOW TO FIND THE RESPONSE COEFFICIENTS, C AND S?

Enforce the initial conditions on the solution:

$$x(0) = x_0 = C\cos 0 + S\sin 0 \quad \Rightarrow \quad C = x_0$$

$$\dot{x}(0) = \dot{x}_0 = -C\omega_n \sin 0 + S\omega_n \cos 0 \quad \Rightarrow \quad S = \frac{\dot{x}_0}{\omega_n}$$

Lec 36 Short Feedback Form: