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ME 274 Lecture 34

Rigid body kinetics: Summary

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04/13/26

Housekeeping/Announcements

***Reminder for Henny to wear a mic during the lecture.

1. **HW 33 (5.M and 5.N) due Today!!**

2. Office hours are changing to ME2008B…
   - Second floor of renovated side of ME.

Kinetics Summary Handout

Kinetics Table (page 352 of the lecture book)

Method	Body model	Fundamental equations
Newton-Euler *(relating forces to accelerations)*	**particle**	$\sum \vec{F} = m\vec{a}$
	rigid body *(G = c.m. and A = any point on body)*	$\sum \vec{F} = m\vec{a}_G$ $\sum \vec{M}_A = I_A\vec{\alpha} + m\vec{r}_{G/A} \times \vec{a}_A$
Work-energy *(relating change in speed to change in position)*	**particle**	$T_1 + V_1 + U_{1\to 2}^{(nc)} = T_2 + V_2$ where $T = \frac{1}{2}mv^2$
	rigid body *(G = c.m. and A = any point on body)*	$T_1 + V_1 + U_{1\to 2}^{(nc)} = T_2 + V_2$ where $T = \frac{1}{2}mv_A^2 + \frac{1}{2}I_A\omega^2 + m\vec{v}_A \cdot \left(\vec{\omega} \times \vec{r}_{G/A}\right)$
Linear impulse-momentum *(relating change in velocity to change in time)*	**particle**	$\int_{t_1}^{t_2} \sum \vec{F}\, dt = m\vec{v}_2 - m\vec{v}_1$
	rigid body *(G = c.m.)*	$\int_{t_1}^{t_2} \sum \vec{F}\, dt = m\vec{v}_{G2} - m\vec{v}_{G1}$
Angular impulse-momentum *(relating change in angular velocity to change in time)*	**particle** *(O = fixed point)*	$\int_{t_1}^{t_2} \sum \vec{M}_O\, dt = \vec{H}_{O2} - \vec{H}_{O1}$ where $\vec{H}_O = m\vec{r}_{P/O} \times \vec{v}_P$
	rigid body *(A = fixed point or c.m.)*	$\int_{t_1}^{t_2} \sum \vec{M}_A\, dt = \vec{H}_{A2} - \vec{H}_{A1}$ where $\vec{H}_A = I_A\vec{\omega}$

Notes on the *four-step* method:

1. ***FBD(s)***
   - Newton/Euler (*N/E*): typically individual FBDs
   - Work/energy (*W/E*), linear impulse momentum (*LIM*) and angular impulse/momentum (*AIM*): typically make it "BIG" (include all moving bodies)
2. ***Kinetics*** (see suggestions in the table to the right as to which method(s) to use)
   - Choose coordinate system(s) based on the *"Given"* and *"Find"*, including consideration of any motion constraints in the system.
   - *N/E*: point G in the Newton equation must be the center of mass for the body
   - *W/E*: mechanical energy is conserved if $U_{1\to 2}^{(nc)} = 0$ (no non-conservative work being done on your system)
   - *LIM*: linear momentum in the *x*-direction is conserved if $\sum F_x = 0$ (no net force in the *x*-direction for your system)
   - *AIM*: angular momentum in the *z*-direction is conserved if $\sum M_{Oz} = 0$ (no net moment in the *z*-direction about point O for your system).For particles, O must be a fixed point. For rigid bodies, O can be either a fixed point or the center of mass.
3. ***Kinematics***
   - *N/E*: typically need *acceleration* kinematics
   - *W/E, LIM and AIM:* typically need *velocity* kinematics
4. ***Solve***

Chapter 5: Planar Rigid Body Kinetics

Newton–Euler (N/E) Equations

- Newton–Euler is a vector-based approach.
- For planar motion, each free body provides up to three scalar equations (two force + one moment).
- Force equations always use the center of mass acceleration.
- Moment equations can be taken about any point and involve mass moment of inertia.
- **These equations relate forces/reactions to accelerations at a specific instant.**

Work–Energy (W/E) Equation

- The work–energy equation is a scalar equation derived from Newton–Euler via path integrals.
- It is expressed using kinetic energy, potential energy, and work done.
- **It is useful for relating changes in speed to changes in position.**
- A system approach is recommended.
- Including multiple bodies helps treat internal forces as workless.

Impulse–Momentum (L.I.M & A.I.M) Equations

- Impulse–momentum equations are vector equations derived from time integration of Newton–Euler.
- For planar motion, they provide up to three components.
- **They are useful for relating changes in velocity to time.**
- A system approach is recommended, especially when momentum is conserved.
- They are commonly used with coefficient of restitution to solve impact problems.

Discussion - Newton-Euler Equation - 3 Special Forms of Euler's Equation where $\vec{r}_{G/A} \times (m\vec{a}_A)$ = 0

$$\boxed{\left(\sum \vec{M}_A\right)_{ext} = I_A\vec{\alpha} + \vec{r}_{G/A} \times (m\vec{a}_A)}$$

1. If you choose A to be the **center of mass, G.** $\vec{r}_{G/A} = \vec{0}$.

$$\sum \vec{M}_G = I_G\vec{\alpha}$$

2. If you choose A to be a **fixed point on the body, O.** $\vec{a}_O = \vec{0}$.

$$\sum \vec{M}_O = I_O\vec{\alpha}$$

3. If you choose a point A which has an acceleration vector that is **parallel to** $\vec{r}_{G/A}$ **.** $\vec{r}_{G/A} \times \vec{a}_A = \vec{0}$,

$$\sum \vec{M}_A = I_A\vec{\alpha}$$

Picking one of these 3 locations will simplify our problem, **it is recommended to do this as often as possible.**

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Discussion – Work/Energy Equation for Rigid Bodies

$$\boxed{T_1 + V_1 + U_{1\to2}^{(nc)} = T_2 + V_2}$$

where $\boxed{T = \frac{1}{2}mv_A^2 + \frac{1}{2}I_A\omega^2 + m\vec{v}_A \cdot \left(\vec{\omega} \times \vec{r}_{G/A}\right)}$

The point P in the above equation for K.E. can be any point on the rigid body:

- **However, typically we pick one of the two,** since it makes the equation above simpler:

1. Center of Mass, G, for the body. Equation reduces to:

$$\boxed{T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2}$$

2. A fixed point, O. Equation above reduces to:

$$\boxed{T = \frac{1}{2}I_O\omega^2}$$

- **Can be the instant center of the body,** we will typically do this
- **Need to use Parallel Axis Theorem,** since its not center of mass, G

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Impulse-Momentum Equations for Planar Rigid Bodies

- LIM and AIM for Planar Rigid Bodies (lectures 32-33)
    - Where A is a **fixed point on the body or center of mass**

$$\boxed{\int_1^2 \sum \vec{F}\, dt = m\vec{v}_{G2} - m\vec{v}_{G1}}$$

$$\boxed{\int_1^2 \sum \vec{M}_A dt = \vec{H}_{A2} - \vec{H}_{A1} = I_A\vec{\omega}_2 - I_A\vec{\omega}_1}$$

Kinetics Table

Method	Body model	Fundamental equations
Newton-Euler *(relating forces to accelerations)*	**particle**	$\sum \vec{F} = m\vec{a}$
	rigid body *(G = c.m. and A = any point on body)*	$\sum \vec{F} = m\vec{a}_G$ $\sum \vec{M}_A = I_A\vec{\alpha} + m\vec{r}_{G/A} \times \vec{a}_A$
Work-energy *(relating change in speed to change in position)*	**particle**	$T_1 + V_1 + U_{1\to 2}^{(nc)} = T_2 + V_2$ $where\ \ T = \frac{1}{2}mv^2$
	rigid body *(G = c.m. and A = any point on body)*	$T_1 + V_1 + U_{1\to 2}^{(nc)} = T_2 + V_2$ $where\ \ T = \frac{1}{2}mv_A^2 + \frac{1}{2}I_A\omega^2 + m\vec{v}_A \cdot (\vec{\omega} \times \vec{r}_{G/A})$
Linear impulse-momentum *(relating change in velocity to change in time)*	**particle**	$\int_{t_1}^{t_2} \sum \vec{F}\, dt = m\vec{v}_2 - m\vec{v}_1$
	rigid body *(G = c.m.)*	$\int_{t_1}^{t_2} \sum \vec{F}\, dt = m\vec{v}_{G2} - m\vec{v}_{G1}$
Angular impulse-momentum *(relating change in angular velocity to change in time)*	**particle** *(O = fixed point)*	$\int_{t_1}^{t_2} \sum \vec{M}_O\, dt = \vec{H}_{O2} - \vec{H}_{O1}$ $where\ \ \vec{H}_O = m\vec{r}_{P/O} \times \vec{v}_P$
	rigid body *(A = fixed point or c.m.)*	$\int_{t_1}^{t_2} \sum \vec{M}_A\, dt = \vec{H}_{A2} - \vec{H}_{A1}$ $where\ \ \vec{H}_A = I_A\vec{\omega}$

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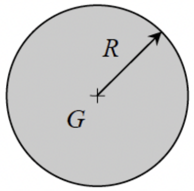
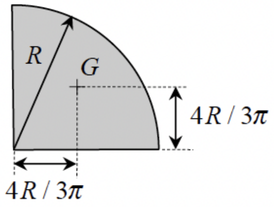
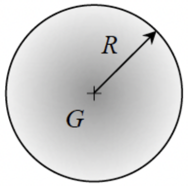
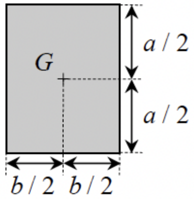
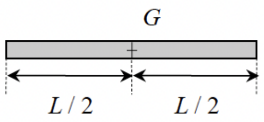
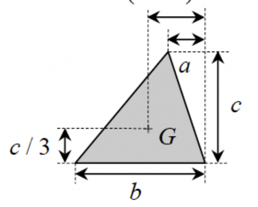
Mass Moment of Inertia of Commonly Used Objects and **when to use Parallel Axis Theorem (P.A.T.)**

1. Often times you will reference a table such as below and get your I_G

2. **Note:** These values below are presented in reference to G.
   - If you are not using your Euler's Equation in terms of G, you will have to use **Parallel Axis Theorem (P.A.T.)**

$$\boxed{I_A = I_G + m d_{AG}^2}$$

3. When do I use P.A.T.? -> What point are you evaluating?

Uniform disk of mass m		$I_G = \frac{1}{2} m R^2$
Quarter circular plate of mass m		$I_G = \left(\frac{9\pi^2 - 64}{18\pi^2}\right) m R^2$
Uniform sphere of mass m		$I_G = \frac{2}{5} m R^2$
Uniform rectangular plate of mass m		$I_G = \frac{1}{12} m \left(a^2 + b^2\right)$
Uniform thin bar of mass m		$I_G = \frac{1}{12} m L^2$
Uniform triangular plate of mass m		$I_G = \frac{1}{18} m \left(a^2 + b^2 + c^2 - ab\right)$

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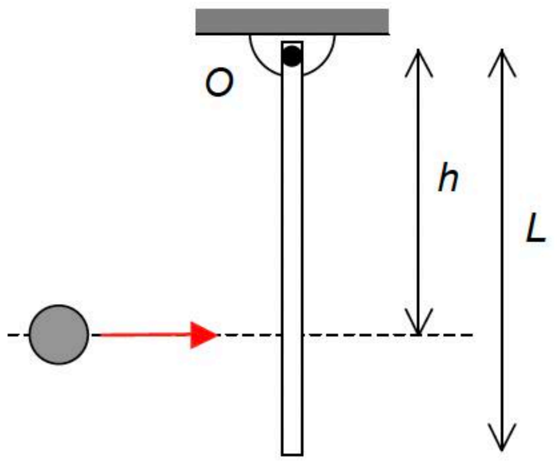
Example 5.A.17

Given: A ball of mass m strikes a homogeneous bar of length L and mass M at a distance h from the pivot point O.

Find: Determine the distance h such that the horizontal reaction force at O on the bar is zero. (This location on the bar is known as the body's "center of percussion". Can you think of a practical application of knowing the location of the center of percussion of a body?)



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Example 5.A.17 p.324

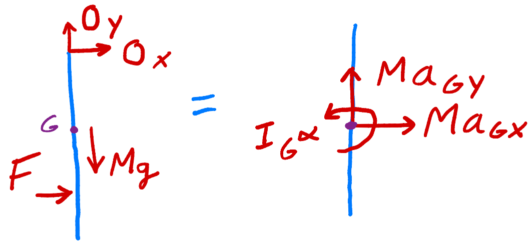
Given: A ball of mass m strikes a homogeneous bar of length L and mass M at a distance h from the pivot point O. R.F.R.

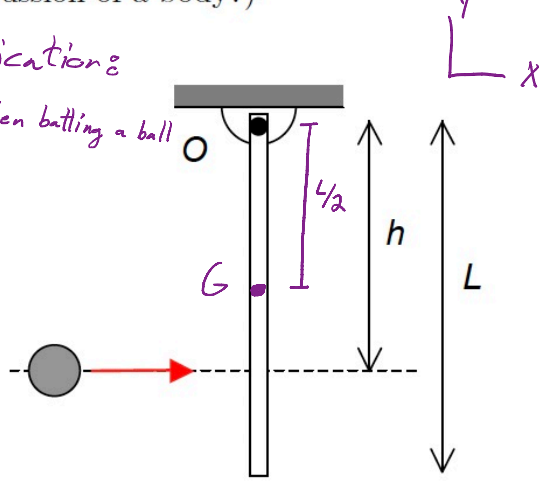
Find: (a) Determine the distance h such that the horizontal reaction force at O on the bar is zero. $h = ?$ (This location on the bar is known as the body's "center of percussion". (b) Can you think of a practical application of knowing the location of the center of percussion of a body?)

(0) N/E

(b) ↳ baseball application:
↳ "sweet spot" when batting a ball

(1) FBD





Kinetics

(2) Newton - x

$\Sigma F_x = Ma_{Gx}$

$\Rightarrow O_x + F = Ma_{Gx}$ (1)

(3) Newton - y

$\Sigma F_y = Ma_{Gy}$

$\Rightarrow O_y - Mg = Ma_{Gy}$ (2)

(4) Euler - @ O. fixed pt

P.A.T. : $I_o = I_G + md_{OG}^2$

$\Sigma M_O = I_O\alpha$

$\Rightarrow Fh = (I_G + M(\frac{L}{2})^2)\alpha$

$= \frac{1}{3}ML^2\alpha$ (3)

(4.5) 5 unkns 3 eqns

Kinematics

(5) GO

$\vec{a}_G = \cancel{\vec{a}_O}^{\text{fixed}} + \vec{\alpha} \times \vec{r}_{G/O} - \cancel{\omega^2 \vec{r}_{G/O}}^{\vec{0}}$ R.F.R.

$\Rightarrow a_{Gx}\hat{i} + a_{Gy}\hat{j} = \vec{0} + \alpha\hat{k} \times -\frac{L}{2}\hat{j} - \vec{0}$

$= \frac{\alpha L}{2}\hat{i}$

$\hat{i}: a_{Gx} = \frac{\alpha L}{2}$ (4)

$\hat{j}: a_{Gy} = 0$ (5)

(5.5) 5 eqns 5 unkns

Solve

(6) (1) & (4)

(1) $\Rightarrow O_x + F = M\frac{\alpha L}{2}$

(7) (2) & (5)

(2) $\Rightarrow O_y - Mg = 0$

(8) (3) solve for α

(3) $\Rightarrow Fh = \frac{1}{3}ML^2\alpha$

$\Rightarrow \alpha = \frac{3Fh}{ML^2}$

(9) (1) & use α. Enforce $O_x = 0$ ($h = ?$ @ no reactions)

(1) $\Rightarrow \cancel{O_x}^{0} + \cancel{F} = \frac{\cancel{M}\cancel{L}}{2}\left(\frac{3\cancel{F}h}{\cancel{M}L^{\cancel{2}}}\right)$

$\Rightarrow 1 = \frac{3h}{2L}$

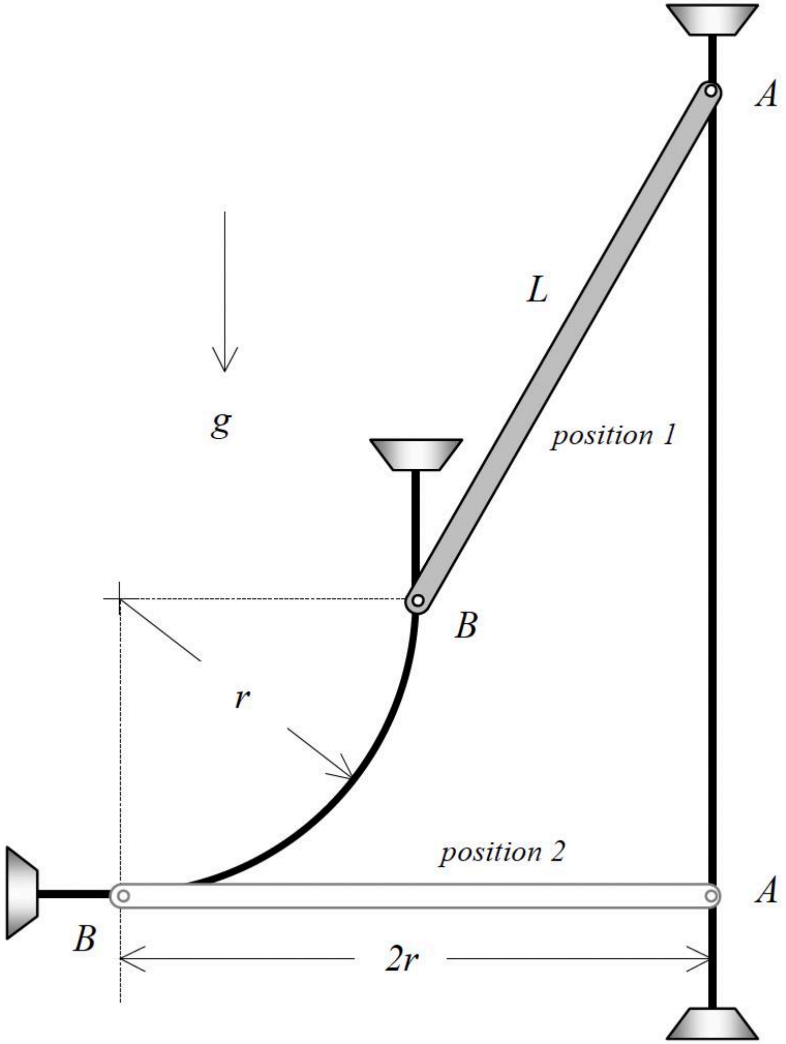
$\Rightarrow h = \frac{2}{3}L$

For the bar, this is location of center of percussion. Stike it 2/3 from fixed pt/center of rotation, you'll get 0 net reaction in x-dir.

Example 5.B.9

Given: Ends A and B of the thin, homogeneous bar AB (having a mass of m and a length of $L = 2r$) are constrained as follows: A moves along a smooth vertical guide and B moves along a smooth semi-circular guide of radius r. The bar is released from rest at position 1.

Find: Find the speed of end A when the bar is horizontal (i.e. is at position 2).



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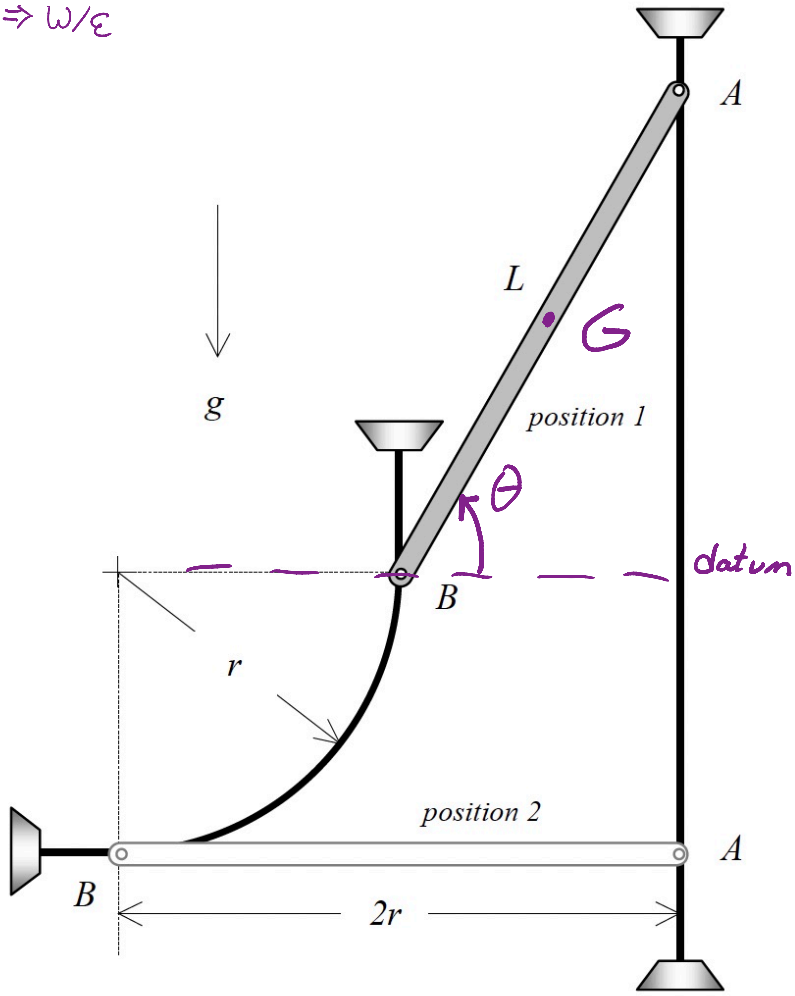
Example 5.B.9 p.338 hw w/ a trebuchet ⇒ W/E problem

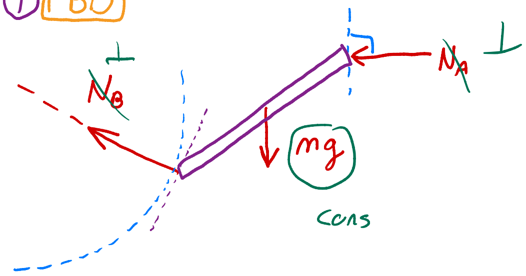
Given: Ends A and B of the thin, homogeneous bar AB (having a mass of m and a length of $L = 2r$) are constrained as follows: A moves along a smooth vertical guide and B moves along a smooth semi-circular guide of radius r. The bar is released from rest at position 1.

Find: Find the speed of end A when the bar is horizontal (i.e. is at position 2). v_A ?

(0) Δ in speed for Δ in position ⇒ W/E



(1) FBD



(2) Kinetics W/E

$$\cancel{T_1}^{0} + V_1 + \cancel{U^{(nc)}_{1\to2}}^{0} = T_2 + V_2$$

$T_1 = 0$; RFR

$V_1 = mg\frac{L}{2}\sin\theta$; datum

$U^{(nc)}_{1\to2} = 0$; mech. e. is conserved

$T_2 = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$

$V_2 = -mgr$; datum

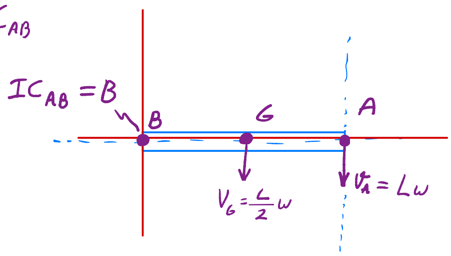
$$\Rightarrow mg\frac{L}{2}\sin\theta = \frac{1}{2}m v_G^2 + \frac{1}{2}I_G\omega^2 - mgr \quad (1)$$

(3) 1 eqn 2 unkns

(4) Kinematics $v_B = 0 \therefore B = IC_{AB}$

$\Rightarrow v_G = \frac{L}{2}\omega \quad (2)$

$\Rightarrow v_A = L\omega \quad (3)$



(5) 3 eqns 3 unkns

(6) Solve

$$mg\frac{L}{2}\sin\theta = \frac{1}{2}m\left(\frac{L}{2}\omega\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega^2 - mgr$$

↳ $\omega =$ ___

↳ $v_A = \frac{L}{2}\omega$ ___

(7) Alternate approach w/ B instead of G

since $B = IC$:

$\Rightarrow T_2 = \frac{1}{2}I_B\omega^2$

w/ $I_B = I_G + m\left(\frac{L}{2}\right)^2 = \frac{1}{3}mL^2$

$\Rightarrow mg\left(r + \frac{L}{2}\sin\theta\right) = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega^2$

↳ $\omega =$ ___

Summary: Particle and Planar Rigid Body Kinetics

WHICH TOOL(s) TO USE?
Put effort up front deciding on which method(s) to use: *Newton-Newton/Euler, work/energy, linear impulse momentum* or *angular impulse momentum*. Use the Kinetics Table in Section 5.D of the lecture book as a guide.

PARTICLE or RIGID BODY?
How is a particle distinguished from a rigid body? For a *particle*, we have:

$$\sum \vec{M}_G = I_G \vec{\alpha} = \vec{0} \quad \text{(EITHER } I_G = 0 \text{ OR } \vec{\alpha} = \vec{0}\text{)}$$

THE FOUR-STEP PLAN: Follow it…it is your friend!

Kinetics Table

Method	Body model	Fundamental equations
Newton-Euler *(relating forces to accelerations)*	**particle**	$\sum \vec{F} = m\vec{a}$
	rigid body *(G = c.m. and A = any point on body)*	$\sum \vec{F} = m\vec{a}_G$ $\sum \vec{M}_A = I_A \vec{\alpha} + m\vec{r}_{G/A} \times \vec{a}_A$
Work-energy *(relating change in speed to change in position)*	**particle**	$T_1 + V_1 + U_{1\to 2}^{(nc)} = T_2 + V_2$ where $T = \frac{1}{2}mv^2$
	rigid body *(G = c.m. and A = any point on body)*	$T_1 + V_1 + U_{1\to 2}^{(nc)} = T_2 + V_2$ where $T = \frac{1}{2}mv_A^2 + \frac{1}{2}I_A\omega^2 + m\vec{v}_A \cdot (\vec{\omega} \times \vec{r}_{G/A})$
Linear impulse-momentum *(relating change in velocity to change in time)*	**particle**	$\int_{t_1}^{t_2} \sum \vec{F}\, dt = m\vec{v}_2 - m\vec{v}_1$
	rigid body *(G = c.m.)*	$\int_{t_1}^{t_2} \sum \vec{F}\, dt = m\vec{v}_{G2} - m\vec{v}_{G1}$
Angular impulse-momentum *(relating change in angular velocity to change in time)*	**particle** *(O = fixed point)*	$\int_{t_1}^{t_2} \sum \vec{M}_O\, dt = \vec{H}_{O2} - \vec{H}_{O1}$ where $\vec{H}_O = m\vec{r}_{P/O} \times \vec{v}_P$
	rigid body *(A = fixed point or c.m.)*	$\int_{t_1}^{t_2} \sum \vec{M}_A\, dt = \vec{H}_{A2} - \vec{H}_{A1}$ where $\vec{H}_A = I_A \vec{\omega}$

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Lec 34 Short Feedback Form: