

# ***ME 274: Basic Mechanics II***

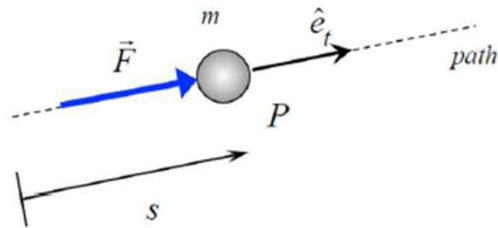
Lecture 19: Particle Kinetics – Work-Energy Equation



School of Mechanical Engineering

## Kinetics for the rectilinear Motion of Particles – Net force dependent on position

For a particle of mass  $m$  traveling along a straight path and experiencing a force  $\vec{F} = F\hat{e}_t$



$$\sum F_t = ma_t \Rightarrow F = m \frac{dv}{dt} \quad \leftarrow \text{need to write in terms of } s$$

If force is a function of position:  $F = F(s)$

$$F = m \frac{dv}{dt} \rightarrow \text{using chain rule} \\ = m \frac{dv}{ds} \left( \frac{ds}{dt} \right) \quad \leftarrow \text{velocity!}$$

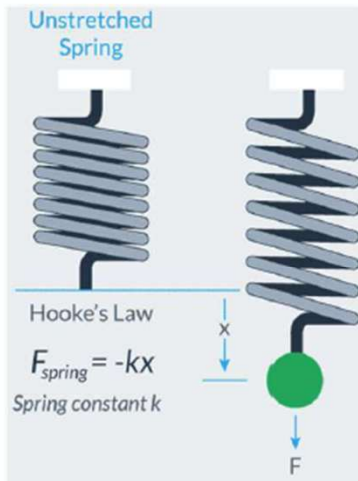
$$F(s) = m v \frac{dv}{ds}$$

$$\int_{s_1}^{s_2} F(s) ds = \int_{v_1}^{v_2} m v dv$$

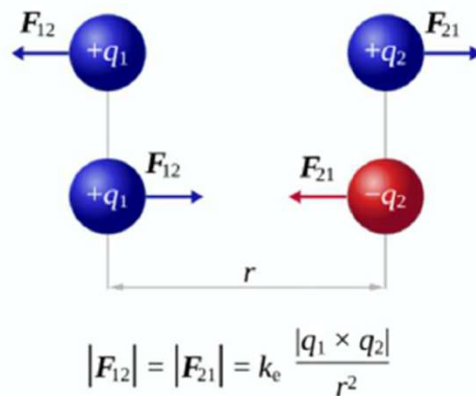
$$\int_{s_1}^{s_2} F(s) ds = \frac{m v^2}{2} \bigg|_{v_1}^{v_2} = \frac{1}{2} m (v_2^2 - v_1^2)$$

Work energy

Ex: spring forces



Ex: magnetic forces



# Newton's law for the path description of motion

$\vec{R}$   $\rightarrow$  resultant force acting on the particle as it moves along the path

$$\vec{R} = \sum \vec{F} = m \vec{a}$$

Remember in the path description  $\vec{a} = \underline{\dot{v} \hat{e}_t} + \underline{\frac{v^2}{\rho} \hat{e}_n}$

Breaking  $\vec{R}$  into components:

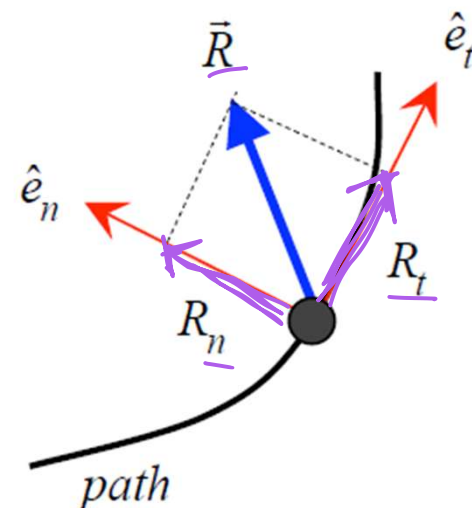
$$\underline{R_t} = m \dot{v} = m \frac{dv}{dt}$$

$$\underline{R_n} = m \frac{v^2}{\rho}$$

What is each component responsible for?

$R_t \rightarrow$  changing speed along path

$R_n \rightarrow$  turning



If we are interested in studying the **change in speed** of a particle, we need only deal with the **tangential components** of the forces acting on the particle  $V(s)$

$$\sum \vec{F} = m\vec{a}$$

projection  $\rightarrow \vec{R} \cdot \hat{e}_t = m\vec{a} \cdot \hat{e}_t$   $\swarrow \vec{a}$  path description

$$(\underline{R_t \hat{e}_t} + \underline{R_n \hat{e}_n}) \cdot \hat{e}_t = m \left( \frac{dv}{dt} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n \right) \cdot \hat{e}_t$$

$$R_t \hat{e}_t \cdot \hat{e}_t + R_n \hat{e}_n \cdot \hat{e}_t = m \frac{dv}{dt} \hat{e}_t \cdot \hat{e}_t + m \frac{v^2}{\rho} \hat{e}_n \cdot \hat{e}_t$$

$$R_t(1) + \cancel{R_n(0)} = m \frac{dv}{dt}(1) + \cancel{\frac{v^2}{\rho}(0)}$$

$$R_t = m \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v \frac{dv}{ds}$$

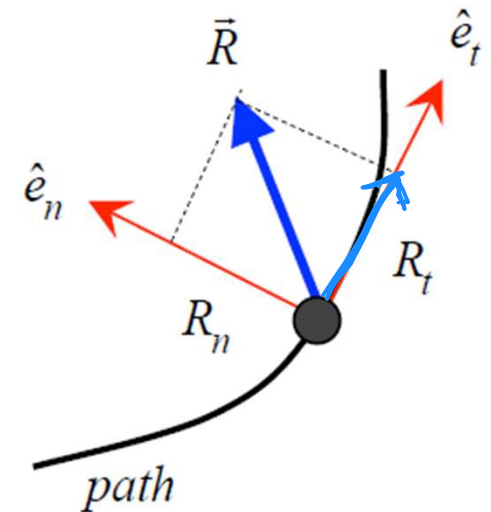
$$R_t = mv \frac{dv}{ds}$$

$$\int_1^2 R_t ds = m \int_1^2 v dv = \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2$$

$$\underline{U_{1 \rightarrow 2} = T_2 - T_1}$$

Work done by external forces = Change in kinetic energy

Kinetic energy  $\rightarrow T = \frac{1}{2} mv^2$



$$U_{1 \rightarrow 2} = \int_1^2 R_t ds$$



## What if our motion/forces are described with the cartesian description?

From our work-energy equation:

$$\underline{U_{1 \rightarrow 2}} = \int_1^2 (\vec{R} \cdot \hat{e}_t) ds$$

To convert to cartesian form, we need to relate  $\hat{e}_t$  to  $\hat{i}, \hat{j}$ , and  $s$  to  $x, y$

$$U_{1 \rightarrow 2} = \int_1^2 (\vec{R} \cdot \underline{\hat{e}_t}) ds$$

$$= \int_1^2 (R_x \hat{i} + R_y \hat{j}) \cdot (\underline{dx \hat{i} + dy \hat{j}})$$

$$= \int_1^2 R_x dx + \int_1^2 R_y dy$$

$$y = f(x)$$

path coords

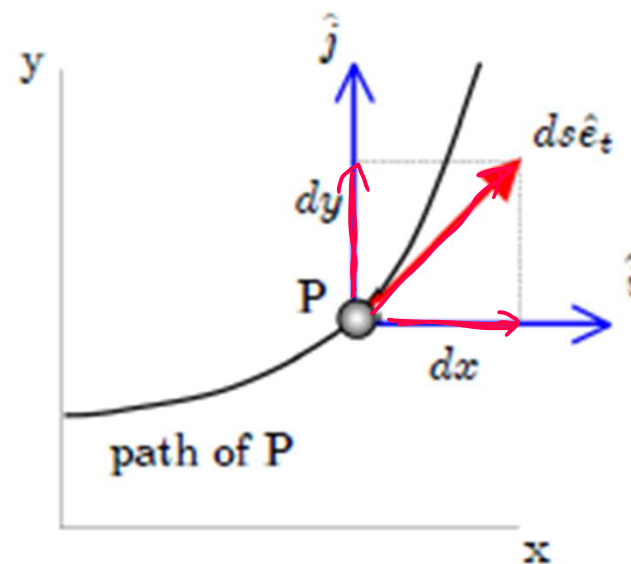
$$\vec{v} = v \hat{e}_t = \frac{ds}{dt} \hat{e}_t$$

cartesian

$$\vec{v} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j}$$

$$\frac{ds}{dt} \hat{e}_t = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j}$$

$$\underline{ds \hat{e}_t} = \underline{dx \hat{i} + dy \hat{j}}$$



# Conservative forces and Potential Energy

In general, the work done by a force  $\vec{F}$  is dependent on the shape/distance of the path traveled.

**Conservative forces** are a class of forces where the work is **path independent** and instead only depend on the **start and end positions**.

**Examples:** spring force, weight

You do not need to integrate conservative forces – instead, take the difference of their potential energy functions between points 1 and 2!

We can rewrite our work energy expression in terms of conservative and non-conservative components:

$$\underline{U_{1 \rightarrow 2}} = \underline{U_{1 \rightarrow 2}^{(c)}} + \underline{U_{1 \rightarrow 2}^{(nc)}} = -(\underline{V_2} - \underline{V_1}) + U_{1 \rightarrow 2}^{(nc)} \rightarrow \underline{T_2 + V_2 = T_1 + V_1 + U_{1 \rightarrow 2}^{(nc)}} \text{ Work energy}$$

## Spring potential energy

$$V_{sp} = \frac{k}{2} (L - L_0)^2$$

- $L$  – current length of spring
- $L_0$  – unstretched spring length
- $k$  – spring constant

Note:  $V_{sp}$  is ALWAYS  $\geq 0$

## Gravitational potential energy

$$V_{gr} = mgh$$

- $h$  – height above/below datum

Note: The sign of  $V_{gr}$  matches the sign of  $h$ ,  
i.e. if the ending position is **above** the  
datum  $V_{gr} > 0$ , If the ending position is  
**below** the datum  $V_{gr} < 0$

potential energy  $V$

## Example - work and potential energy of a spring force

Spring force:  $F_{sp} = k(L - L_0)$

polar form  $\vec{F}_{sp} = -k(L - L_0)\hat{e}_r$

Work done by spring force:

$$U_{1 \rightarrow 2} = \int_1^2 (\vec{F}_{sp} \cdot \hat{e}_t) ds$$

Convert between path and polar descriptions:

polar :  $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

path :  $\vec{v} = \frac{ds}{dt}\hat{e}_t$

$$r = L$$

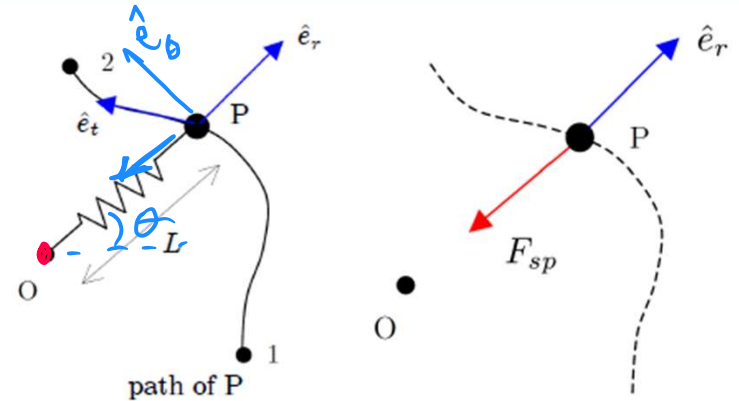
$$\dot{r} = \frac{dL}{dt}$$

$$\hat{e}_t ds = dL\hat{e}_r + Ld\theta\hat{e}_\theta$$

$$U_{1 \rightarrow 2} = \int_1^2 [-k(L - L_0)\hat{e}_r \cdot (dL\hat{e}_r + Ld\theta\hat{e}_\theta)] = -k \int_1^2 (L - L_0) dL$$

$$= -\frac{k}{2} [(L_2 - L_0)^2 - (L_1 - L_0)^2]$$

$$V_{sp} = \frac{1}{2} k (\Delta L)^2 = -(\hat{V}_2 - \hat{V}_1)$$



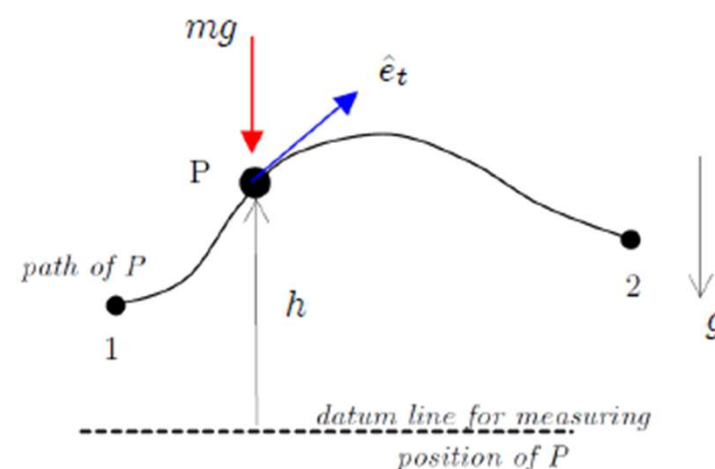
## Example - work and potential energy of a weight force

Weight force:  $\vec{F}_w = -mg\hat{j}$

Work done by weight force:  $U_{1 \rightarrow 2} = \int_1^2 (\vec{F}_w \cdot \hat{e}_t) ds$

Convert between path and cartesian descriptions:

$$\begin{aligned} U_{1 \rightarrow 2} &= \int_1^2 (-mg\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) \\ &= -mg \int_1^2 dy = -mg(h_2 - h_1) \\ &= -(\bar{V}_2 - \bar{V}_1) \end{aligned}$$



### Example 4.B.1

**Given:** The 6 kg particle P and the attached light rod (of length  $L = 2$  m) rotate in a vertical plane about a fixed axis passing through O. The assembly is released from rest at  $\theta = 0$  and moves under the action of the  $F = 100$  N force (which is maintained normal to the rod).

**Find:** Determine the speed of the particle P when  $\theta = 90^\circ$ .

speed after position change  $\rightarrow$  work energy

2) Kinetic

$$T_1 + V_1 + U_{1 \rightarrow 2} = T_2 + V_2$$

$$T_1 = 0 \leftarrow \text{(RFR)}$$

$$V_1 = 0 \leftarrow \text{at datum}$$

$$U_{1 \rightarrow 2} = \int_1^2 \vec{F} \cdot d\vec{r} = F(0.4L)\left(\frac{\pi}{2}\right)$$

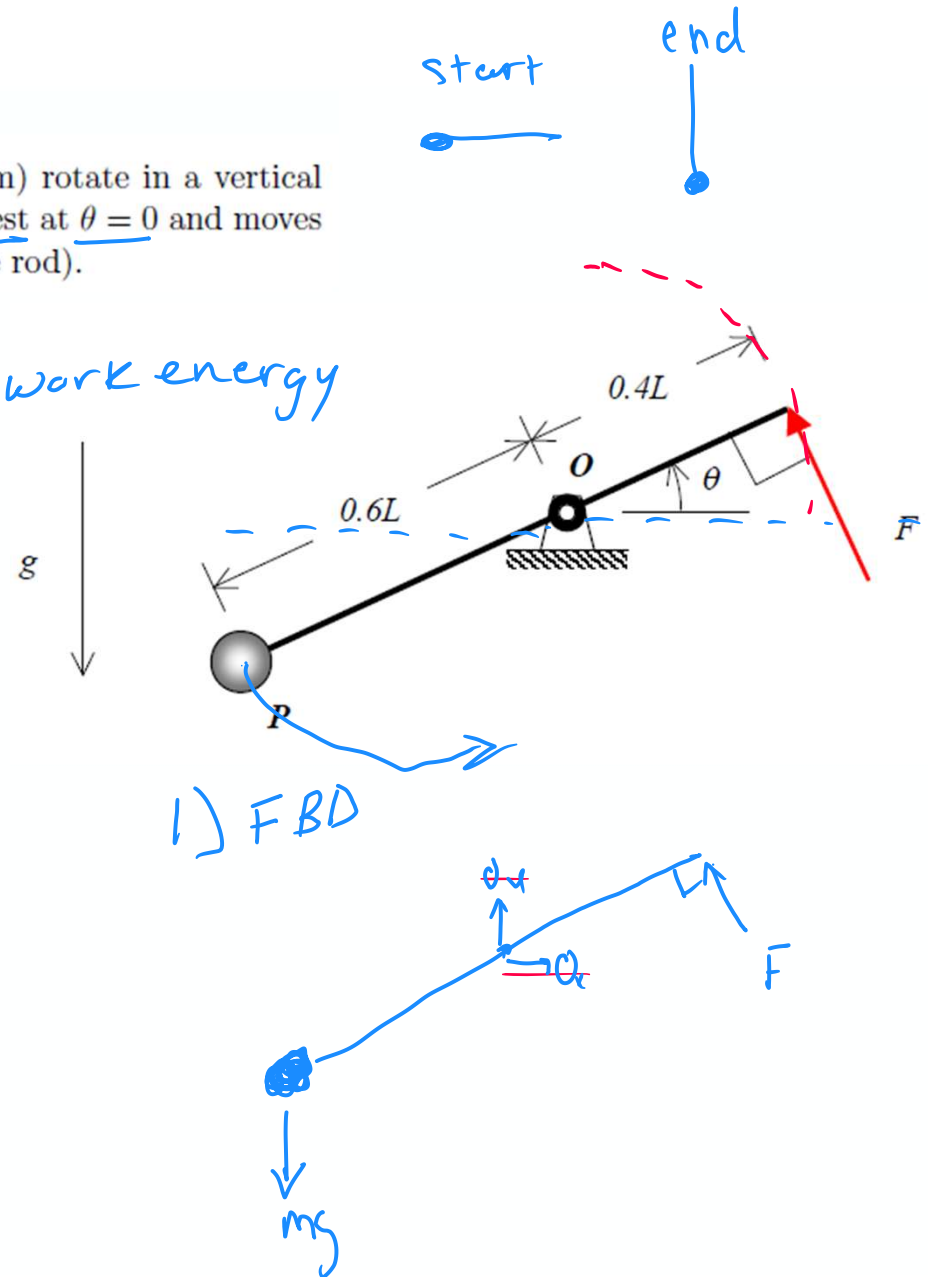
$$T_2 = \frac{1}{2} m v_2^2$$

$$V_2 = -mg(0.6L)$$

Solving:

$$0 + 0 + 2\pi L F = \frac{1}{2} m v_2^2 - mg(0.6L)$$

$\rightarrow$  solve  $v_2$



### Example 4.B.2

**Given:** A block of mass  $m$  is released from rest on a rough, inclined surface (with a coefficient of kinetic friction  $\mu_k$ ) and slides toward an uncompressed spring of stiffness  $k$ .

**Find:** Determine the maximum deflection of the spring.

$$2) T_1 + V_1 + U_{1 \rightarrow 2}^{nc} = T_2 + V_2$$

$$T_1 = 0 \text{ (RFR)}$$

$$V_1 = 0 \text{ at datum}$$

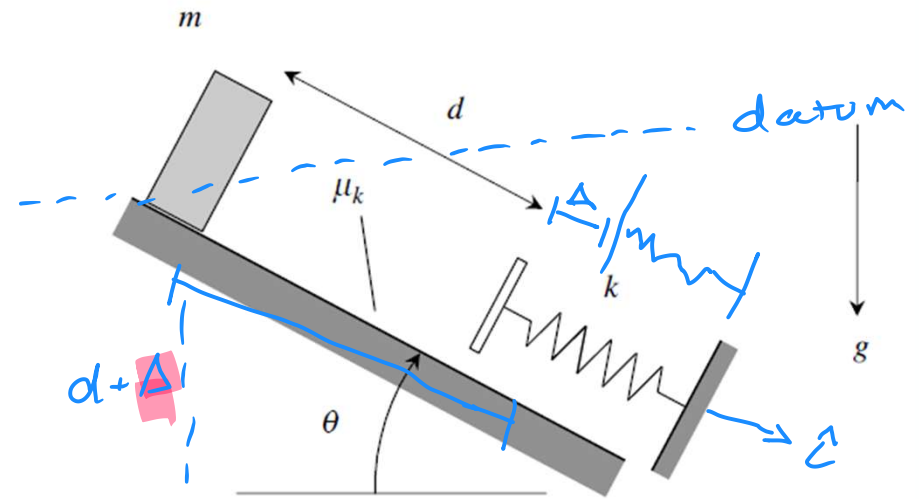
$$U_{1 \rightarrow 2}^{nc} = \int_0^{\delta} \vec{F} \cdot d\vec{r} = -f \hat{i} \cdot (d + \Delta) \hat{i} \\ = -f(d + \Delta)$$

$$T_2 = 0 \leftarrow \text{at max deflection, velocity is instantaneously } 0$$

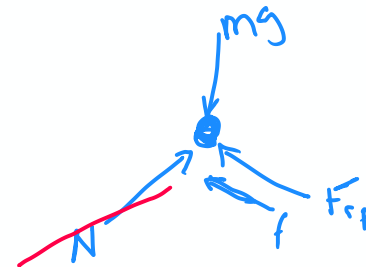
$$V_2 = -mg(d + \Delta)\sin\theta + \frac{1}{2}k\Delta^2$$

$$0 + 0 - f(d + \Delta) = 0 - mg(d + \Delta)\sin\theta + \frac{1}{2}k\Delta^2$$

Solve for  $\Delta$



1) FBD



# ***ME 274: Basic Mechanics II***

Lecture 20: Particle Kinetics – Work-Energy



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# Overview: Work-Energy Equation

Q: When do we know to use the work-energy equation in a kinematic analysis?

A: - relating speed to a change in position  
 - know energy is conserved

Components of the work-energy equation:  $T_1 + V_1 + U_{1 \rightarrow 2}^{(nc)} = T_2 + V_2$

$\underbrace{\hspace{10em}}_{\text{initial position}} \quad \underbrace{\hspace{10em}}_{\text{final position}}$

work to get from 1 to 2

$T = \frac{1}{2}mv^2$  = particle kinetic energy

$V = V_{sp} + V_{gr}$  = potential energy resulting from conservative (path independent) forces

- $V_{sp} = \frac{1}{2}k(L - L_0)^2 \rightarrow V_{sp} > 0$
- $V_{gr} = mgh \rightarrow V_{gr} > 0$  or  $< 0$ , depending on datum

$U_{1 \rightarrow 2}^{(nc)} = \int_1^2 (\vec{R}_t) ds$  = work done by  $\vec{R}$  from 1 to 2  $R \Rightarrow$  resultant force

- Path description:  $U_{1 \rightarrow 2} = \int_1^2 (\vec{R} \cdot \underline{\hat{e}_t}) ds$
- Cartesian description:  $U_{1 \rightarrow 2} = \int_1^2 (R_x \hat{i} + R_y \hat{j}) \cdot (\underline{dx \hat{i} + dy \hat{j}})$
- Polar description:  $U_{1 \rightarrow 2} = \int_1^2 (R_r \hat{e}_r + R_\theta \hat{e}_\theta) \cdot (\underline{dr \hat{e}_r + r d\theta \hat{e}_\theta})$

non-conservative  $\rightarrow$  path dependent



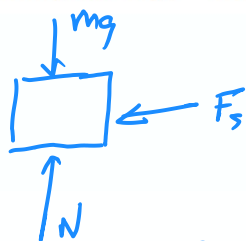
### Example 4.B.4

**Given:** Block C having a weight  $mg$  is traveling with a speed of  $v_0$  when it strikes a fixed wall. A spring of stiffness  $k$  is attached to the front of the block. The spring eventually contacts the wall, resulting in a maximum spring compression of  $\Delta_{max}$ .

**Find:** Determine the stiffness  $k$  of the spring.

Use the following parameters in your analysis:  $mg = 3200$  lb,  $v_0 = 10$  ft/sec and  $\Delta_{max} = 6$  in.

1) FBD



initial position:

$$T_1 = \frac{1}{2} m v_0^2 \quad V_1 = 0$$

$$T_1 + V_1 + U_{1 \rightarrow 2}^{nc} = T_2 + V_2$$

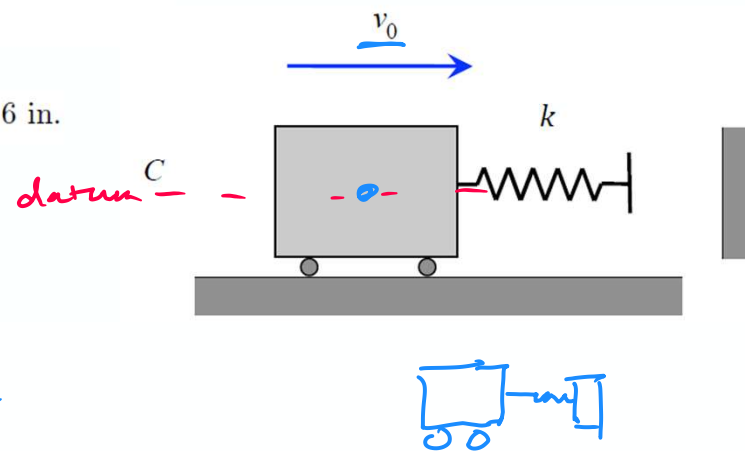
Non-conservative work:  $U_{1 \rightarrow 2}^{nc} = 0 \leftarrow \text{mechanical energy conserved}$

final position:

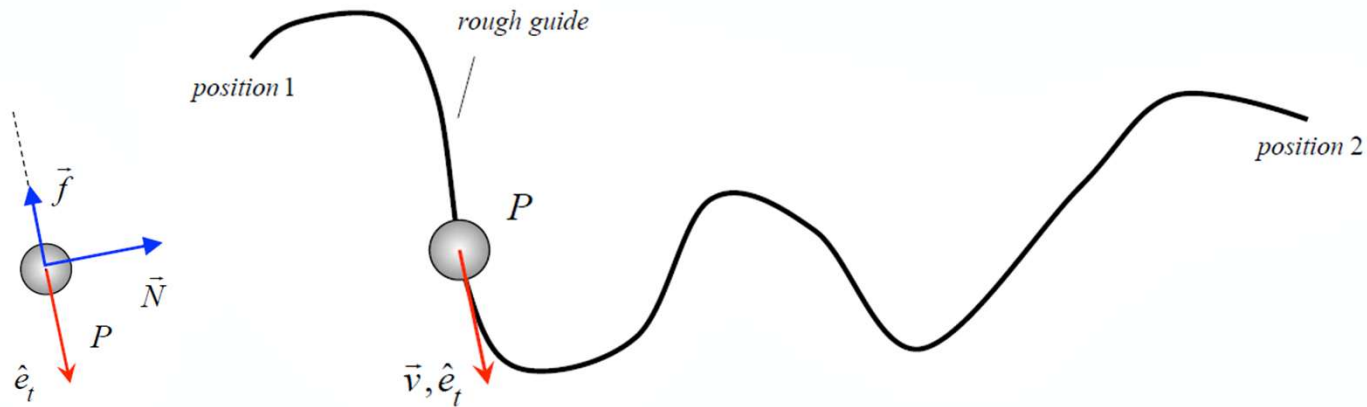
$$T_2 = 0 \quad V_2 = \frac{1}{2} k \Delta_{max}^2$$

Work-energy:

$$\frac{1}{2} m v_0^2 = \frac{1}{2} k \Delta_{max}^2 \quad \rightarrow \quad k = \frac{m v_0^2}{\Delta_{max}^2}$$



**CHALLENGE QUESTION:** A particle slides on a rough, arbitrarily-shaped guide. What can we say about the sign of the work done by the friction force acting on the particle as it moves from position 1 to position 2? Similarly, what is the work done by the normal force acting on the particle?

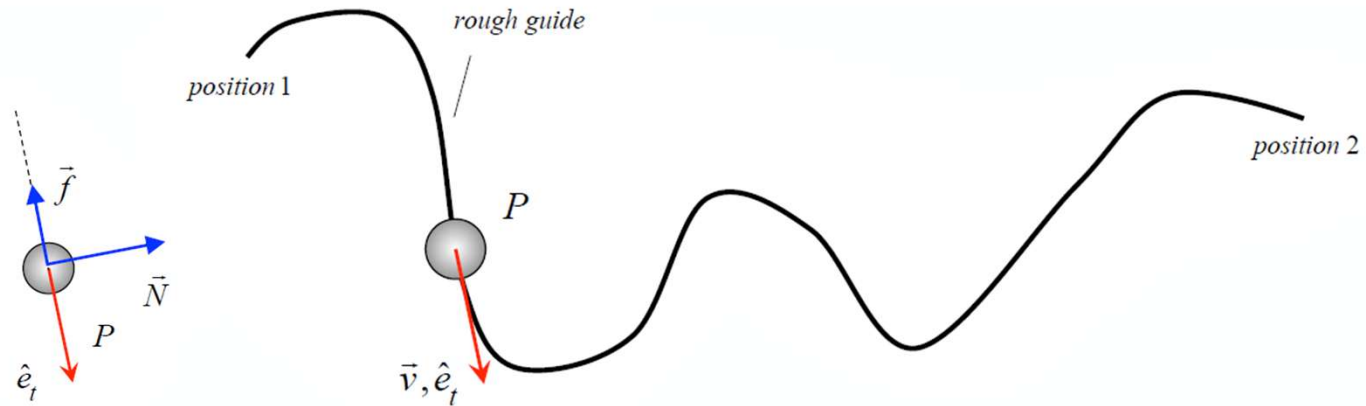


remember :  $U_{1 \rightarrow 2}^{Nc} = \int_1^2 (\vec{R} \cdot \hat{e}_t) ds$

$$\vec{f} = -f \hat{e}_t, \quad -\underbrace{\hat{e}_t \cdot \hat{e}_t}_1 = -f < 0$$

friction opposes motion  
 $\therefore$  opposite direction of  $\vec{v}$

**CHALLENGE QUESTION:** A particle slides on a rough, arbitrarily-shaped guide. What can we say about the sign of the work done by the friction force acting on the particle as it moves from position 1 to position 2? Similarly, what is the work done by the normal force acting on the particle?



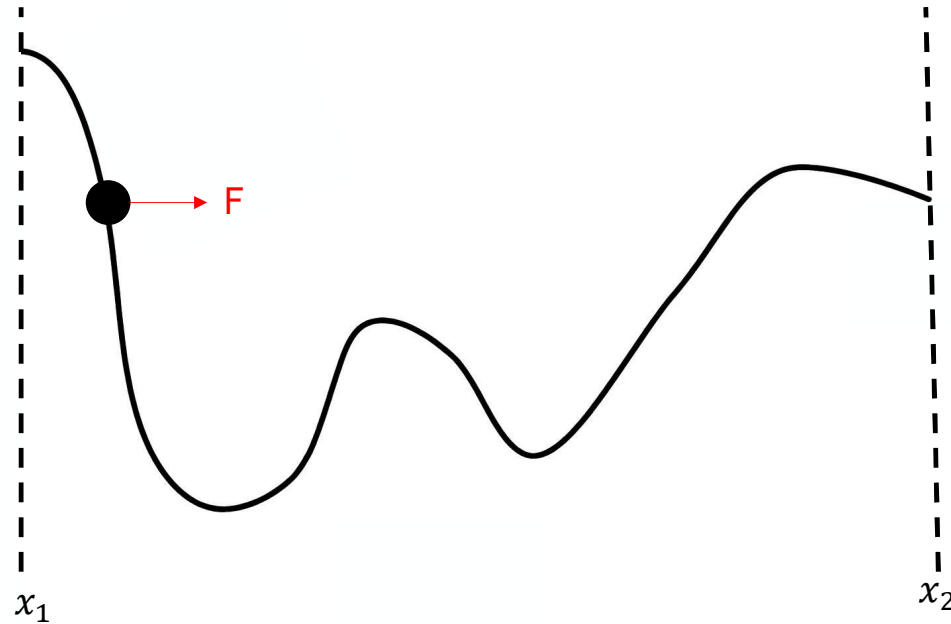
Only the tangential component of the resultant force  $\vec{R}$  contributes to the work done by  $\vec{F}$ :

$$U_{1 \rightarrow 2} = \int_1^2 (\vec{R} \cdot \hat{e}_t) ds$$

If a force is perpendicular to the path of the particle on which it acts, then the force does NO work on the particle.

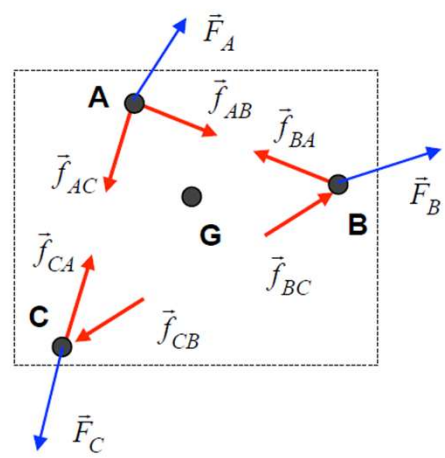
Normal force always perpendicular  $\rightarrow$  will never contribute to work

**Challenge question:** The particle is being pulled along a smooth guide by a force  $\vec{F} = F\hat{i}$ .  
 What is the magnitude of  $U_{1 \rightarrow 2}^{(nc)}$ ?

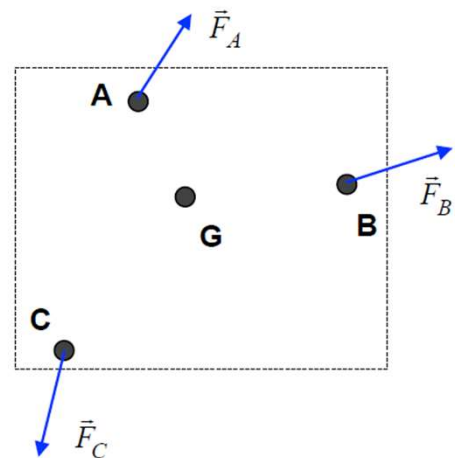


$$\begin{aligned}
 U_{1 \rightarrow 2}^{(nc)} &= \int_1^2 (\vec{F} \cdot \hat{e}_t) ds = \int_1^2 F \hat{i} \cdot (dx \hat{i} + dy \hat{j}) \\
 &= \int_{x_1}^{x_2} F dx = F(x_2 - x_1)
 \end{aligned}$$

# Work - Energy Equation for Systems of Particles



all forces acting on system



"external" forces acting on system

There will only be a single work-energy equation, regardless of the number of particles in the system!

*if you have more than 1 unknown, use kinematics*

$$T_1 + V_1 + U_{1\rightarrow 2}^{(nc)} = T_2 + V_2$$

Each term will be the sum for all particles in the system:

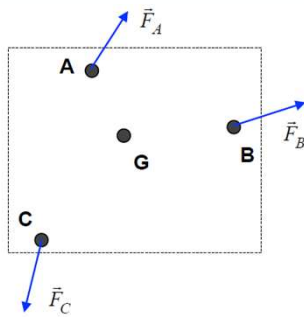
$$T = \sum_{j=1}^N T_j = \frac{1}{2} \sum_{j=1}^N m_j v_j^2$$

$$V = \sum_{j=1}^N V_j$$

$U_{1\rightarrow 2}^{(nc)}$  - Total nonconservative work done on the system

## Calculating $U_{1 \rightarrow 2}^{(nc)}$ for systems of particles

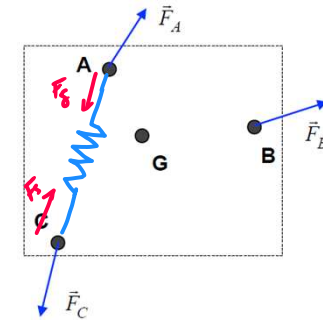
Work done by external forces



"external" forces acting on system

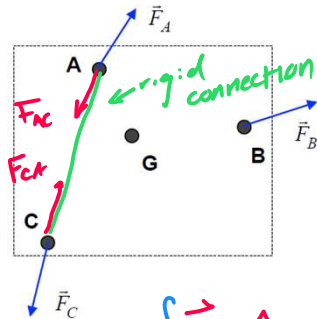
$$U_{1 \rightarrow 2}^{(nc)} = \int_1^2 ((\vec{F}_A + \vec{F}_B + \vec{F}_C) \cdot \hat{e}_t) ds$$

Work done by internal forces: conservative forces



include in  $V$  term, not in integral

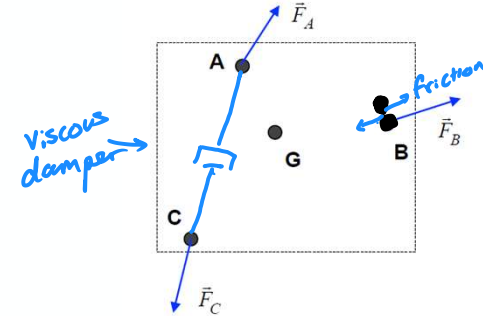
Work done by internal forces: rigid connections



$$\int \vec{F}_{AC} \cdot \hat{e}_i = - \int \vec{F}_{CA} \cdot \hat{e}_i$$

integrals cancel  $\rightarrow$  Net work = 0

Work done by internal forces: dissipative forces

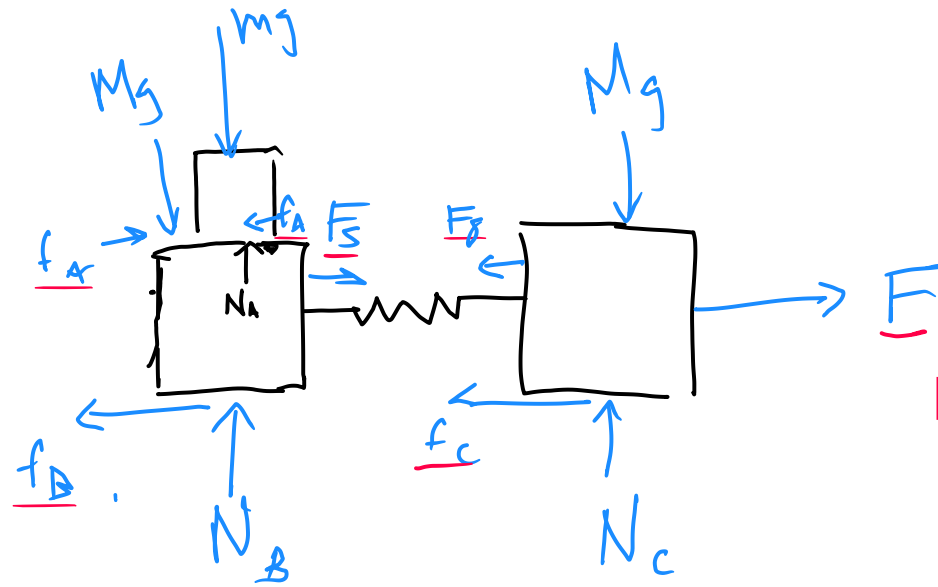
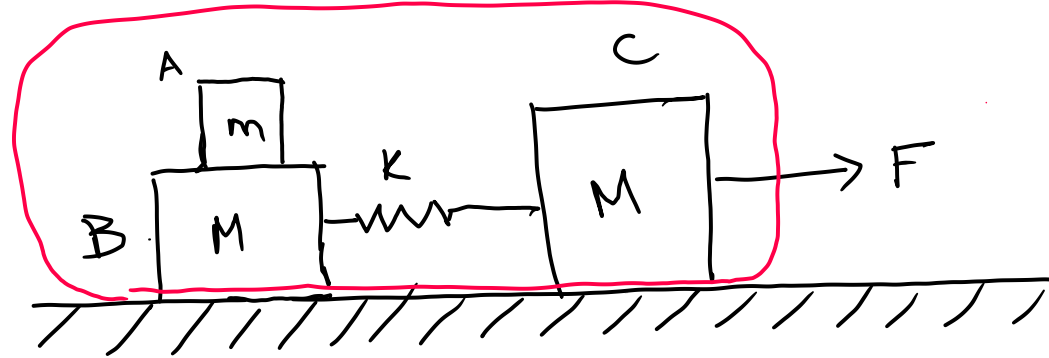


Energy leaves system  
(mechanical energy not conserved)  
 $\rightarrow$  must include in work integral  
(if possible)

**Rule of thumb:** make your system as large as possible to make workless forces internal. Make friction/dissipative forces external by your choice of system.

Example: two blocks connected by a spring pulled along a rough surface by force  $F$ , block A sliding on top of B

System : blocks A, B, C  
+ spring



How do we include each force in our Work-energy equation?

$F \rightarrow$  external  $\rightarrow$  in  $U_{1 \rightarrow 2}$

$F_s \rightarrow$  internal, conservative  $\rightarrow$  in  $\bar{V}$

$mg, Mg, N_A, N_B, N_C \rightarrow$  perpendicular to path  $\Rightarrow$  0 work

$f_A \rightarrow$  internal, dissipative  $\rightarrow$  in  $U_{1 \rightarrow 2}$

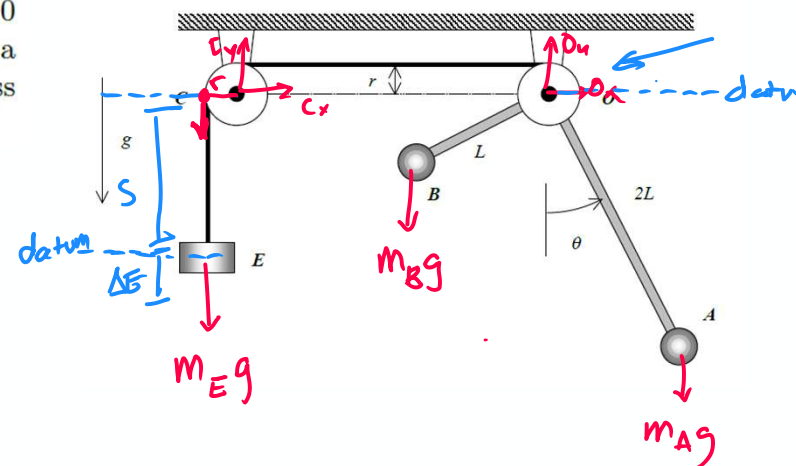
$f_B, f_C \rightarrow$  external dissipative  $\rightarrow$  in  $U_{1 \rightarrow 2}$

## Example 4.B.6

**Given:** The system shown below is made up of particles A, B and E (having masses of 5 kg, 20 kg and 80 kg, respectively). Lightweight bars OB and OA are welded together such that there is a right angle between the two bars, and these bars are welded to the pulley at O. Consider the mass of the pulleys to be negligible. The system is released from rest when  $\theta = 0$ .

**Find:** Determine the speed of particle E when  $\theta = 90^\circ$ .

step 1) FBD



1) Kinetics

$$T_1 + V_1 + U_{1 \rightarrow 2}^{nc} = T_2 + V_2$$

position 1

$$T_1 = 0 \text{ (RFR)}$$

$$V_1 = -2m_A g L$$

$$U_{1 \rightarrow 2}^{nc} = 0$$

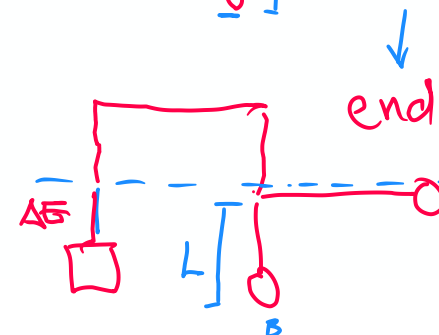
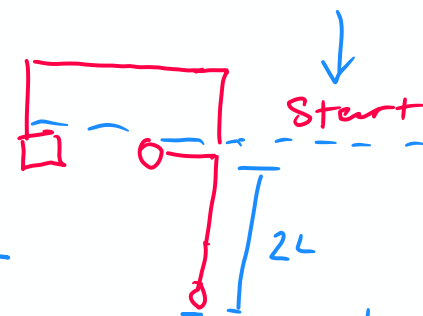
Work - Energy

$$-2m_A g L = \frac{1}{2} m_E V_E^2 + \frac{1}{2} m_B V_B^2 + \frac{1}{2} m_A V_A^2 - m_E g \Delta E - m_B g L$$

position 2

$$T_2 = \frac{1}{2} m_E V_E^2 + \frac{1}{2} m_B V_B^2 + \frac{1}{2} m_A V_A^2$$

$$V_2 = -m_E g \Delta E - m_B g L$$



3) Kinetics

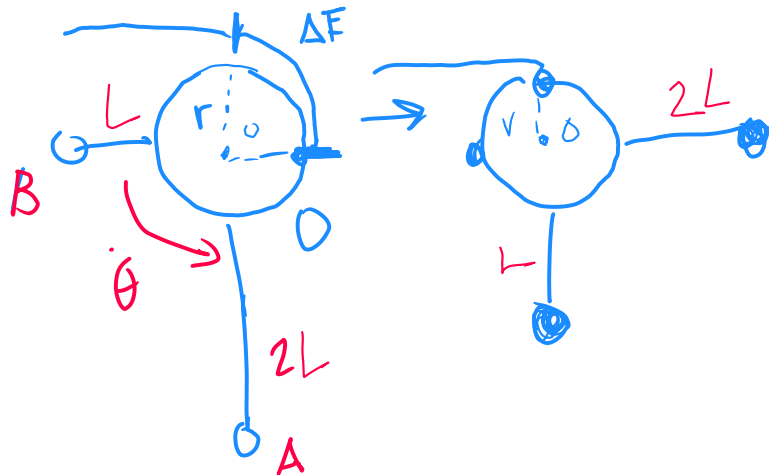


# Work - Energy

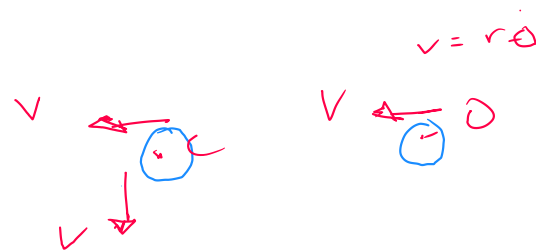
$$-2m_A g L = \frac{1}{2} m_E \underbrace{V_E^2}_{r\dot{\theta}} + \frac{1}{2} m_B \underbrace{V_B^2}_{L\dot{\theta}} + \frac{1}{2} m_A \underbrace{V_A^2}_{2L\dot{\theta}} - mg\Delta E - m_B g L$$

1 eqn, 4 unknowns  $\rightarrow$  use kinematics!

Solve for  $\Delta E \rightarrow$  the change in height of E is a result of the cable unspooling as pulley O rotates through  $\frac{1}{4}$  turn



$$\Delta E = \frac{\pi r}{2}$$



relate  $V_A, V_B, V_E \rightarrow$  pulleys O & C are connected at top by inextensible cable  $\rightarrow$  have the same velocity &  $\omega$  because of matching radii

$$\underline{V_B = L\dot{\theta}}$$

$$V_A = 2L\dot{\theta}$$

$$V_E = r\dot{\theta}$$

Substitute into Work-Energy eqn,

Solve for  $\dot{\theta}$ , then solve for  $V_E$

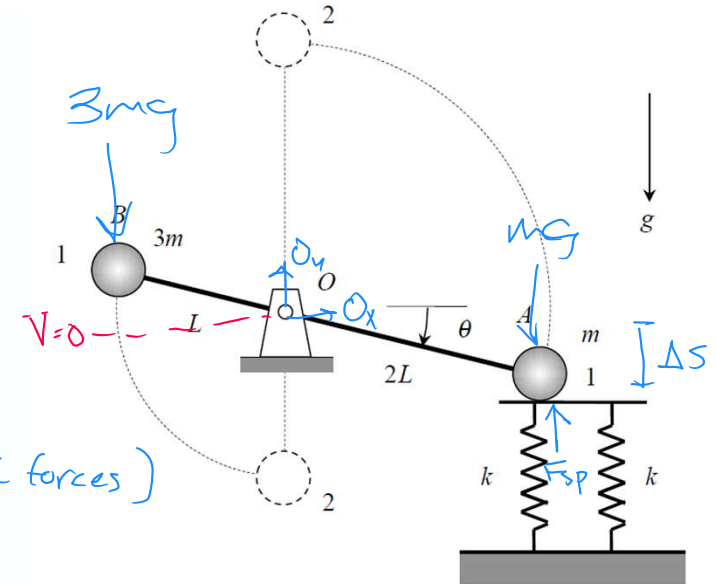
use  $|v| = r\omega$   
IC rule

### Example 4.B.7

**Given:** Particles A and B, having masses of  $m$  and  $2m$ , respectively, are connected by rigid bar AB, with AB having negligible mass. Bar AB is pinned to ground with a pin joint at O. This system is released from rest at position 1 with  $\theta = \theta_1$ , with A in contact with a pair of identical springs, as shown in the figure. Each spring has a stiffness of  $k$ , and the springs are unstretched when  $\theta = 0$ . Assume the dimensions of the particles to be negligible.

**Find:** Determine the speeds of particles A and B at position 2, where in position 2 particle A is directly above O.

1) FBD



2) Kinetics

$$T_1 + V_1 + U_{1 \rightarrow 2} = T_2 + V_2$$

$$T_1 = 0 \text{ (RFR)}$$

$$V_1 = \underbrace{3mgL \sin \theta_1}_{V_{gr}} - \underbrace{2mgL \sin \theta_1}_{V_{sp}} + \underbrace{2 \left( \frac{1}{2} k (2L \sin \theta_1)^2 \right)}_{V_{sp}}$$

$$U_{1 \rightarrow 2}^{nc} = 0 \text{ (mechanical energy conserved, no external NC forces)}$$

$$T_2 = \frac{1}{2} m v_A^2 + \frac{3}{2} m v_B^2$$

$$V_2 = 2mgL - 3mgL$$

$$\Rightarrow 3mgL \sin \theta_1 - 2mgL \sin \theta_1 + k(2L \sin \theta_1)^2 = \frac{1}{2} m v_A^2 + \frac{3}{2} m v_B^2 + 2mgL - 3mgL$$

1 eqn, 2 unknown  $\Rightarrow$  kinematics

$$\text{kinematics: } v_A = 2L\dot{\theta} \quad v_B = L\dot{\theta} \Rightarrow v_A = 2v_B$$

$\rightarrow$  Substitute into work energy & solve!

# ***ME 274: Basic Mechanics II***

Lecture 21: Particle Kinetics – Linear Impulse-Momentum



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## Kinetics: Four-Step Problem Solving Method

### 1) FBD

- Draw appropriate FBD (note- you may have to draw more than 1)
- Choose coordinate system (cartesian, path, polar)

### 2) Kinetics $\sum F = ma$

- Sum forces and break into components
- Choose appropriate method to solve the problem – we will learn these in upcoming sections!
  - Newton/Euler
  - Work/Energy  $\rightarrow v$  as it varies  $s$  &  $F(s)$
  - Linear impulse/Momentum  $\rightarrow v$  //  $t$  &  $F(t)$
  - Angular impulse/Momentum

### 3) Kinematics

- Perform a kinematic analysis of the system using techniques we have developed in the previous chapters.
- Use your kinetics equations from step 2 to determine what information you need to solve the problem

### 4) Solve

- Count the number of equations and unknowns. Do they match?
- If not:
  - Draw more FBDs
  - Do additional kinematic analysis

# Linear impulse momentum - Analyzing changes in velocity resulting in forces over an elapsed time

Remember our resultant force  $\vec{R}$ :

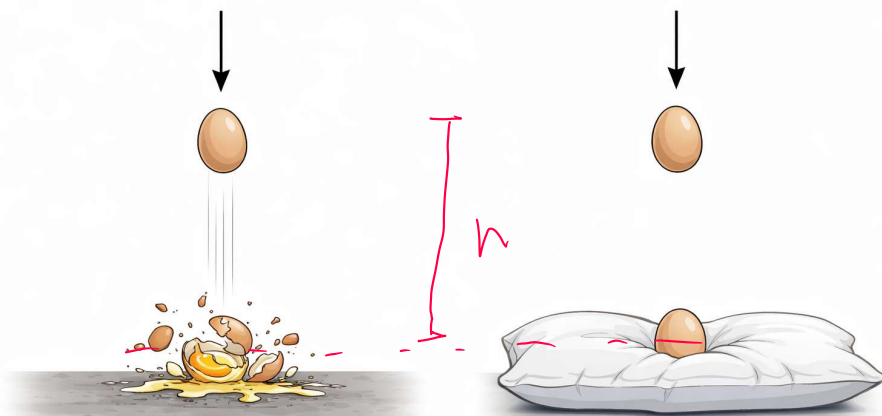
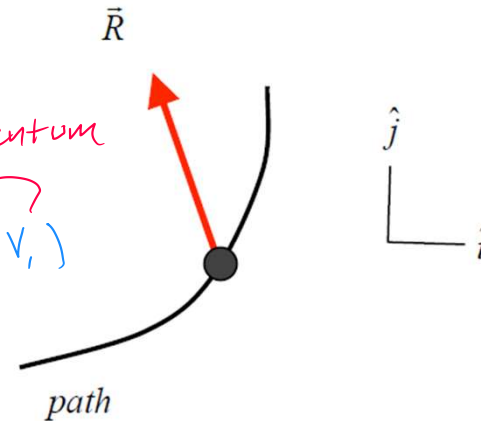
$$\vec{R} = \sum \vec{F} = m\vec{a} = m \frac{dv}{dt}$$

Separate & integrate:

$$\int_{t_1}^{t_2} \vec{R} dt = \int_{v_1}^{v_2} m dv \Rightarrow$$

$$\int_{t_1}^{t_2} \vec{R} dt = m(v_2 - v_1)$$

impulse                      momentum

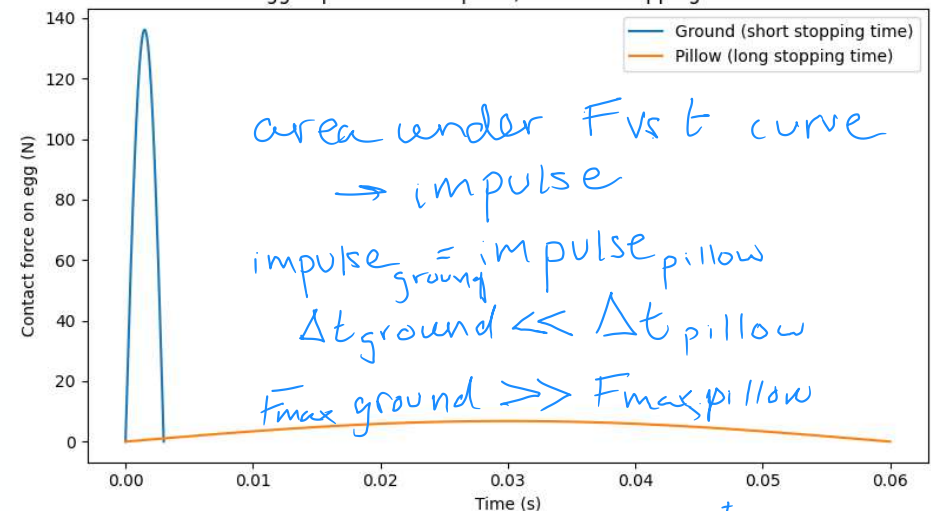


Ground

Pillow

$v_1 \rightarrow$  right before impact  $= \sqrt{2gh}$   
 (you can solve using W-E)  
 $v_2 = 0 \rightarrow$  egg fully stopped

Egg impact: same impulse, different stopping times



$\Delta p = m(v_2 - v_1) = \int_{t_1}^{t_2} \vec{R} dt$   
 change in momentum  
 is the same for both cases  
 what is different?  $\rightarrow \Delta t$

**Important considerations for Linear Impulse-Momentum:**  $\underline{m\vec{v}_2} = \underline{m\vec{v}_1} + \int_1^2 \underline{\vec{R} dt}$

$m\vec{v}$  - linear momentum of the particle

$\int_1^2 \vec{R} dt$  - impulse from the net force acting on the particle

- Linear impulse-momentum relates the change in linear momentum to the impulse acting on the particle.
- Note that **impulse is a function of time**, so this effectively gives the **change in particle velocity with respect to the time** a force is acting on that particle.
- Also note  $m\vec{v}_2 = m\vec{v}_1 + \int_1^2 \vec{R} dt$  is a **vector** equation (in contrast to work-energy)  $\rightarrow$  we can resolve it into components!

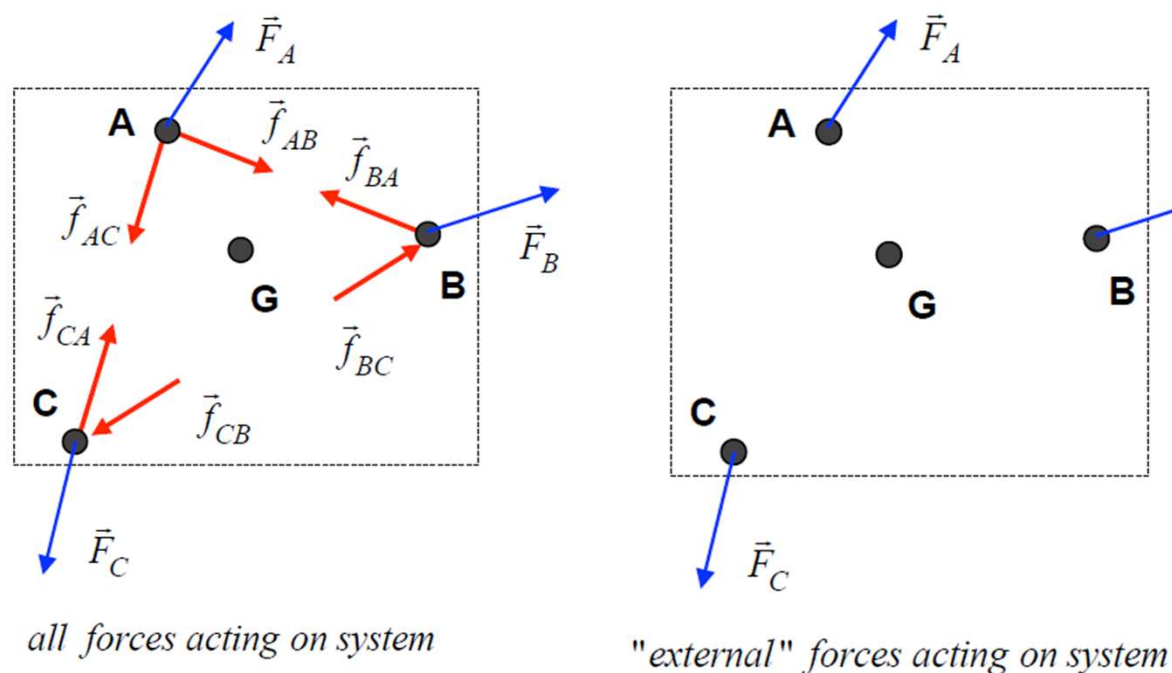
i.e. in the cartesian description:

*2 scalar eqn*

$$\begin{aligned} mv_{x2} &= mv_{x1} + \int_1^2 R_x dt \\ mv_{y2} &= mv_{y1} + \int_1^2 R_y dt \end{aligned}$$

- If there is no resultant force, the linear momentum is said to be conserved:  $mv_{x2} = mv_{x1}$
- Conservation of momentum does not imply conservation of energy (or vice versa)

# Linear Impulse-Momentum Equations for a System of Particles



Solving for impulse and momentum for each particle:

$$\begin{aligned} A \quad & \left( m_A \vec{v}_{A2} = m_A \vec{v}_{A1} + \int_1^2 [\vec{F}_A + \vec{f}_{AB} + \vec{f}_{AC}] dt \right) \\ B \quad & \left( m_B \vec{v}_{B2} = m_B \vec{v}_{B1} + \int_1^2 [\vec{F}_B + \vec{f}_{BA} + \vec{f}_{BC}] dt \right) \\ & = m_B \vec{v}_{B1} + \int_1^2 [\vec{F}_B - \vec{f}_{AB} + \vec{f}_{BC}] dt \\ C \quad & \left( m_C \vec{v}_{C2} = m_C \vec{v}_{C1} + \int_1^2 [\vec{F}_C + \vec{f}_{CA} + \vec{f}_{CB}] dt \right) \\ & = m_C \vec{v}_{C1} + \int_1^2 [\vec{F}_C - \vec{f}_{AC} - \vec{f}_{BC}] dt \end{aligned}$$

Summing for entire system:

$$m_A \vec{v}_{A2} + m_B \vec{v}_{B2} + m_C \vec{v}_{C2} = m_A \vec{v}_{A1} + m_B \vec{v}_{B1} + m_C \vec{v}_{C1} + \int_1^2 [\vec{F}_A + \vec{F}_B + \vec{F}_C] dt \Rightarrow \sum m_j \vec{v}_{j2} = \sum m_j \vec{v}_{j1} + \int_1^2 \left[ \sum \vec{F}_j^{ext} \right] dt$$

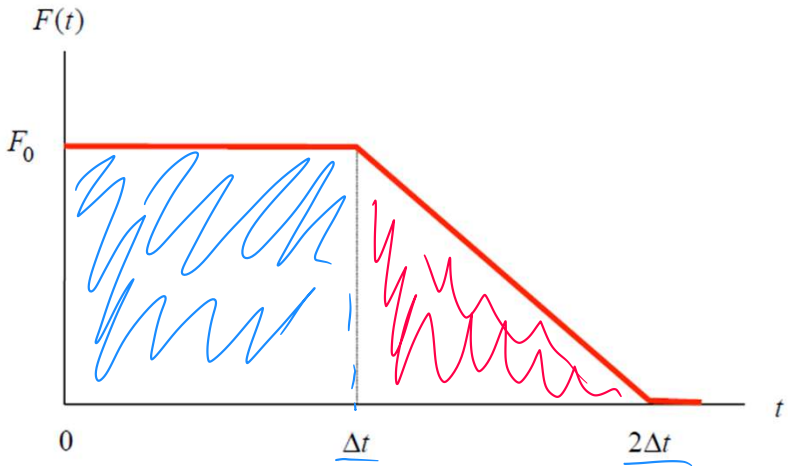
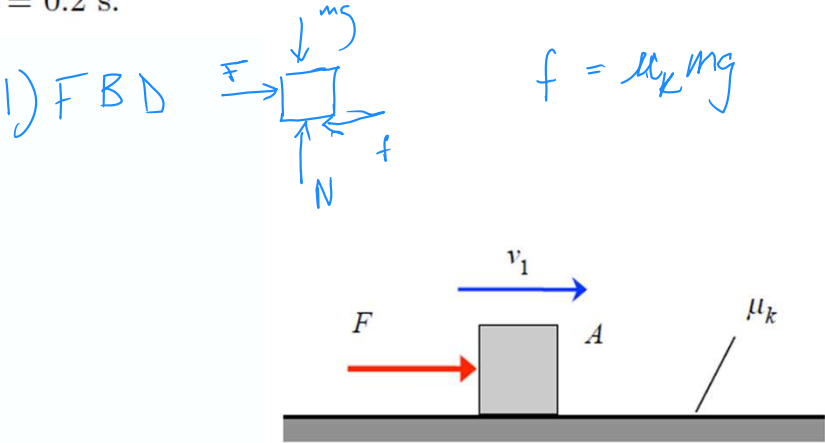
Key takeaway: make system as large as possible to include as many internal forces as you can!

# Example 4.C.1

**Find:** Determine:

- (a) The speed of the block at  $t = \Delta t$ ; and
- (b) The speed of the block at  $t = 2\Delta t$ .

Use the following parameters in your analysis:  $m = 8 \text{ kg}$ ,  $F_0 = 20 \text{ N}$ ,  $v_1 = 15 \text{ m/s}$ ,  $\mu_k = 0.15$  and  $\Delta t = 0.2 \text{ s}$ .



Kinetics  $\rightarrow$  L I M

$$mv_2 = mv_1 + \int_{t_1}^{t_2} \sum F_x dt$$

$$= mv_1 + \int_{t_1}^{t_2} (F(t) - \mu_k mg) dt$$

area under the curve

$$0 \rightarrow \Delta t$$

$$\int_0^{\Delta t} F_0 dt = F_0 \Delta t$$

$$0 \rightarrow 2\Delta t$$

$$F_0 \Delta t + \underbrace{\int_{\Delta t}^{2\Delta t} F(t) dt}_{\frac{1}{2} F_0 \Delta t} = \frac{3}{2} F_0 \Delta t$$



integrate friction  
force

$$\rightarrow \int_0^{\Delta t} \mu_k mg dt = \mu_k mg \Delta t$$

$$\int_0^{2\Delta t} \mu_k mg dt = 2\Delta t \mu_k mg$$

a)  $mv_2 = mv_1 + F_0 \Delta t - \mu_k mg \Delta t \rightarrow \text{solve for } v_2$

b)  $mv_2 = mv_1 + \frac{3}{2} F_0 \Delta t - 2\mu_k mg \Delta t \rightarrow \text{solve for } v_2$

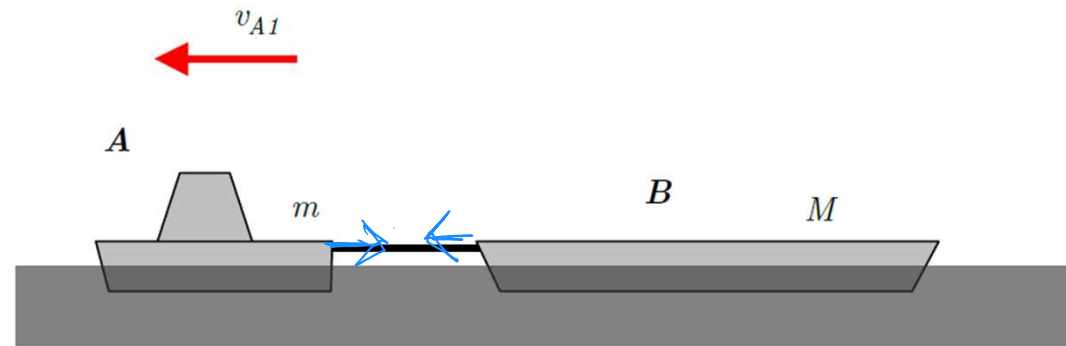
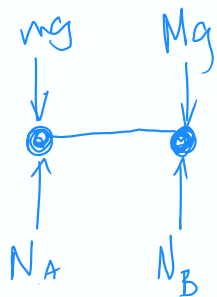
### Example 4.C.3

**Given:** A tug having a mass of  $m$  is towing a coal barge (mass  $M$ ) with a speed of  $v_{A1}$ . During a short period of time  $\Delta t$  when the engines of the tug are turned off, the stern winch takes in the towing cable at a rate of  $v_{B/A} = v_B - v_A$ .

Ignore the water resistance on the tug and barge during this time period. Also, assume that the towing cable remains taut at all times.

**Find:** Determine:

- The speed of the tug during this period of time; and
- The average value of the tension in the towing cable during this time.



a) LIM for combined tug/barge system  $\rightarrow$  internal forces cancel

$$m v_{A1} + M v_{A1} = m v_{A2} + M v_{B2}$$

$$(m+M) v_{A1} = m v_{A2} + M (v_{B/A} + v_{A2}) \rightarrow v_{A2} = v_{A1} + \frac{M v_{B/A}}{(m+M)}$$

b) tension  $\rightarrow$  just tug boat as system

$$\underline{m v_{A2}} = m v_{A1} + \int_0^{t_1} T dt$$

$$m \left( v_{A1} + \frac{M v_{B/A}}{(m+M)} \right) = m v_{A1} + T \Delta t$$

$$T = \frac{M}{\Delta t} \left( \frac{M v_{B/A}}{(m+M)} \right)$$

# ***ME 274: Basic Mechanics II***

Lecture 21: Particle Kinetics – Linear Impulse-Momentum



School of Mechanical Engineering

## ***Linear Impulse-Momentum Key Takeaways:***

LIM equation  $m\vec{v}_2 = m\vec{v}_1 + \int_1^2 \vec{R} dt$

$m\vec{v}$  - linear momentum of the particle

$\int_1^2 \vec{R} dt$  - impulse from the net force acting on the particle

- Linear impulse-momentum relates the change in linear momentum to the impulse (force applied over time) acting on the particle.
- a **vector** equation → we can resolve it into components!

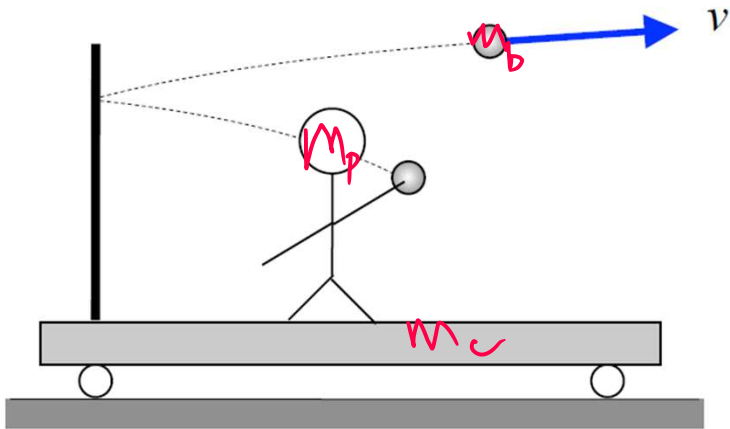
$$mv_{x2} = mv_{x1} + \int_1^2 R_x dt$$
$$mv_{y2} = mv_{y1} + \int_1^2 R_y dt$$

- If there is no resultant force, the linear momentum is said to be **conserved**:  $mv_{x2} = mv_{x1}$
- Conservation of momentum does not imply conservation of energy (or vice versa)
- For systems of multiple particles, **internal forces cancel** → make your system as large as feasibly possible!

**Question C4.9**

You are on a cart that is initially at rest on a smooth track. You throw a ball at a partition that is rigidly mounted on the cart. If the ball bounces off the partition as shown in the figure, then at the instant shown in the figure:

- (a) The cart is moving to the right
- (b) The cart is stationary
- (c) The cart is moving to the left
- (d) More information is needed about the impact of the ball with the partition in order to answer this question



$$@ t_1 \quad (m_b + m_p + m_c) V = 0$$

$$@ t_2 \quad m_b v_b + (m_p + m_c) V_{p+c} = 0$$

$$(m_p + m_c) V_{p+c} = -m_b v_b$$

$V_{p+c}$  opposite direction  
to  $v_b$

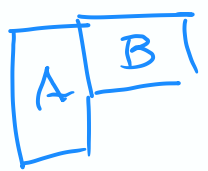
### Example 4.C.7

**Given:** Cars A and B (having weights of  $W_A$  and  $W_B$ , respectively) strike each other with speeds of  $v_{A1}$  and  $v_{B1}$ . After a short collision time, the cars stick together.

**Find:** Determine the  $x$  and  $y$  components of velocity for the cars after the collision.

Use the following parameters in your analysis:  $W_A = 3000$  lb,  $W_B = 4000$  lb,  $v_{A1} = 40$  mph and  $v_{B1} = 25$  mph.

FBD



→ no forces →  
linear momentum  
conserved in  $x$  &  $y$

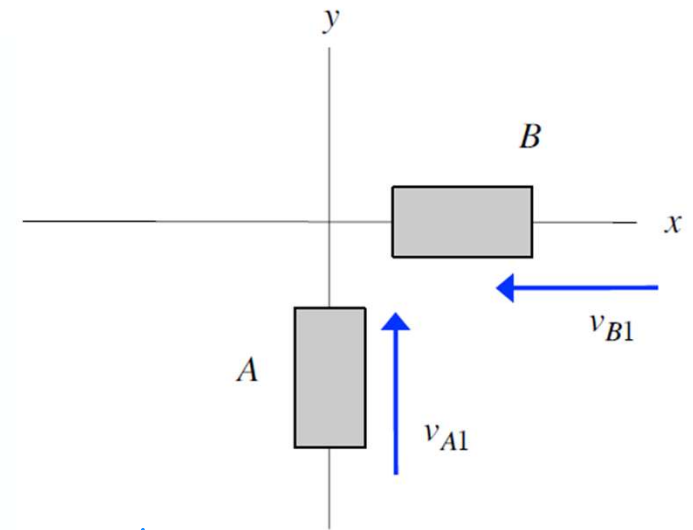
kinetics

LIM  $x$ -dir:

$$-\frac{W_B}{g} v_{B1} = \left(\frac{W_B + W_A}{g}\right) v_{2x} \Rightarrow v_{2x} = \frac{-W_B v_{B1}}{W_B + W_A}$$

LIM  $y$ -dir:

$$\frac{W_A}{g} v_{A1} = \left(\frac{W_B + W_A}{g}\right) v_{2y} \Rightarrow v_{2y} = \frac{W_A v_{A1}}{W_B + W_A}$$



### Example 4.C.8

**Given:** Block A (having a weight of  $W$ ) is suspended by a cord from fixed point O. Bullet B (having a weight of  $w$ ) strikes the stationary block A with a speed of  $v_{B1}$ . On impact, the bullet sticks to block A.

**Find:** Determine:

- The maximum elevation angle  $\theta$  of the cord after impact; and
- The energy lost during the impact of B with A.

Use the following parameters in your analysis:  $w = 0.2$  lb,  $W = 75$  lb,  $L = 5$  ft and  $v_{B1} = 1800$  ft/s.

FBD



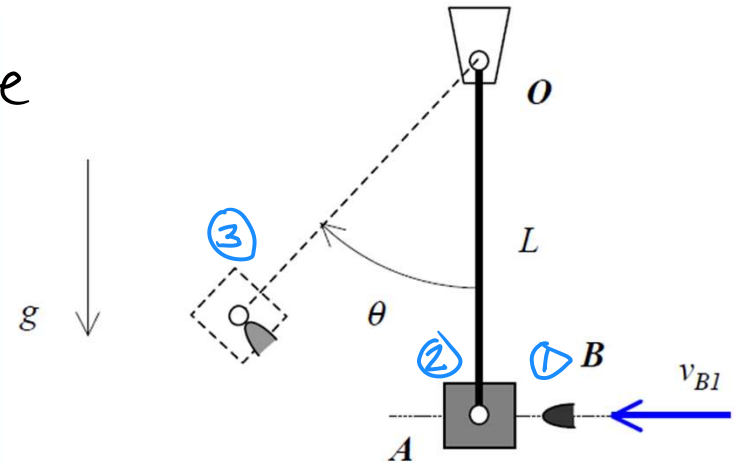
cord cannot carry a transverse load

(short time frame)

1-2  $\rightarrow$  impact, LHM

$$\frac{w}{g} v_{B1} = \frac{w+W}{g} v_{2x}$$

$$v_2 = \frac{w v_{B1}}{(w+W)}$$





2-3  $\rightarrow$  change in velocity, change in Position  $\rightarrow$  W-E

$$T_2 + V_2 + U_{2 \rightarrow 3}^N = T_3 + V_3$$

$$T_2 = \frac{1}{2} \left( \frac{W+w}{g} \right) V_2^2 = \frac{1}{2} \frac{(wV_{B1})^2}{g(w+W)}$$

$$V_2 = -(W+w)L$$

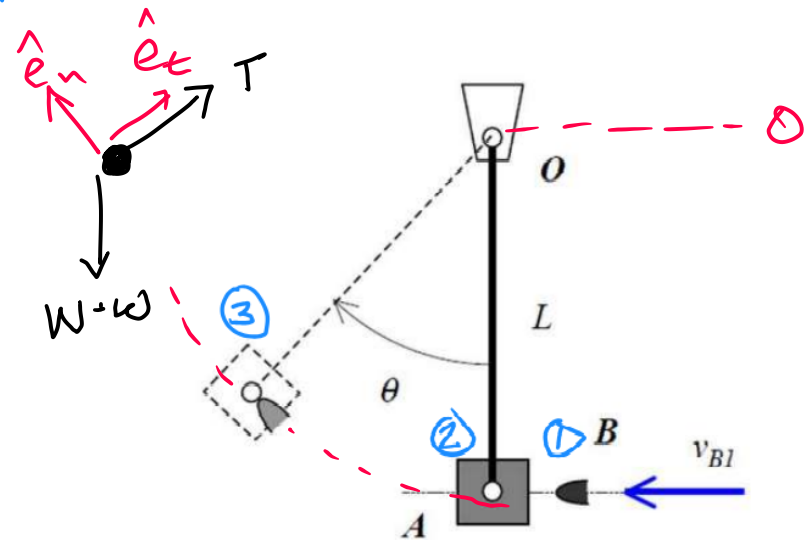
$U_{2 \rightarrow 3}^N = 0 \rightarrow$  tension always Normal

$$T_3 = 0$$

$$V_3 = -(W+w)L \cos \theta_{\max}$$

$$\frac{1}{2} \frac{(wV_{B1})^2}{g(w+W)} - (W+w)L = -(W+w)L \cos \theta_{\max}$$

$\rightarrow$  solve for  $\theta_{\max}$



## Energy Loss from 1 $\rightarrow$ 2

total energy at each time = Kinetic + potential

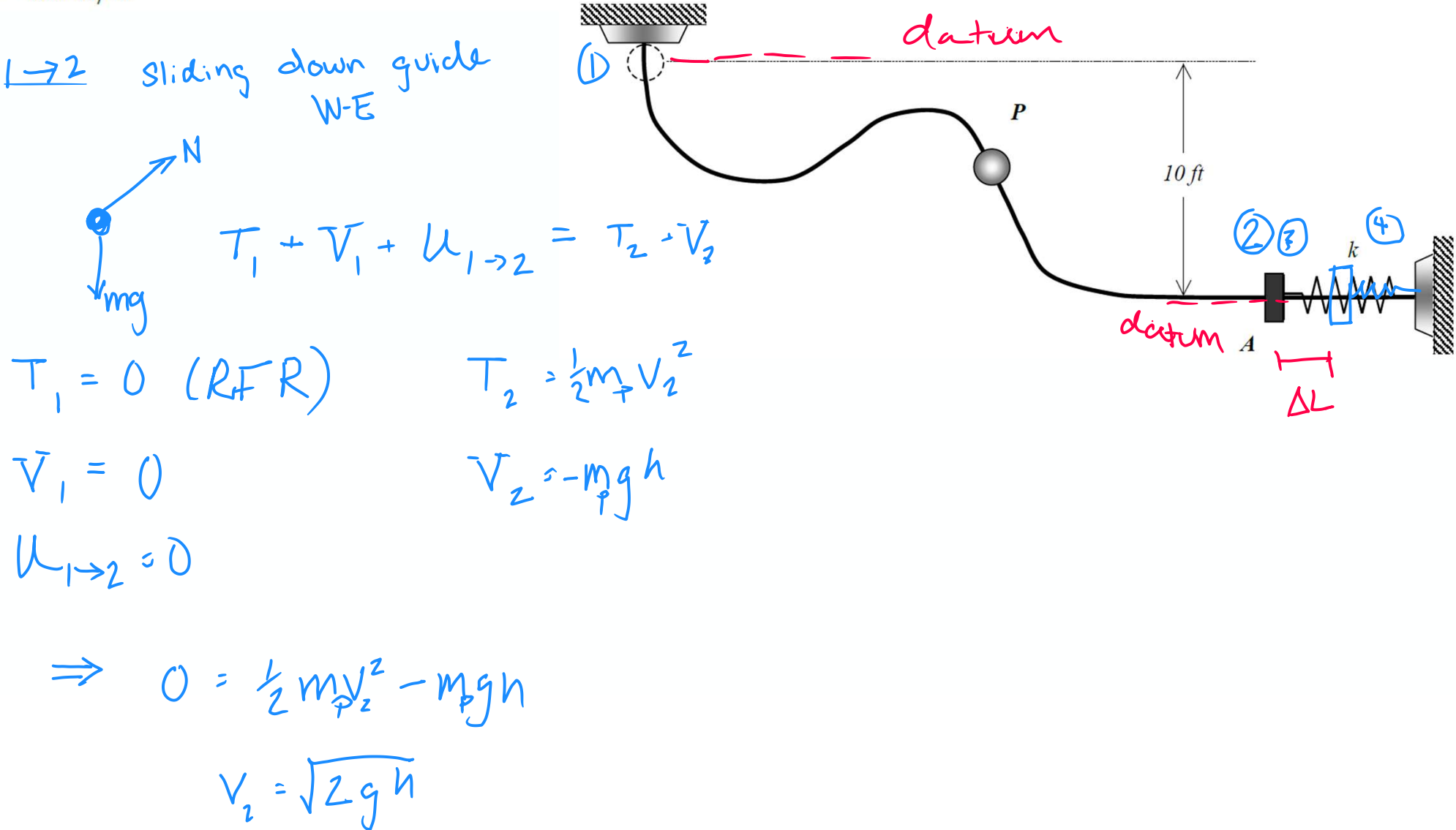
$$\Delta \text{Energy} = (T_2 + \cancel{V_2}) - (T_1 + \cancel{V_1})$$

$$= \frac{1}{2} \frac{(\omega V_{B1})^2}{g(\omega + \omega)} - \frac{1}{2} \frac{\omega}{g} V_{B1}^2$$

Example 4.C.6

**Given:** Particle P (weighing 10 lb) is released from rest and slides down a smooth, curved rod and sticks to block A (weighing 5 lb).

**Find:** Determine the maximum deflection of the spring attached to A, if the spring has a stiffness of  $k = 100 \text{ lb/ft}$ .



2 → 3 impact event → LIM

assume short duration → no spring compression → no resultant force, linear momentum conserved

before impact (only p moving)

$$m_p v_2 + \int F_s dt = (m_p + m_A) v_3$$

after impact

$$v_3 = \frac{m_p v_2}{(m_p + m_A)} = \frac{m_p \sqrt{2gh}}{(m_p + m_A)}$$

3 → 4 spring compression → WE

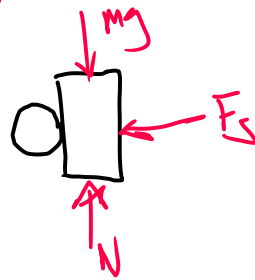
position 3

$$T_3 = \frac{1}{2} (m_p + m_A) v_3^2$$

$$= \frac{m_p^2 gh}{m_p + m_A}$$

$$V_3 = 0$$

$$U_{3 \rightarrow 4}^{nc} = 0$$



position 4

$$T_4 = 0$$

$$V_4 = \frac{1}{2} k (\Delta L)^2$$

Work - Energy

$$\frac{m_p^2 gh}{m_p + m_A} = \frac{1}{2} k \Delta L$$

$$\Delta L = \frac{2 m_p^2 gh}{k (m_p + m_A)}$$