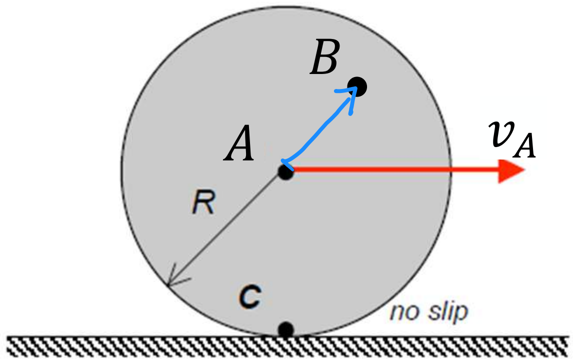
ME 274: Basic Mechanics II

Lecture 11: Moving Reference Frame Kinematics

PURDUE UNIVERSITY | School of Mechanical Engineering

Previously: Rigid body kinematics



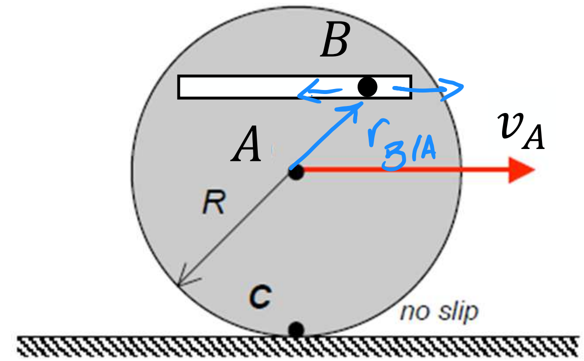
- Points A and B are on the same body
- $r = |\vec{r}_{B/A}|$=CONSTANT and $\dot{r} = 0$

Velocity and acceleration of point B with respect to point A:

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right] + \vec{\alpha} \times \vec{r}_{B/A}$$

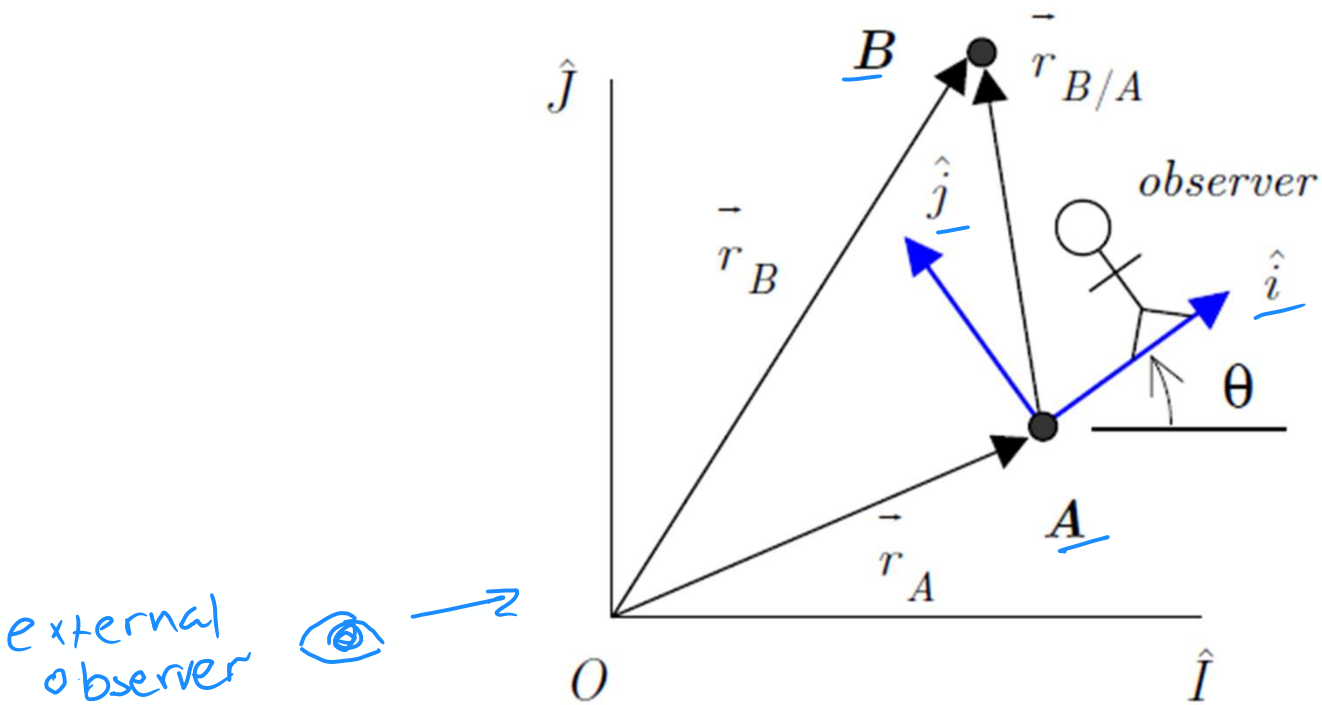
Today: Moving reference frame kinematics

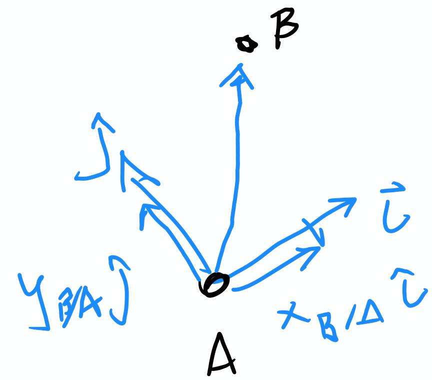


- Points A and B are NOT on the same body
- $r = |\vec{r}_{B/A}| \neq$ CONSTANT and $\dot{r} \neq 0$

How do we relate the motion of the two bodies?

Moving Reference Frames





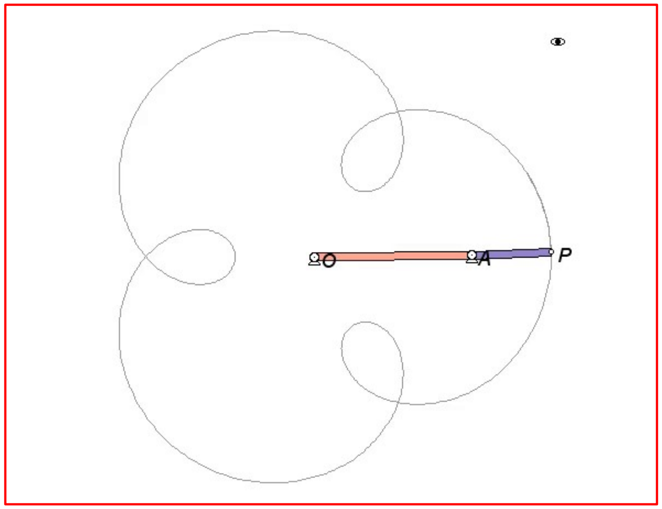
- Stationary axis ⟶ XYZ axis ⟶ $\hat{I}, \hat{J}, \hat{K}$ unit vectors
- Moving reference frame (translating and rotating) ⟶ xyz axis ⟶ $\hat{\imath}, \hat{\jmath}, \hat{k}$ unit vectors
- The observer at point A describes the position of point B as:

$$\vec{r}_{B/A} = x_{B/A}\hat{i} + y_{B/A}\hat{j}$$

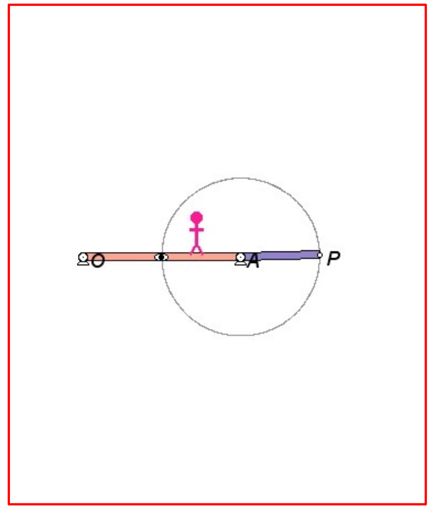
- NOTE: the vectors $\hat{\imath}, \hat{\jmath}$ rotate with the observer – their orientations change with time!

Animation: Views from different reference frames

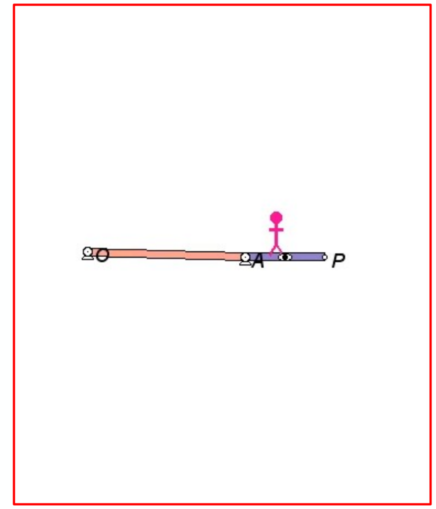
External Observer



Observer on OA



Observer on AP



→ too complex

→ simplified

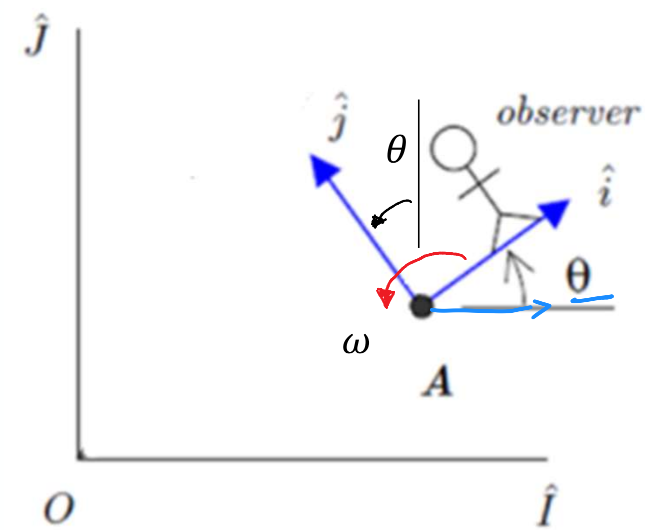
→ no motion

Question: Which observer do you think would be most useful in calculating the velocity and acceleration of point P?

Angular Velocity of the Rotating Reference Frame

- θ is angle between the x and X axis and the y and Y axis.
- The angular speed of the observer is given by the time rate of change of the angle θ

$$\frac{d\theta}{dt} = \dot{\theta} \quad , \quad \vec{\omega} = \dot{\theta}\vec{k}$$



- $\hat{\imath}, \hat{\jmath}$ unit vectors are a constant length, but change direction ⟶ time derivatives $\neq \mathbf{0}$

  Writing $\hat{\imath}$ and $\hat{\jmath}$ in terms of their components in the $\hat{I}$ and $\hat{J}$ direction:

$$\hat{i} = \cos\theta\,\hat{I} + \sin\theta\,\hat{J}$$

$$\hat{j} = -\sin\theta\,\hat{I} + \cos\theta\,\hat{J}$$

$$\frac{d\hat{i}}{dt} = -\dot{\theta}\sin\theta\,\hat{I} + \dot{\theta}\cos\theta\,\hat{J} = \dot{\theta}\hat{j}$$

$$\frac{d\hat{j}}{dt} = -\dot{\theta}\cos\theta\,\hat{I} - \dot{\theta}\sin\theta\,\hat{J} = -\dot{\theta}\hat{i}$$

- Using the right-hand rule we know $\hat{\imath} = -\hat{k} \times \hat{\jmath}$, $\hat{\jmath} = \hat{k} \times \hat{\imath}$

$$\frac{d\hat{i}}{dt} = \dot{\theta}(\hat{k} \times \hat{i}) = \vec{\omega} \times \hat{i}$$

$$\frac{d\hat{j}}{dt} = -\dot{\theta}(-\hat{k} \times \hat{j}) = \vec{\omega} \times \hat{j}$$

Velocity Equation – 2D

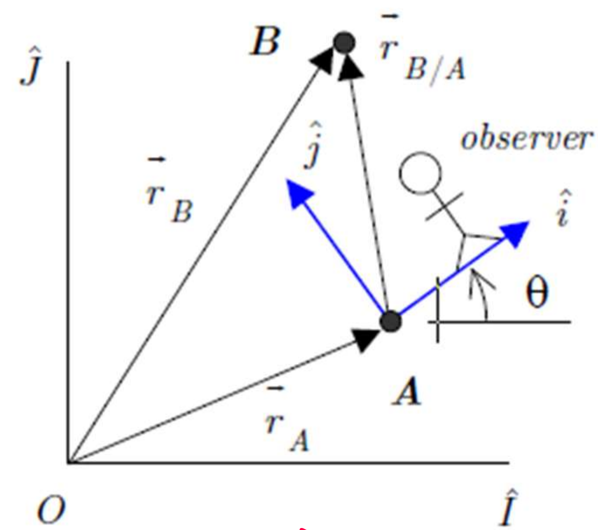
The position of point B written with respect to point A:

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$ > fixed ref. frame

The vector $\vec{r}_{B/A}$ written in terms of the observer's xy coordinates:

$$\vec{r}_{B/A} = x\hat{i} + y\hat{j}$$

Differentiating with respect to time (remember $\hat{i},\hat{j}$ vary with time):



$(V_{B/A})_{rel}$ $\hat{i}$ $\hat{j}$

$$\frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\vec{r}_{B/A}}{dt} \quad \Longrightarrow$$

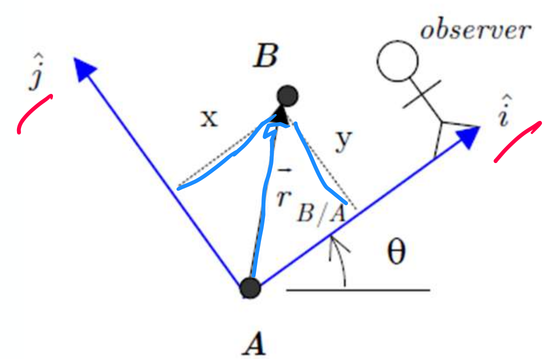
$$\vec{v}_B = \vec{v}_A + \frac{d}{dt}\left(x\hat{i} + y\hat{j}\right)$$ $\vec{\omega}\times\hat{i}$

$$= \vec{v}_A + \frac{dx}{dt}\hat{i} + x\frac{d\hat{i}}{dt} + \frac{dy}{dt}\hat{j} + y\frac{d\hat{j}}{dt}$$ $\vec{\omega}\times\hat{j}$

$$= \vec{v}_A + \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + x\left(\vec{\omega}\times\hat{i}\right) + y\left(\vec{\omega}\times\hat{j}\right)$$

$$= \vec{v}_A + \dot{x}\hat{i} + \dot{y}\hat{j} + \vec{\omega}\times\left(x\hat{i} + y\hat{j}\right)$$ ← distribute

$\vec{V}_B$ $$= \vec{v}_A + \underbrace{\left(\vec{v}_{B/A}\right)_{rel}}_{} + \vec{\omega}\times\vec{r}_{B/A}$$



$V_{B/A} = \hat{I}\ \hat{J}$

$$(\vec{V}_{B/A})_{rel} = \dot{x}\hat{i} + \dot{y}\hat{j} \neq \vec{V}_{B/A}$$

where $\vec{\omega}$ is the angular velocity of the observer at A and $\left(\vec{v}_{B/A}\right)_{rel}$ is the "velocity of B as seen by the observer at A".

Acceleration Equation – 2D

Starting with our equation for the velocity of point B:

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

And differentiating with respect to time:

$$\frac{d\vec{v}_B}{dt} = \frac{d\vec{v}_A}{dt} + \frac{d}{dt}\left(\vec{v}_{B/A}\right)_{rel} + \frac{d}{dt}\left(\vec{\omega} \times \vec{r}_{B/A}\right) \qquad \Longrightarrow$$

product rule

$$\vec{a}_B = \vec{a}_A + \frac{d}{dt}\left(\vec{v}_{B/A}\right)_{rel} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \frac{d\vec{r}_{B/A}}{dt}$$

previous slide

$$= \vec{a}_A + \frac{d}{dt}\left(\dot{x}\hat{i} + \dot{y}\hat{j}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \omega \times \vec{r}_{B/A}\right]$$

product rule

$$= \vec{a}_A + \left(\ddot{x}\hat{i} + \ddot{y}\hat{j}\right) + \left(\dot{x}\frac{d\hat{i}}{dt} + \dot{y}\frac{d\hat{j}}{dt}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right]$$

$$= \vec{a}_A + \left(\ddot{x}\hat{i} + \ddot{y}\hat{j}\right) + \dot{x}\left(\vec{\omega} \times \hat{i}\right) + \dot{y}\left(\vec{\omega} \times \hat{j}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \omega \times \vec{r}_{B/A}\right]$$

distributive prop.

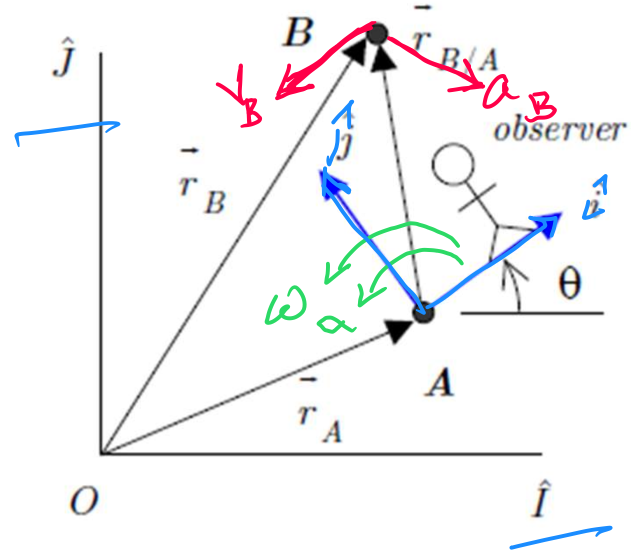
$\vec{\omega} \times (\dot{x}\hat{i} + \dot{y}\hat{j})$

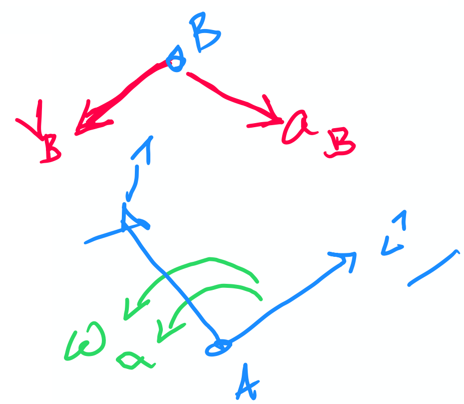
$$= \vec{a}_A + \left(\ddot{x}\hat{i} + \ddot{y}\hat{j}\right) + \vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \omega \times \vec{r}_{B/A}\right]$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$$

What do each of these terms represent?

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$
$$= \vec{v}_A + \vec{v}_{B/A}$$
$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$$
$$= \vec{a}_A + \vec{a}_{B/A}$$





- $\vec{v}_A$ and $\vec{v}_B$ are the velocities of points B and A as seen by a fixed observer.
- $\vec{a}_A$ and $\vec{a}_B$ are the accelerations of points B and A as seen by a fixed observer.
- $\vec{\omega}$ is angular velocity of the moving observer.
- $\vec{\alpha}$ is the angular acceleration of the moving observer.
- $\left(\vec{v}_{B/A}\right)_{rel}$ is the "velocity of point B as seen by the moving observer at A". In order to write down this term, you must clearly understand how the moving observer views the motion of B.
- $\left(\vec{a}_{B/A}\right)_{rel}$ is the "acceleration of point B as seen by the moving observer at A". In order to write down this term, you must clearly understand how the moving observer views the motion of B.
- The term $2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel}$ is known as the "Coriolis" component of acceleration. The term arises due to an observed velocity of B by the moving observer, when the observer has a non-zero angular velocity. We will introduce a number of physical examples that demonstrate the significance of this term during lecture.

$-\vec{V}_B \hat{i} = (V_{B/A})_{rel}$

$a_{Bx} \hat{i} + a_{By} \hat{j}$
$= (a_{B/A})_{rel}$

CHALLENGE QUESTIONS: What are the differences between $\vec{v}_{B/A}$ and $\left(\vec{v}_{B/A}\right)_{rel}$? And between $\vec{a}_{B/A}$ and $\left(\vec{a}_{B/A}\right)_{rel}$? Are they ever the same?

Starting with the equations we derived earlier:

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$$

$$\left(\vec{V}_{B/A}\right)_{rel} = \underbrace{\vec{V}_B - \vec{V}_A} - \vec{\omega} \times \vec{r}_{B/A}$$

$$\left(\vec{V}_{B/A}\right)_{rel} = \vec{V}_{B/A} + \cancel{\vec{\omega} \times \vec{r}}_{B/A} \; 0$$

Discussion – The "How To" with Moving Reference Frame Kinematics Equations

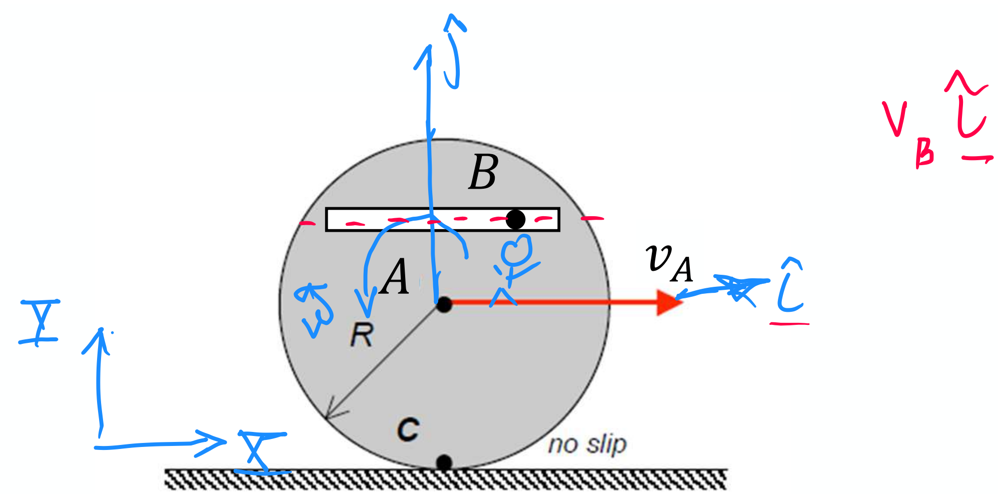
1. Choose your moving reference frame (observer). It is recommended that you draw a stick figure of your observer on this frame to remind yourself of your choice of reference frame. Note that point A must be on this reference frame.

2. Draw your choice of xyz axes for the moving reference frame. State in words to what the xyz axes are attached. Also show your choice of stationary XYZ axes for the problem.

3. Determine the angular velocity $\vec{\omega}$ of the moving reference frame. [Note: this represents the ANGULAR MOTION OF THE OBSERVER, and not what the observer sees.]

4. Determine the angular acceleration $\vec{\alpha}$ of the moving reference frame.

5. Imagine yourself as the observer on the moving reference frame. Answer the question: How do I see point B move if I am that observer? Based on this answer, write down $\left(\vec{v}_{B/A}\right)_{rel}$ and $\left(\vec{a}_{B/A}\right)_{rel}$. [Note: this is the MOTION THAT THE OBSERVER SEES, not the motion of the observer.]



Example 3.A.1

Given: The disk shown below rolls without slipping on a horizontal surface. At the instant shown, the center O is moving to the right with a speed of $v_0 = 5$ m/s with this speed decreasing at a rate of 2 m/s^2. Also for this instant, the particle P is at a position of $x_p = 0.2$ m with $\dot{x}_p = 2$ m/s = *constant*, where x_p is measured relative to the xyz coordinate system that is attached to the disk.

$a_0 = -2$

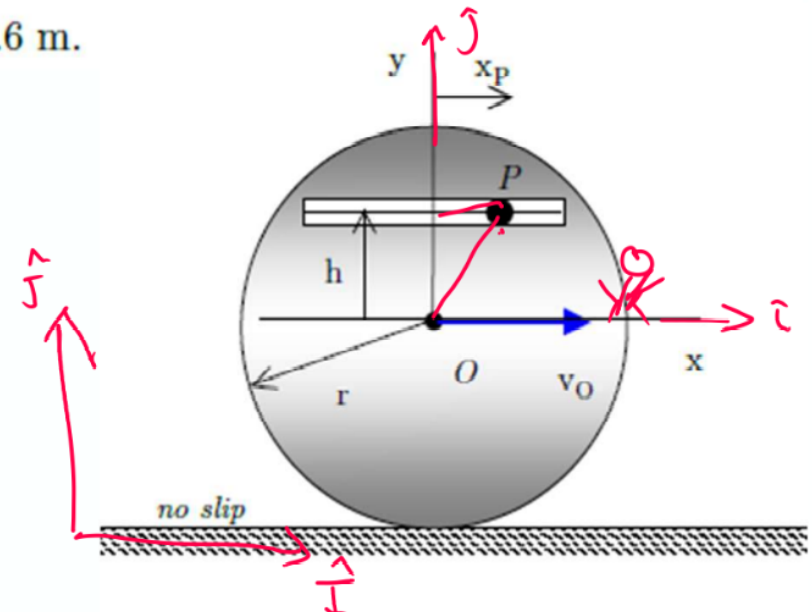
$\ddot{x}_p = 0$

Use the following parameters in your analysis: $h = 0.2$ m and $r = 0.6$ m.

Find: Determine:

(a) The velocity of particle P; and

(b) The acceleration of particle P.



$$\vec{V}_p = \vec{V}_o + (\vec{V}_{p/o})_{rel} + \vec{\omega} \times r_{p/o}$$

ang vel of observer

$$\vec{\omega} = -\frac{V_0}{r}\hat{K}$$ ← rolling w/o slip

$$r_{p/o} = x_p\hat{\imath} + h\hat{\jmath}$$

$$(V_{p/o})_{rel} = \dot{x}_p\hat{\imath}$$

$$V_p = V_0\hat{I} + \dot{x}_p\hat{\imath} + \left(-\frac{V_0}{r}\hat{k}\right)\times\left(x_p\hat{\imath} + h\hat{\jmath}\right)$$

$$= V_0\hat{I} + \dot{x}_p\hat{\imath} - \frac{V_0}{r}x_p\hat{\jmath} + \frac{V_0 h}{r}\hat{\imath}$$

convert to one coordinate set → $\hat{\imath} = \hat{I}$ $\quad \hat{\jmath} = \hat{J}$

$$V_p = \left(V_0 + \dot{x}_p + \frac{V_0 h}{r}\right)\hat{\imath} - \frac{V_0 x_p}{r}\hat{\jmath}$$

solving for α:

$$a_c = a_o + \alpha \times r_{c/o} - \omega^2 r_{c/o}$$

$$= a_o\hat{\imath} + \alpha\hat{k}\times(-r\hat{J}) - \left(\frac{V_0}{r}\right)^2(-r\hat{J})$$

$$\hat{\imath}: \; 0 = a_0 + r\alpha \rightarrow \alpha = -\frac{a_0}{r}$$

acceleration: $\dot{x}_p = \text{const}$

$$\vec{a}_p = \vec{a}_o + \cancel{(\vec{a}_{p/o})_{rel}}^{\,0} + \vec{\alpha} \times \vec{r}_{p/o} + 2\vec{\omega} \times (\vec{v}_{p/o})_{rel} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{p/o})$$

$$= a_o\hat{i} + \left(-\frac{a_o}{r}\hat{k}\right) \times (x_p\hat{i} + h\hat{j}) + 2\left(-\frac{v_o}{r}\hat{k}\right) \times (\dot{x}\hat{i}) - \left(\frac{v_o}{r}\right)^2 (x_p\hat{i} + h\hat{j})$$

$$= a_o\hat{i} - \frac{a_o x_p}{r}\hat{j} + \frac{a_o h}{r}\hat{i} - \frac{2v_o\dot{x}}{r}\hat{j} - \frac{v_o^2 x_p}{r^2}\hat{i} - \frac{v_o^2 h}{r^2}\hat{j}$$

$$a_p = \left(a_o + \frac{a_o h}{r} - \frac{v_o^2 x_p}{r^2}\right)\hat{i} + \left(-\frac{a_o x_p}{r} - \frac{2v_o x}{r} - \frac{v_o^2 h}{r^2}\right)\hat{j}$$

Example 3.A.3

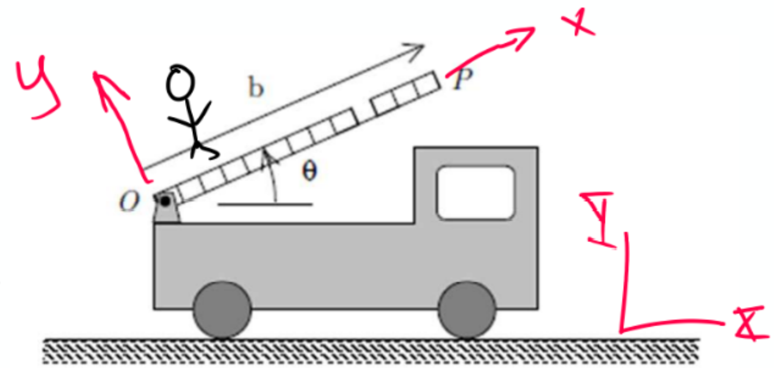
Given: The fire truck moves forward at a constant speed of 50 ft/s. The ladder is being raised at a constant rate of $\dot{\theta} = 0.3$ rad/s. In addition, the ladder is being extended at a constant rate of $\dot{b} = 2$ ft/s.

Find: The acceleration of end P of the ladder when $b = 25$ ft and $\theta = 30°$.

$$\vec{a}_P = \cancel{\vec{a}_O}^{\,0} + \cancel{(\vec{a}_{P/O})_{rel}}^{\,0} + \cancel{\vec{\alpha}\times\vec{r}_{P/O}}^{\,0} + 2\vec{\omega}\times(\vec{v}_{P/O})_{rel} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{P/O})$$

$$= 0 + 0 + 0 + 2\dot{\theta}\hat{k}\times(\dot{b}\hat{i}) - \dot{\theta}^2 b\hat{i}$$

$$\vec{a}_P = 2\dot{\theta}\dot{b}\hat{j} - \dot{\theta}^2 b\hat{i}$$



ME 274: Basic Mechanics II

Lecture 12: Moving Reference Frame Kinematics

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Announcements

Exam 1 info:

- Exam 1, Thurs. 2/12, 8 – 9:30, MA 175
- Students with exam accommodations: HIKS G980D, 6:30PM – reminder to email Dr. Krousgrill
- Exam review session Tuesday 2/10, 7pm over Zoom (link on course website on “Exams” page)

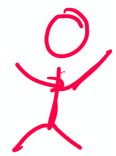
Homework and class updates:

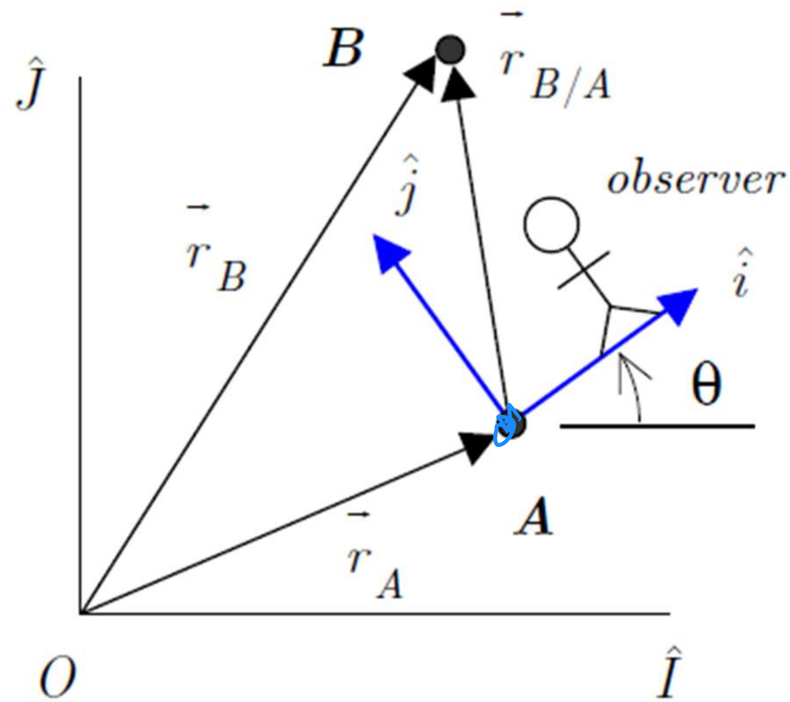
- No class Friday, 2/13 to account for evening exam
- HW released today (2.C, 2.D) **due Friday 2/13**
- HW released on Wednesday (2.E, 2.F) **due Monday 2/16**
- **Reminder to use correct HW formatting!**
- Office hrs: T 9-10:30, W 3:30-4:20, Th 3-4:30

Moving Reference Frames – Overview

- Stationary axis ⟶ XYZ axis ⟶ point B as seen by an outside observer
- Moving reference frame ⟶ xyz axis ⟶point B as seen from an observer at point A
- The angular speed of the observer is given by the time rate of change of the angle θ between the fixed and moving reference frame ⟶$\vec{\omega} = \dot{\theta}\hat{k}$

$\vec{\alpha} = \ddot{\theta}\hat{k}$

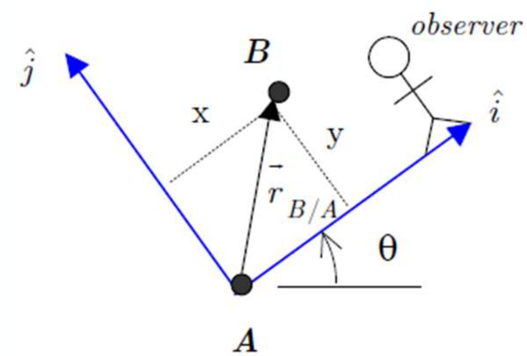




Relating velocity and acceleration of pts. B and A :

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + \boxed{2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel}} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$$



Where $\left(\vec{v}_{B/A}\right)_{rel}$,$\left(\vec{a}_{B/A}\right)_{rel}$ are the velocity and acceleration of point B as observed from A.

$(\vec{V}_{B/A})_{rel} = \dot{x}\hat{i} + \dot{y}\hat{j}$

Remember pt A must be on the same reference frame as the observer!

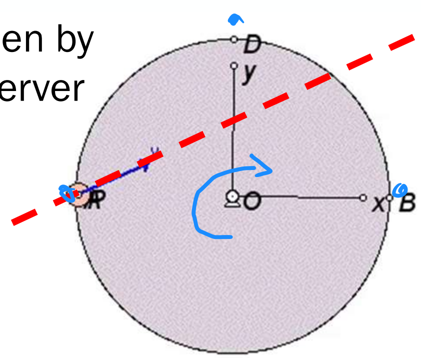
$(\vec{a}_{B/A})_{rel} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$

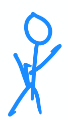
The Coriolis Component of Acceleration

- The Coriolis component arises when a point has a nonzero velocity relative to a rotating reference frame.
- Although the point may be moving in a straight line with constant velocity in an inertial frame, the rotating observer sees the direction of that relative velocity continuously change as the reference axes rotate.

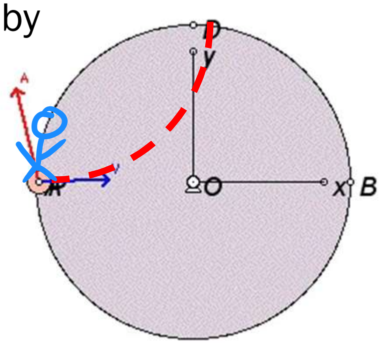
Ex 1: Merry-go-round

Motion of P seen by stationary observer

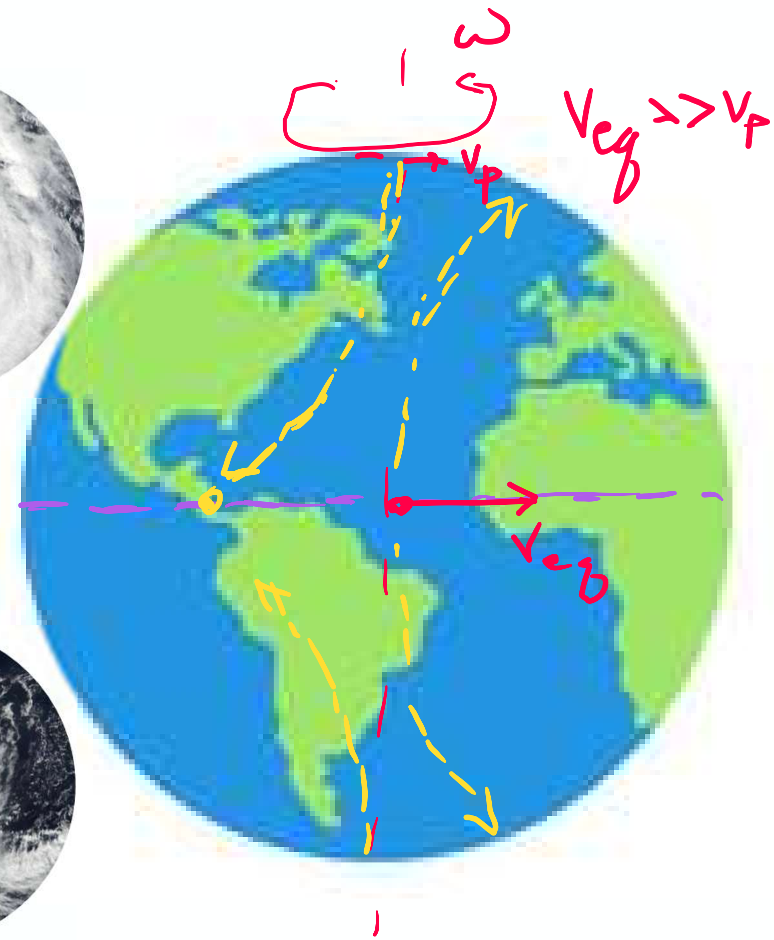




Motion of P seen by moving observer



Ex 2: Weather Systems



Question C3.3

Sprinkler arm OA is pinned to a cart at point O. The cart moves to the right with a speed of v_{cart} with $\dot{v}_{cart} = 2$ ft/s^2 $=$ *constant*. Fluid flows through the sprinkler arm at a rate of $\dot{d}$ with $\ddot{d} =$ -3 ft/s^2 $=$ *constant*. The sprinkler arm is being raised at a constant rate of $\dot{\theta} = 4$ rad/s. An observer and xyz coordinate system are attached to the sprinkler arm, as shown in the figure below. The following equation is to be used to find the acceleration of a pellet P that flows with the fluid in the arm:

$$\vec{a}_P = \vec{a}_O + \left(\vec{a}_{P/O}\right)_{rel} + \vec{\alpha} \times \vec{r}_{P/O} + 2\vec{\omega} \times \left(\vec{v}_{P/O}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{P/O}\right]$$

Provide numerical values for the following terms when: $d = 3$ ft, $v_{cart} = 3$ ft/s, $\dot{d} = 5$ ft/s and $\theta = 90°$.

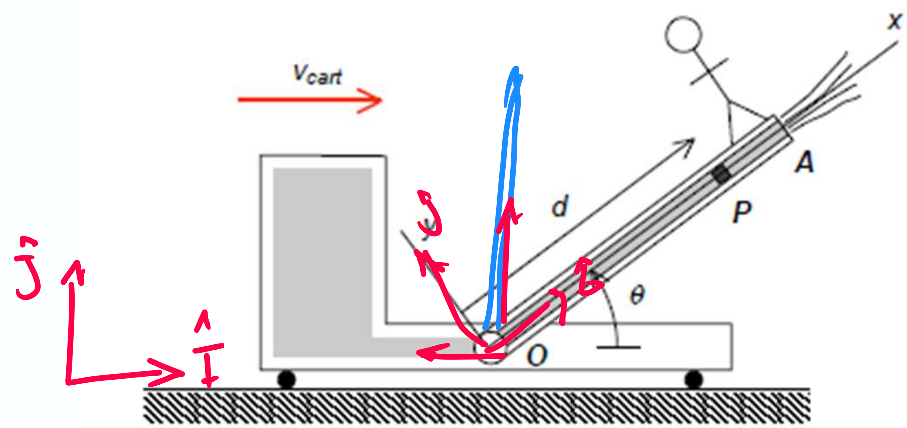
$\vec{a}_O = 2\hat{I} \rightarrow -2\hat{j}$

$\vec{\omega} = 4\hat{k}$

$\vec{\alpha} = \vec{0}$

$\left(\vec{v}_{P/O}\right)_{rel} = \dot{d}\,\hat{i} = 5\hat{i}$

$\left(\vec{a}_{P/O}\right)_{rel} = -3\hat{i}$

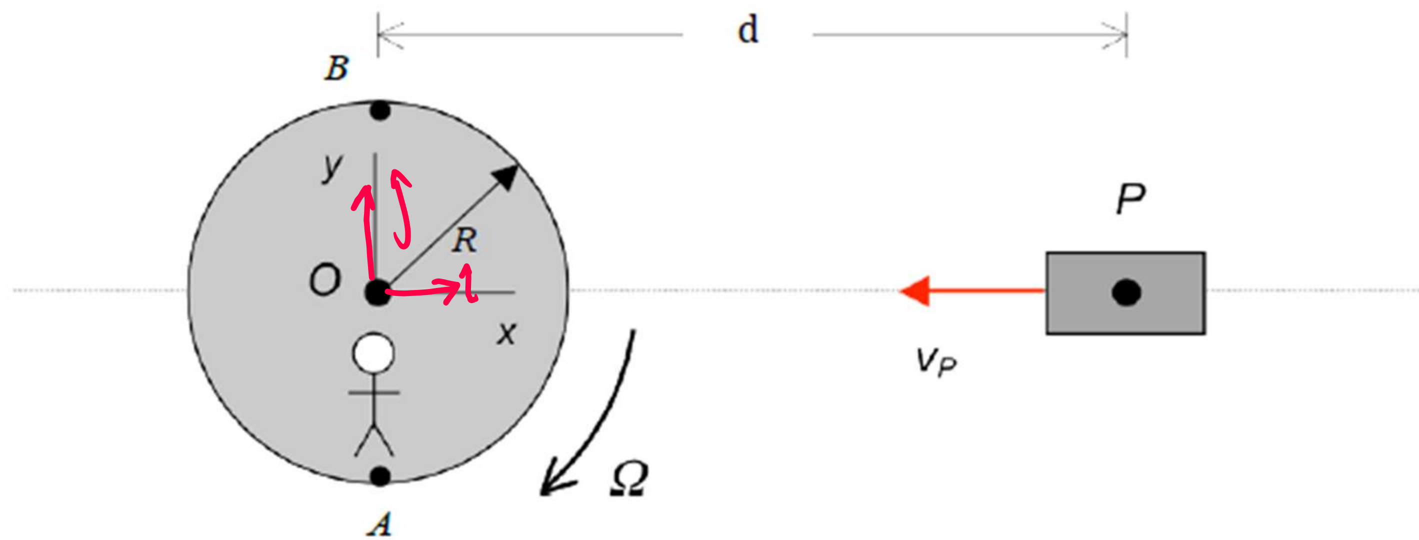


$\hat{i} = \hat{J}$ $\quad \hat{j} = -\hat{I}$

Question C3.2

A disk is pinned to ground at its center O with the disk rotating clockwise at a constant rate of $\Omega = 3$ rad/s. Block P is traveling to the left along a straight path toward O with a constant speed of $v_P = 20$ ft/s. Determine the acceleration of P as seen by an observer on the disk when P is at a distance of 50 ft from O.

$\hookrightarrow (a_{P/O})_{rel}$



$$\vec{a}_P = \vec{a}_O + \left(\vec{a}_{P/O}\right)_{rel} + \vec{\alpha} \times \vec{r}_{P/O} + 2\vec{\omega} \times \left(\vec{v}_{P/O}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{P/O}\right]$$

$$(\vec{a}_{P/O})_{rel} = \cancel{\vec{a}_P}^{\,0} - \vec{a}_O - \cancel{\vec{\alpha} \times \vec{r}_{P/O}} - 2\vec{\omega} \times (\vec{V}_{P/O})_{rel} - \omega^2 \vec{r}_{P/O}$$

$$= 0 - 0 - 0 + 2\Omega\hat{k} \times \underline{(V_{P/O})_{rel}} - \Omega^2 d\,\hat{i}$$

$$\vec{V}_P = \vec{V}_O + (\vec{V}_{P/O})_{rel} + \vec{\omega} \times \vec{r}_{P/O}$$

$$\Rightarrow (\vec{V}_{P/O})_{rel} = \vec{V}_P - \cancel{\vec{V}_O}^{\,0} - \vec{\omega} \times \vec{r}_{P/O}$$

$$= -\vec{V}_p \hat{\imath} + \Omega \hat{k} \times (d\hat{\imath})$$

$$(\vec{V}_{p/o})_{rel} = -V_p \hat{\imath} + \Omega d \hat{\jmath}$$

$$(a_{p/o})_{rel} = 2\Omega \hat{k} \times (-V_p \hat{\imath} + \Omega d \hat{\jmath}) - \Omega^2 d \hat{\imath}$$

$$= -2\Omega V_p \hat{\jmath} - 2\Omega^2 d \hat{\imath} - \Omega^2 d \hat{\imath}$$

$$= -3\Omega^2 d \hat{\imath} - 2\Omega V_p \hat{\jmath}$$

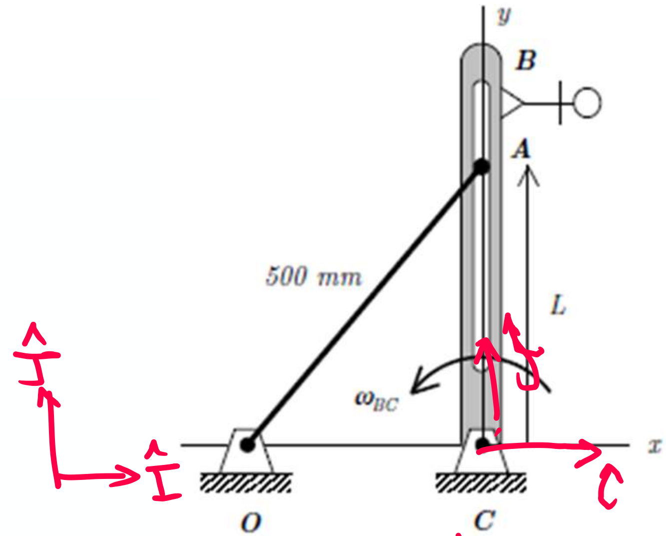
Example 3.A.5

Given: The mechanism shown below is made up of links OA and BC. Links OA and BC are pinned to ground at points O and C, respectively. Pin A of link OA is able to slide within a slot that is cut in link BC, as shown. Pins O and C are on the same horizontal line. Let the xyz axes be attached to link BC. At the instant shown:

- Link BC is vertical with $L = 400$ mm; and
- Link BC is rotating counterclockwise at a rate of $\omega = 6$ rad/s.

Find: Determine:

(a) The angular velocity of link OA at the instant shown; and
(b) The value for $\dot{L}$ at the instant shown.



$$d = \sqrt{500^2 - L^2}$$

1) link OA

$$\vec{V}_A = \cancel{\vec{V}_O}^{\,0} + \vec{\omega} \times \vec{r}_{A/O}$$

$$= \omega_{OA}\hat{K} \times (d\hat{I} + L\hat{J})$$

$$= \omega_{OA} d\hat{J} - \omega_{OA} L\hat{I}$$

2) A relative to C

$$\vec{V}_A = \cancel{\vec{V}_C} + (\vec{V}_{A/C})_{rel} + \vec{\omega}_{BC} \times \vec{r}_{A/C}$$

$$= \dot{L}\hat{j} + \omega_{BC}\hat{k} \times L\hat{j}$$

set =

$$= \dot{L}\hat{j} - \omega_{BC} L \hat{i}$$

$$\dot{L}\hat{J} - \omega_{BC} L \hat{I} = \omega_{OA} d \hat{j} - \omega_{OA} L \hat{i}$$

here, $\hat{J} = \hat{j}$ $\quad \hat{I} = \hat{i}$

Split into components

$\hat{i}: \; -\omega_{BC} L = -\omega_{OA} L \rightarrow \omega_{BC} = \omega_{OA}$

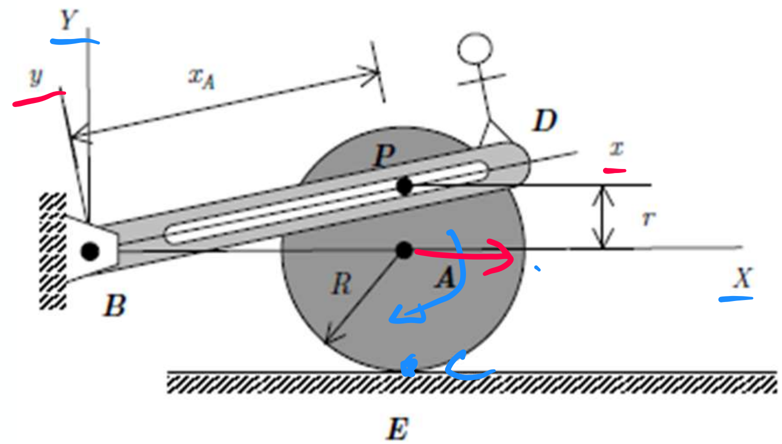
$\hat{j}: \; \dot{L} = \omega_{OA} d$

Example 3.A.7

Given: The disk rolls without slipping to the right with a constant angular speed of ω_d. At the instant shown, pin P is directly above the center A of the disk.

Find: Determine:
(a) The angular acceleration of the disk; and
(b) The acceleration of P as seen by an observer on arm BD.

Use the following parameters in your analysis: $\omega_d = 20$ rad/s (clockwise), $x_A = 0.48$ m, $r = 0.14$ m and $R = 0.2$ m.



velocity of pt P.

$$\vec{V}_P = \cancel{\vec{V}_C}^{\,0} + \vec{\omega} \times \vec{r}_{P/C}$$
$$= -\omega_d \hat{K} \times (R+r)\hat{J}$$
$$= (R+r)\omega_d \hat{I}$$

relate P to B

$$\vec{V}_P = \cancel{\vec{V}_B}^{\,0} + (V_{P/B})_{rel} + \vec{\omega}_{BD} \times r_{P/B}$$
$$= \dot{x}_A \hat{i} + \omega_{BD} \hat{k} \times x_A \hat{i}$$
$$= \dot{x}_A \hat{i} + \omega_{BD} x_A \hat{j}$$

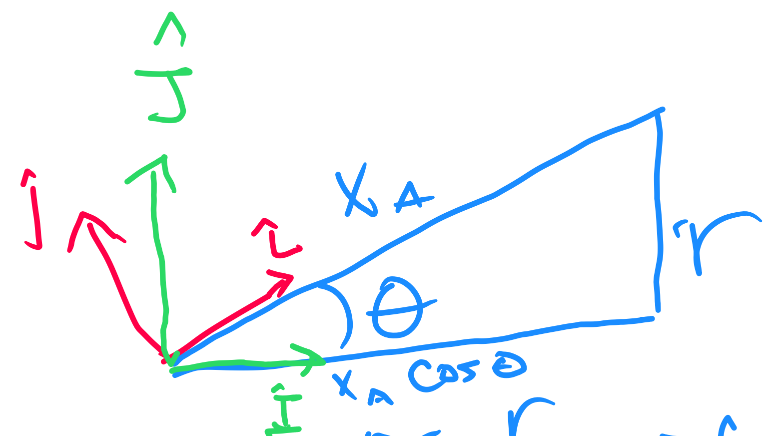
Set eqns =

$$(R+r)\omega_d \hat{I} = \dot{x}_A \hat{i} + \omega_{BD} x_A \hat{j} \quad *$$

write $\hat{i}, \hat{j}$ in terms of $\hat{I}$ & $\hat{J}$

$$\hat{i} = \cos\theta \hat{I} + \sin\theta \hat{J}$$

$$\hat{j} = -\sin\theta \hat{I} + \cos\theta \hat{J}$$



$\sin\theta = \frac{r}{x_A} \rightarrow$ find θ

Sub in to * & split up components

$\hat{I}$: $(R+r)\omega_d = \dot{x}\cos\theta - \omega_{BD} x_A \sin\theta$

$\hat{J}$: $0 = \dot{x}\sin\theta + \omega_{BD} x_A \cos\theta$

Solve for $\dot{x}$ & ω_{BD}

acceleration of pt P

$\vec{\alpha}_D = \vec{0}$

$\vec{V}_A = \text{const}$

$\vec{a}_A = 0$

$$\vec{a}_P = \cancel{\vec{a}_A} + \cancel{\vec{\alpha}_A} \times \vec{r}_{P/A} - \omega_A^2 \vec{r}_{P/A}$$

$$= -\omega_D^2 r \hat{J}$$

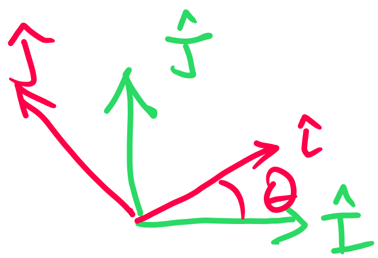
relate acceleration of P + B

$$\vec{a}_P = \vec{a}_B + (a_{P/B})_{rel} + \vec{\alpha}_{BD} \times \vec{r}_{P/B} + 2\vec{\omega} \times (v_{P/B})_{rel} - \omega^2 r_{P/B}$$

$$= 0 + \ddot{x}_A \hat{i} + \alpha_{BD} \hat{k} \times x_A \hat{i} + 2\omega_{BD} \hat{k} \times \dot{x}_A \hat{i} - \omega_{BD}^2 x_A \hat{i}$$

$$= \ddot{x}_A \hat{i} + \alpha_{BD} x_A \hat{j} + 2\omega_{BD} \dot{x} \hat{j} - \omega_{BD}^2 x_A \hat{i}$$

set $\vec{a}_P = \vec{a}_P$



$$\hat{J} = \sin\theta \hat{i} + \cos\theta \hat{j}$$

$$-\omega_D^2 r(\sin\theta \hat{i} + \cos\theta \hat{j}) = \ddot{x}_A \hat{i} + \alpha_{BD} x_A \hat{j} + 2\omega_{BD} \dot{x} \hat{j} - \omega_{BD}^2 x_A \hat{i}$$

Break into components

$\hat{i}$: $-\omega_D^2 r \sin\theta = \ddot{x}_A - \omega_{BD}^2 x_A$

$\hat{j}$: $-\omega_D^2 r \cos\theta = \alpha_{BD} x_A + 2\omega_{BD} \dot{x}$

Solve for α_{BD}, $\ddot{x}_A$

ME 274: Basic Mechanics II

Lecture 13: Moving Reference Frame Kinematics -3D

PURDUE UNIVERSITY | School of Mechanical Engineering

Announcements

Exam 1 info:

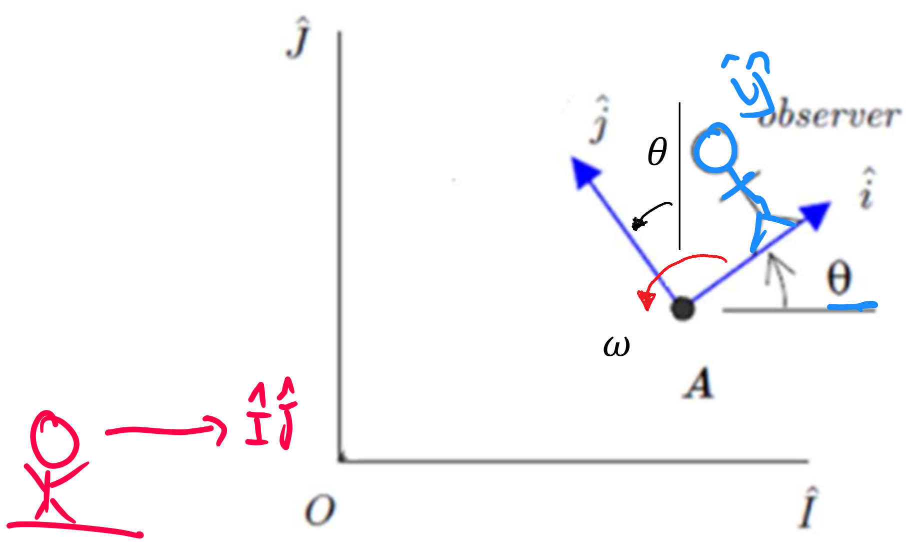
- Exam 1, Thurs. 2/12, 8 – 9:30, MA 175
- Arrive 15 minutes early so we can start on time
- You will have time after the exam to scan and submit.

Homework and class updates:

- No class Friday, 2/13 to account for evening exam
- HW released Monday (2.C, 2.D) **due Friday 2/13**
- HW released on Wednesday (2.E, 2.F) **due Monday 2/16**
- **Reminder to use correct HW formatting!**
- Office hrs change: Th 12-1:30

How do our expressions for velocity and acceleration change when our system is moving in 3 dimensions?

2D Rotating Reference Frames



Fixed X Y Z

moving x y z

- Angular velocity measured around a single axis (usually $\hat{k}$)

$$\vec{\omega} = \dot{\theta}\hat{k}$$

- Unit vectors $\hat{\imath}, \hat{\jmath}$ change with time

$$\frac{d\hat{\imath}}{dt} = \vec{\omega} \times \hat{\imath}, \qquad \frac{d\hat{\jmath}}{dt} = \vec{\omega} \times \hat{\jmath}$$

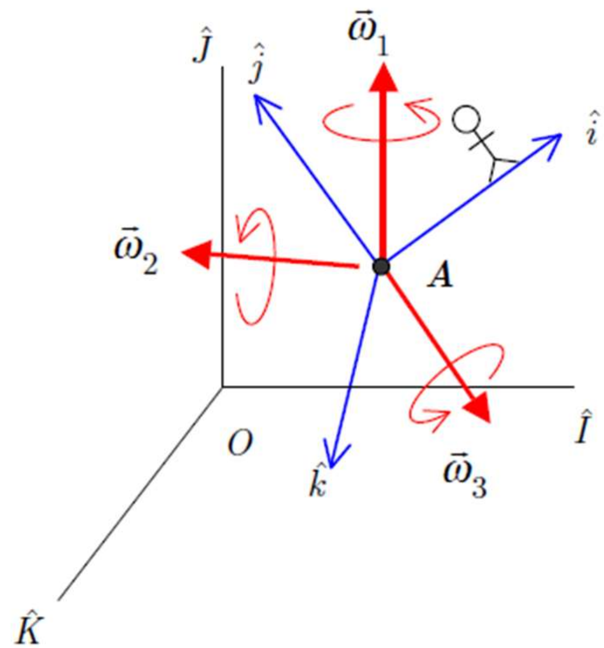
- Velocity: $\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$

$(V_{B/A})_{rel} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$

- Acceleration: $\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$

$(a_{B/A})_{rel} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

3D Rotating Reference Frames



- Angular velocity around a moving axis → not aligned with $\hat{i}, \hat{j}, \hat{k}$
- Multiple components each representing a different axis of rotation, $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \ldots$
- The total angular velocity of the observer is found by vector addition of all the components:

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3 + \cdots$$

- Unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ change with time

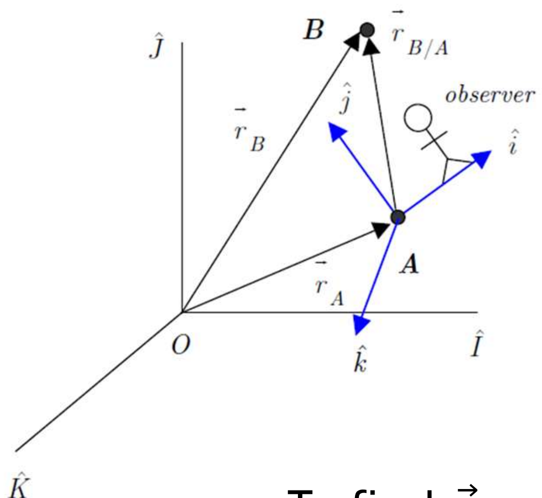
$$\frac{d\hat{\imath}}{dt} = \omega \times \hat{\imath}, \qquad \frac{d\hat{\jmath}}{dt} = \omega \times \hat{\jmath}, \qquad \frac{d\hat{k}}{dt} = \omega \times \hat{k}$$

Some rotations about fixed Axis → $\omega_1 \hat{K}, \omega_2 \hat{I}, \omega \hat{J}$
Some about moving Axis ⇒ $\omega_1 \hat{k}, \omega_2 \hat{\imath}, \omega_3 \hat{\jmath}$

Velocity Equation – 3D

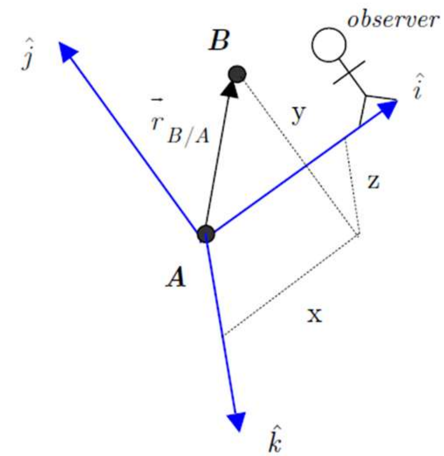
In the fixed reference frame, the position of pt. B is related to the position of pt. A using the relative position vector:

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$



In the moving reference frame, $\vec{r}_{B/A}$ can be described using xyz coordinates:

$$\vec{r}_{B/A} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$



To find $\vec{v}_B$, differentiate $\vec{r}_B$ with respect to time:

$$\frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\vec{r}_{B/A}}{dt} \qquad \Longrightarrow$$

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) \\
&= \vec{v}_A + \frac{dx}{dt}\hat{i} + x\frac{d\hat{i}}{dt} + \frac{dy}{dt}\hat{j} + y\frac{d\hat{j}}{dt} + \frac{dz}{dt}\hat{k} + z\frac{d\hat{k}}{dt} \\
&= \vec{v}_A + \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} + x\left(\vec{\omega}\times\hat{i}\right) + y\left(\vec{\omega}\times\hat{j}\right) + z\left(\vec{\omega}\times\hat{k}\right) \\
&= \vec{v}_A + \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} + \vec{\omega}\times\left(x\hat{i} + y\hat{j} + z\hat{k}\right) \\
\vec{V}_B &= \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega}\times\vec{r}_{B/A}
\end{aligned}$$

← same as 2D!

$\omega_1 \hat{I} + \omega_2 \hat{J} + \cdots$

$\vec{\omega} = \omega_1 + \omega_2 + \omega_3 \cdots$

$\neq \dot{\theta}\hat{K}$

$(V_{B/A})_{rel} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$

Acceleration Equation – 3D

Acceleration of point B is found by taking the time derivative of the velocity equation

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\frac{d\vec{v}_B}{dt} = \frac{d\vec{v}_A}{dt} + \frac{d}{dt}\left(\vec{v}_{B/A}\right)_{rel} + \frac{d}{dt}\left(\vec{\omega} \times \vec{r}_{B/A}\right) \qquad \Longrightarrow$$

$$\begin{aligned}
\vec{a}_B &= \vec{a}_A + \frac{d}{dt}\left(\vec{v}_{B/A}\right)_{rel} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \frac{d\vec{r}_{B/A}}{dt} \\
&= \vec{a}_A + \frac{d}{dt}\left(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \omega \times \vec{r}_{B/A}\right] \\
&= \vec{a}_A + \left(\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}\right) + \left(\dot{x}\underline{\frac{d\hat{i}}{dt}} + \dot{y}\underline{\frac{d\hat{j}}{dt}} + \dot{z}\underline{\frac{d\hat{k}}{dt}}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right] \\
&= \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \dot{x}\left(\vec{\omega} \times \hat{i}\right) + \dot{y}\left(\vec{\omega} \times \hat{j}\right) + \dot{z}\left(\vec{\omega} \times \hat{k}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right] \\
&= \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \underline{\vec{\omega}} \times \left(\vec{v}_{B/A}\right)_{rel} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right] \\
\vec{a}_B &= \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \underline{\vec{\alpha}} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \underbrace{\vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]}_{2D \to -\vec{\omega}^2 r \neq 3D}
\end{aligned}$$

← same as 2D

$\left(\vec{a}_{B/A}\right)_{rel} = \underline{\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}}$ → the acceleration of B as seen by the moving observer

$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$ → angular acceleration of the observer → change is how α, ω defined

$\neq \ddot{\theta}$

Considerations for determining angular acceleration of the rotating reference frame:

By definition, $\vec{\boldsymbol{\alpha}} = \frac{d\vec{\boldsymbol{\omega}}}{dt}$

In 2D problems, the moving reference frame rotates about a fixed axis (generally $\hat{K}$).

The time derivative of the unit vector therefore is:

$$\frac{d\hat{K}}{dt} = \vec{0}$$

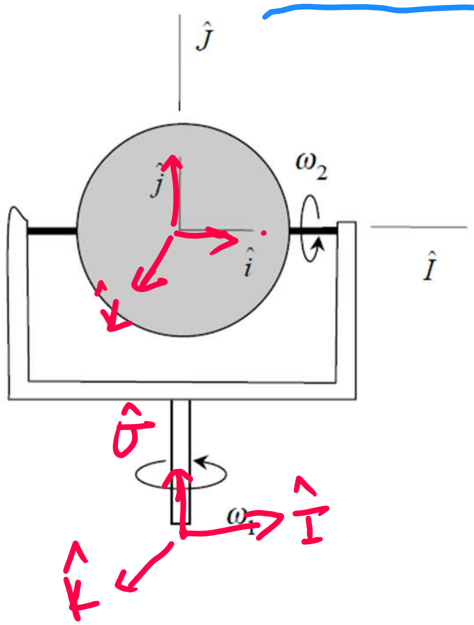
If the angular velocity of the moving reference frame is given by $\vec{\omega} = \Omega\hat{K}$ angular acceleration is found as:

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \dot{\Omega}\hat{K} + \Omega\cancel{\frac{d\hat{K}}{dt}} = 0 \quad \Rightarrow \vec{\alpha} = \dot{\Omega}\hat{K}$$

always for 2D

In 3D problems, the moving reference frame rotates about both fixed and moving axes for example:

$$\vec{\omega} = \omega_1\hat{J} + \omega_2\hat{i}$$



$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

$$\frac{d\vec{\omega}}{dt} = \dot{\omega}_1\hat{J} + \omega_1\cancel{\frac{d\hat{J}}{dt}} + \dot{\omega}_2\hat{i} + \omega_2\frac{d\hat{i}}{dt} \quad *$$

$$\frac{d\hat{J}}{dt} \rightarrow \text{fixed} = 0 \qquad \frac{d\hat{i}}{dt} = \vec{\omega}\times\hat{i}$$

$$\hat{J} = \hat{j}$$

$$\vec{\omega} \times \hat{i} = (\omega_1 \hat{j} + \omega_2 \hat{i})\hat{i} = \omega_1(\hat{j}\times\hat{i}) + \cancel{\omega_2(\hat{i}\times\hat{i})}^{0}$$

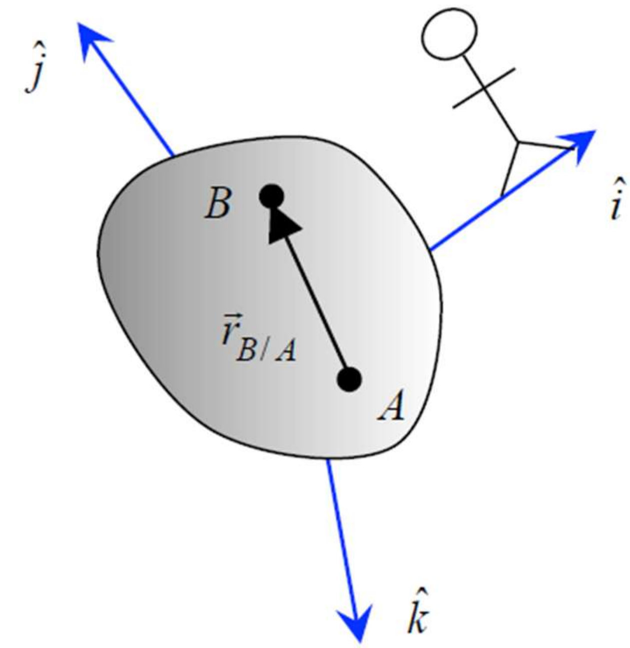
$$= \omega_1(\hat{j}\times\hat{i}) = -\omega_1 \hat{k} \; *$$

$$\underline{\vec{\alpha}} = \dot{\omega}_1 \hat{j} + \dot{\omega}_2 \hat{i} - \omega_2 \omega_1 \hat{k}$$

Challenge: in 2D if $\vec{\omega}$ = const, $\vec{\alpha} = 0$
true in 3D?

No! → in ex. if $\dot{\omega}_1, \dot{\omega}_2 = 0$ $\alpha = -\omega_2 \omega_1 \hat{k}$

3D Rigid Body Kinematics Equations



Remember on rigid bodies $|\vec{r}_{B/A}|=const.$ $\rightarrow$

$$(\vec{V}_{B/A})_{rel} = (\vec{a}_{B/A})_{rel} = \vec{0}$$

We can reduce our moving reference frame expressions for velocity and acceleration:

$$\vec{v}_B = \vec{v}_A + \cancel{(\vec{v}_{B/A})_{rel}}^{\,0} + \vec{\omega} \times \vec{r}_{B/A} \qquad \vec{V}_B = \vec{V}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \cancel{(\vec{a}_{B/A})_{rel}}^{\,0} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \cancel{(\vec{v}_{B/A})_{rel}}^{\,0} + \vec{\omega} \times [\vec{\omega} \times \vec{r}_{B/A}] \qquad \vec{a}_B = \vec{a}_A + \alpha \times r_{B/A} + \vec{\omega} \times \vec{\omega} \times r_{B/A}$$

eqns. same as 2D $\rightarrow$ change in now $\vec{\omega}, \vec{\alpha}$

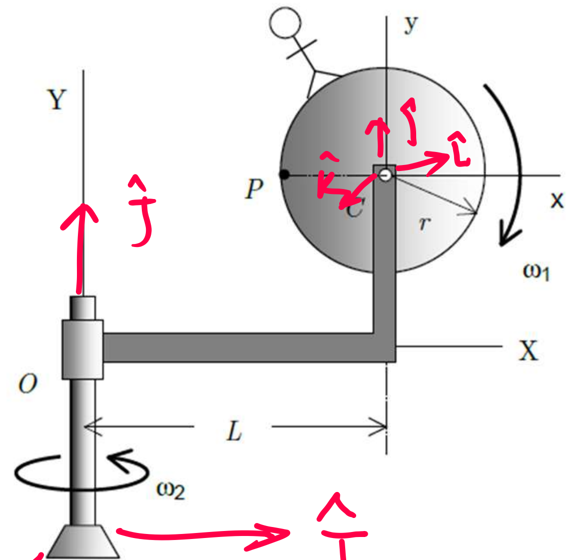
$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \ldots \qquad \vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

Example 3.B.3

Given: A disk rotates with a constant rate of $\omega_1 = 20$ rad/s with respect to the arm OC as the arm OC rotates about a fixed vertical axis with a constant rate of $\omega_2 = 5$ rad/s. The observer and the xyz axes are attached to the disk, while the XYZ axes are fixed. At this instant, the XYZ and xyz axes are aligned. → $\hat{i} = \hat{I}$ $\hat{j} = \hat{J}$ $\hat{k} = \hat{K}$

Find: Determine:

(a) The angular velocity of the observer at the instant shown; and
(b) The angular acceleration of the observer at the instant shown.



$$\vec{\omega} = \omega_1 \hat{k} + \omega_2 \hat{J}$$
$$= \omega_1 \hat{K} + \omega_2 \hat{J}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \cancel{\dot{\omega}_1 \hat{k}} + \omega_1 \frac{d\hat{k}}{dt} + \cancel{\dot{\omega}_2 \hat{J}} + \cancel{\omega_2 \frac{d\hat{J}}{dt}}$$

0 0

$$\vec{\alpha} = \omega_1 (\omega \times \hat{k})$$
$$= \omega_1 (\omega_1 K + \omega_2 J) \times \hat{k}$$
$$\hat{\alpha} = -\omega_1 \omega_2 \hat{i}$$

$\hat{k} \times \hat{k} = 0$

$\hat{J} = \hat{j}$

Example 3.B.4

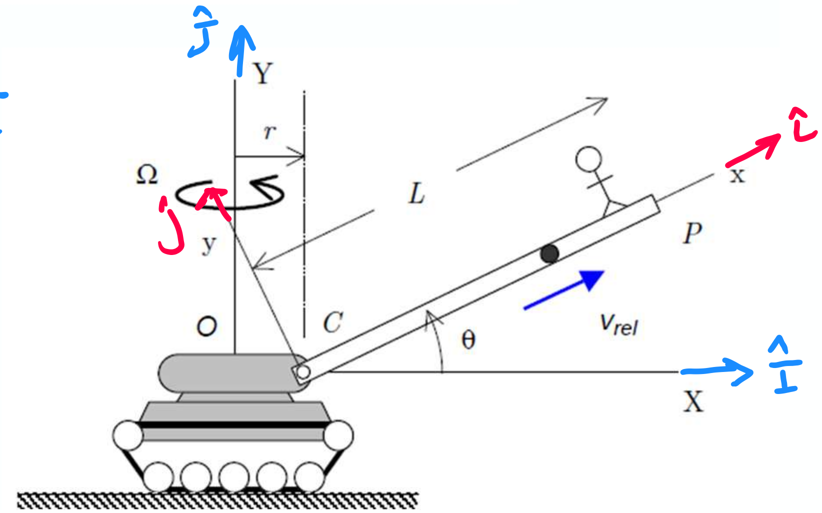
Given: The turret on a tank is rotating about a fixed vertical axis at a constant rate of $\Omega = 0.4$ rad/s. The barrel is being raised at a constant rate of $\dot{\theta} = 0.4$ rad/s . A cannon shell is fired with a constant muzzle speed of $v_{rel} = 200$ ft/s relative to the barrel. The observer and the xyz axes are attached to the barrel, while the XYZ axes are fixed. Here, $r = 3$ ft and $L = 15$ ft.

Find:

(a) The angular velocity of the barrel at the instant shown; $\vec{\omega}$

(b) The angular acceleration of the barrel at the instant shown; and $\vec{\alpha}$

(c) The acceleration of the shell as it leaves the barrel at P. $\vec{a}_P$



a) $\vec{\omega} = \Omega \hat{j} + \dot{\theta} \hat{k}$

$= \Omega \hat{J} + \dot{\theta} \hat{K}$

b) $\vec{\alpha} = \cancel{\dot{\Omega} \hat{J}} + \cancel{\Omega \frac{d\hat{J}}{dt}} + \cancel{\ddot{\theta} \hat{k}} + \dot{\theta} \frac{d\hat{k}}{dt}$

$= \dot{\theta} (\vec{\omega} \times \vec{k})$

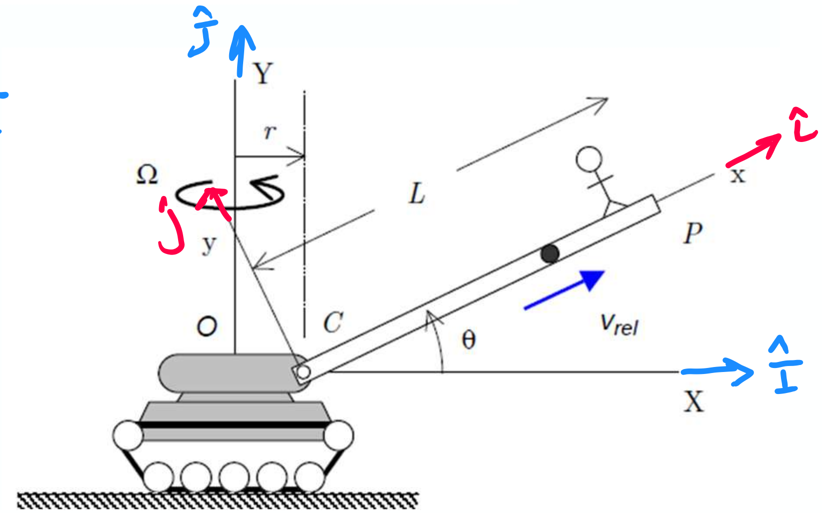
$= \dot{\theta} [(\Omega \hat{J} + \dot{\theta} \hat{k}) \times \hat{k}] = \dot{\theta} \Omega \hat{J} \times \hat{K} = \dot{\theta} \Omega \hat{I}$

Example 3.B.4

Given: The turret on a tank is rotating about a fixed vertical axis at a constant rate of $\Omega = 0.4$ rad/s. The barrel is being raised at a constant rate of $\dot{\theta} = 0.4$ rad/s . A cannon shell is fired with a constant muzzle speed of $v_{rel} = 200$ ft/s relative to the barrel. The observer and the xyz axes are attached to the barrel, while the XYZ axes are fixed. Here, $r = 3$ ft and $L = 15$ ft.

Find:

(a) The angular velocity of the barrel at the instant shown; $\vec{\omega}$
(b) The angular acceleration of the barrel at the instant shown; and $\vec{\alpha}$
(c) The acceleration of the shell as it leaves the barrel at P. $\vec{a}_P$



$$\vec{a}_P = \vec{a}_C + \cancel{(\vec{a}_{P/C})_{rel}}^{\,0} + \vec{\alpha}\times\vec{r}_{P/C} + 2\vec{\omega}\times\underbrace{(\vec{v}_{P/C})_{rel}}_{}{}^{v_{rel}\hat{\imath}} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{P/C})$$

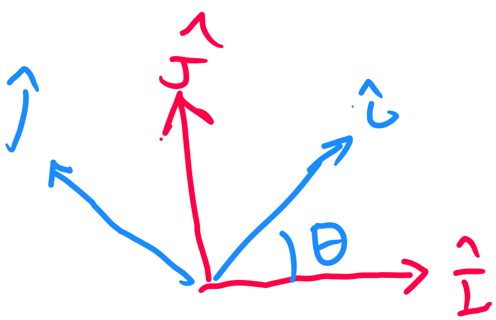
Rigid body OC:

$$\vec{a}_C = \cancel{\vec{a}_O} + \cancel{\vec{\alpha}_{OC}}\times\vec{r}_{C/O} + \omega_{CO}\times(\omega_{CO}\times r_{C/O})$$

$$= \Omega\hat{J}\times(\Omega\hat{J}\times r\hat{I}) = \Omega\hat{J}\times -\Omega r\hat{K} = -\Omega^2 r\,\hat{I}$$

$$\vec{a}_p = -\Omega^2 r \hat{I} + \dot{\theta}\Omega \hat{I} \times L\hat{i} + 2(\Omega\hat{J} + \dot{\theta}\hat{k}) \times V_{rel}\hat{i}$$
$$+ (\Omega\hat{J} + \dot{\theta}\hat{k}) \times [(\Omega\hat{J} + \dot{\theta}\hat{K}) \times L\hat{i}]$$

Convert to consistent vector system



$$\hat{i} = \cos\theta \hat{I} + \sin\theta \hat{J}$$
$$\hat{k} = \hat{K}$$

$$\vec{a}_p = -\Omega^2 r \hat{I} + \dot{\theta}\Omega \hat{I} \times L(\cos\theta \hat{I} + \sin\theta \hat{J})$$
$$+ 2(\Omega\hat{J} + \dot{\theta}\hat{K}) \times V_{rel}(\cos\theta \hat{I} + \sin\theta \hat{J})$$
$$+ (\Omega\hat{J} + \dot{\theta}\hat{K}) \times (\Omega\hat{J} + \dot{\theta}\hat{K}) \times (L\cos\theta \hat{I} + L\sin\theta \hat{J})$$

$$= -\Omega^2 r \hat{I} + \dot{\theta}\Omega L \sin\theta \hat{K} - 2\Omega V_{rel}\cos\theta \hat{K} + 2\dot{\theta} V_{rel}\cos\theta \hat{J}$$
$$- 2\dot{\theta}\sin\theta \hat{I} - (\Omega\hat{J} + \dot{\theta}\hat{K}) \times (-\Omega L\cos\theta \hat{K} + \dot{\theta} L\cos\theta \hat{J} - \dot{\theta} L\sin\theta \hat{I})$$

$$= -\Omega^2 r\,\hat{I} + \dot{\theta}\,\Omega L\sin\theta\,\hat{K} - 2\Omega v_{rel}\cos\theta\,\hat{K} + 2\dot{\theta} v_{rel}\cos\theta\,\hat{J}$$

$$-2\dot{\theta} v_{rel}\sin\theta\,\hat{I} - \Omega^2 L\cos\theta\,\hat{I} + \Omega\dot{\theta} L\sin\theta\,\hat{K} - \dot{\theta}^2 L\cos\theta\,\hat{I}$$

$$-\dot{\theta} L\sin\theta\,\hat{J}$$

ME 274: Basic Mechanics II

Lecture 14: Moving Reference Frame Kinematics -3D

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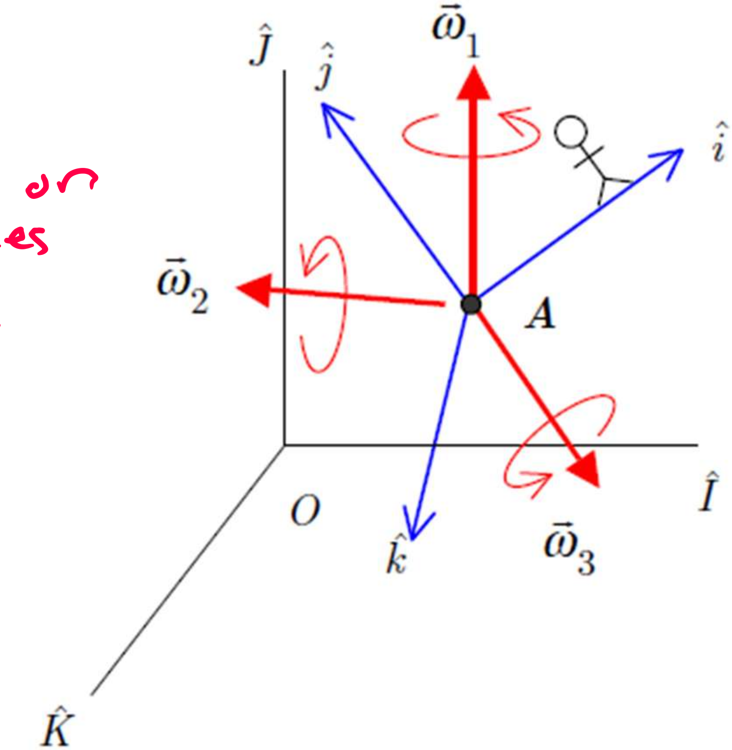
Announcements

- HW 2.E, 2.F **due tonight**
- **Reminder to use correct HW formatting!**
- Office hours ME 2008A: T 9:30-11, W 3:30-4:20, Th 12-1:30

3D Rotating Reference Frames

- 2D rotation about fixed axis $(\hat{k})$ → 3D rotating around a moving axis → not aligned w/ $\hat{i}\,\hat{j}\,\hat{k}$
- Multiple components each representing a different axis of rotation, $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \ldots$ → can include fixed or moving axes
- The total angular velocity of the observer is found by vector addition of all the components: can have 3-vectors
$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3 + \cdots$$
- Rotation can be about either a fixed or moving axis.
- Unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ change with time:



$$\frac{d\hat{\imath}}{dt} = \vec{\omega} \times \hat{\imath}, \qquad \frac{d\hat{\jmath}}{dt} = \vec{\omega} \times \hat{\jmath}, \qquad \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}$$

- Same velocity and acceleration equations as 2D!

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$
$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$$

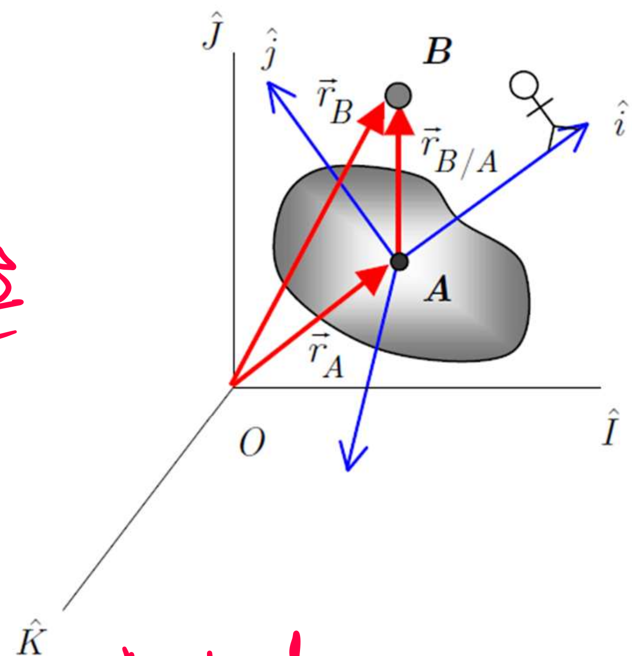
So what is different between 2D and 3D problems?

how α + ω are defined!
$\omega \neq \dot{\theta}\hat{k}$, $\alpha \neq \ddot{\theta}\hat{k}$

ω = constant
→ $\vec{\alpha} \neq \vec{0}$

Question C3.6

Consider an observer who is riding along on a moving (translating and rotating) rigid body. We wish to use the observation of this person in describing the motion of some point B, which is not fixed to the body, in the following moving reference frame acceleration equation.



Answer the following questions in words:

- What is the meaning of $\vec{\omega}$? total angular velocity of rotating reference
- What is the meaning of $\vec{\alpha}$?
  angular accel. of rotating ref. $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$
- What is the meaning of $(\vec{v}_{B/A})_{rel}$?
  velocity of B as observed from A
- What is the meaning of $(\vec{a}_{B/A})_{rel}$?
  accel. of B as observed from A.
- What restrictions, if any, are on the choice of point A?
  must be on the body that rot. ref. frame attached
- What is the difference in meaning between $\vec{a}_{B/A}$ and $(\vec{a}_{B/A})_{rel}$?

$\vec{a}_{B/A} \rightarrow$ rel. acceleration as seen from the fixed ref frame

$(a_{B/A})_{rel} \Rightarrow$ rel. acceleration as seen from moving reference

Question C3.4

The vertical shaft OA rotates about a fixed axis with a constant rate of $\Omega = 8$ rad/s. The arm AB is pinned to OA and is being raised at a constant rate of $\dot{\theta} = 10$ rad/s. An observer and xyz axes are attached to AB. The XYZ axes are stationary. What is the angular acceleration vector for arm AB when $\theta = 90°$?

$$\vec{\omega} = -\Omega \hat{J} + \dot{\theta} \hat{k}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \cancel{-\dot{\Omega}\hat{J}} \cancel{- \Omega \dot{\hat{J}}} + \cancel{\ddot{\theta}\hat{k}} + \dot{\theta}\dot{\hat{k}}$$

$$= \dot{\theta}\dot{\hat{k}} = \dot{\theta}(\vec{\omega} \times \hat{k})$$

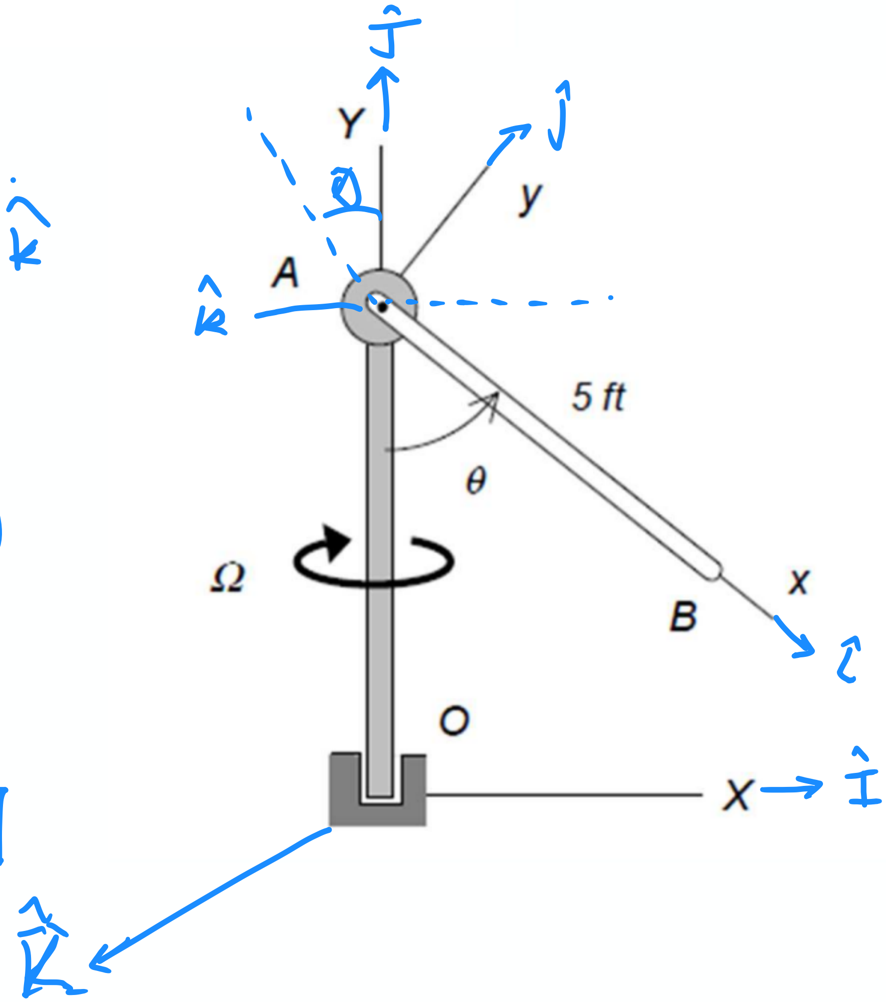
$$= \dot{\theta}\left[(-\Omega \hat{J} + \dot{\theta}\hat{k}) \times \hat{k}\right)$$

$$= \dot{\theta}(-\Omega \hat{J} \times \hat{k})$$

$$\hat{J} = -\cos\theta\, \hat{i} + \sin\theta\, \hat{j}$$

$$\vec{\alpha} = -\dot{\theta}\Omega\left[(-\cos\theta\, \hat{i} + \sin\theta\, \hat{j}) \times \hat{k}\right]$$

$$= -\dot{\theta}\Omega(\cos\theta\, \hat{j} + \sin\theta\, \hat{i})$$



Example 3.B.5

Given: The disk of a gyroscope rotates about its own axis at a constant rate of $\omega_2 = 600$ rev/min. The gimbal support is rotating at a constant rate of $\omega_1 = 10$ rad/s about a fixed vertical axis. The observer and the xyz axes are attached to the disk. The XYZ axes are fixed in space.

Find: Determine:

(a) The angular velocity of the observer at the instant shown; and
(b) The angular acceleration of the observer at the instant shown.

a) $\vec{\omega} = \omega_1 \hat{J} + \omega_2 \hat{i}$

$= \omega_1 \hat{j} + \omega_2 \hat{i}$

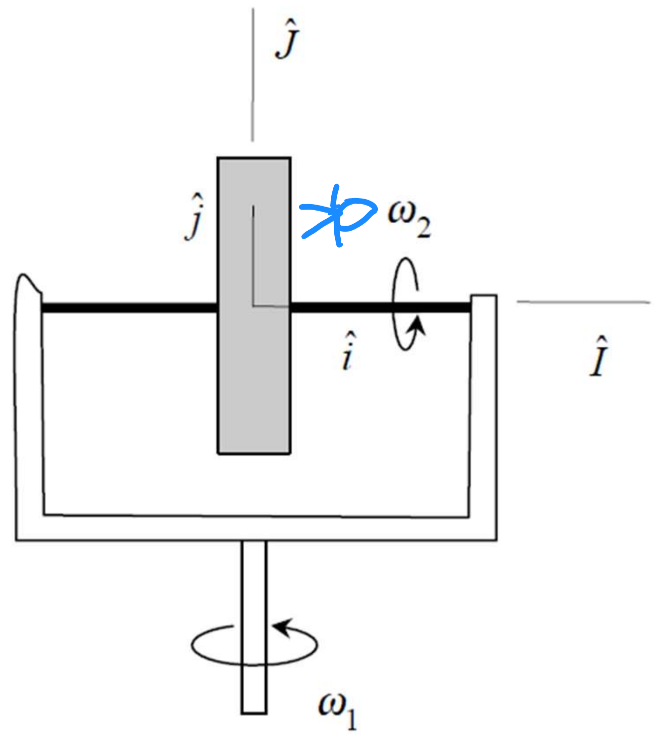
b) $\vec{\alpha} = \dot{\vec{\omega}} = \cancel{\dot{\omega}_1 \hat{J}} + \overset{0}{\cancel{\omega_1 \dot{\hat{J}} + \dot{\omega}_2 \hat{i}}} + \omega_2 \dot{\hat{i}}$

$\dot{\hat{i}} = \vec{\omega} \times \hat{i} = (\omega_1 \hat{J} + \omega_2 \hat{i}) \times \hat{i}$

$= -\omega_1 \hat{k}$

$\vec{\alpha} = -\omega_2 \omega_1 \hat{k}$

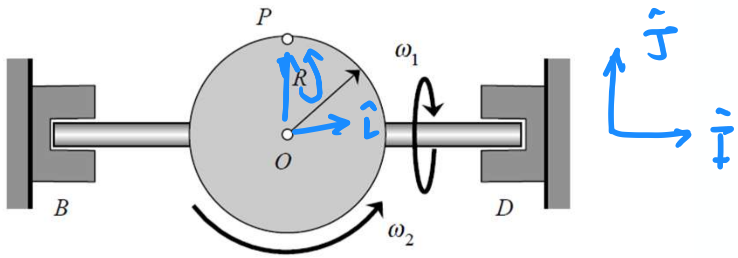
$\hat{J} = \hat{j}$



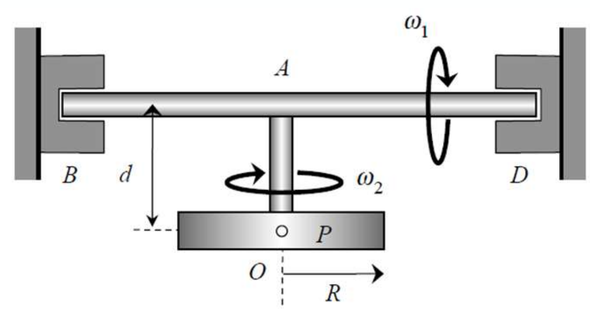
Example 3.B.6

Given: Shaft BD rotates about a fixed axis with a constant rate of ω_1. Shaft OA is rigidly attached to shaft BD with OA being perpendicular to BD. A disk rotates about shaft OA with a constant rate of ω_2 relative to OA.

Find: The acceleration of point P on the edge of the disk for the position shown.



side view



top view

$$\vec{\omega} = -\omega_1 \hat{I} + \omega_2 \hat{k}$$

$$\vec{\alpha} = \cancel{-\dot{\omega}_1 \hat{I}} \cancel{- \omega_1 \dot{\hat{I}}} + \cancel{\dot{\omega}_2 \hat{k}} + \omega_2 \dot{\hat{k}}$$

$$\vec{\alpha} = \omega_2 (\omega \times \hat{k})$$

$$= \omega_2 [(-\omega_1 \hat{I} + \omega_2 \hat{k}) \times \hat{k}]$$

$$= +\omega_1 \omega_2 \hat{j}$$

$$\vec{a}_P = \vec{a}_O + \cancel{(\vec{a}_{P/O})_{rel}} + \vec{\alpha} \times \vec{r}_{P/O} + \cancel{2\vec{\omega} \times (\vec{v}_{P/O})_{rel}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O})$$

$$\vec{a}_O = \cancel{\vec{a}_A} + \cancel{\vec{\alpha}_{OA} \times \vec{r}_{O/A}} + \vec{\omega}_{OA} \times (\vec{\omega}_{OA} \times \vec{r}_{O/A})$$

$$= -\omega_1 \hat{i} \times (-\omega_1 \hat{i} \times d \hat{K}) = \underline{-\omega_1^2 d \hat{K}}$$

$$\vec{a}_P = -\omega_1^2 d \hat{K} + \omega_1 \omega_2 \hat{j} \times R\hat{j} + (-\omega_1 \hat{i} + \omega_2 \hat{k}) \times [(-\omega_1 \hat{i} + \omega_2 \hat{k}) \times R\hat{j}]$$

$$= -\omega_1^2 d\,\hat{k} + (-\omega_1 \hat{i} + \omega_2 \hat{k}) \times (-\omega_1 R \hat{k} + \omega_2 R \hat{i})$$

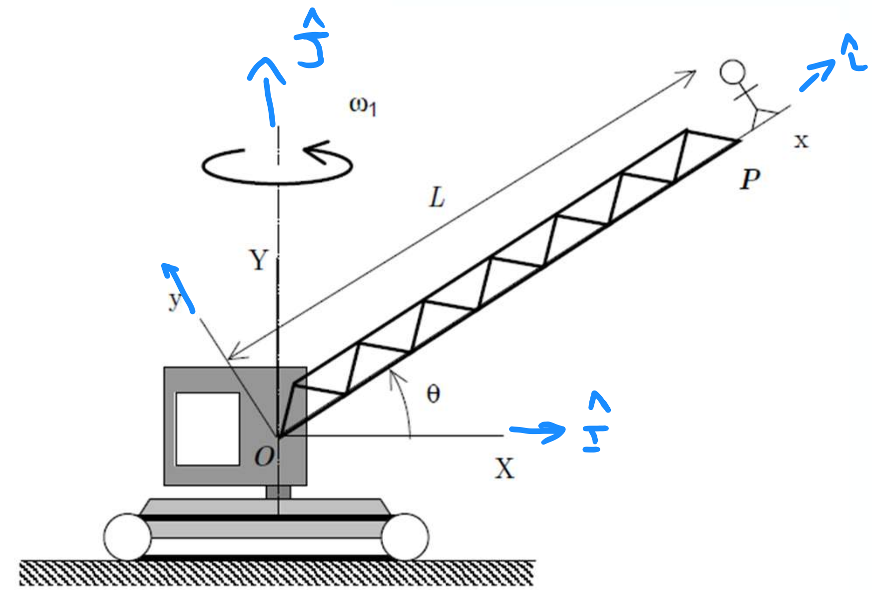
$$\vec{a}_p = -\omega_1^2 d\,\hat{k} - \omega_1^2 R \hat{j} - \omega_2^2 R \hat{j}$$

Example 3.B.7

Given: $\omega_1 = 0.30$ rad/s $=$ *constant*, $\dot{\theta} = 0.5$ rad/s $=$ *constant* and $L = 12$ m.

Find: When $\theta = 30°$, determine:

(a) The angular velocity of boom OP;
(b) The angular acceleration of boom OP;
(c) The velocity of end P of the boom; and
(d) The acceleration of end P of the boom.



$$\vec{\omega} = \omega_1 \hat{J} + \dot{\theta}\hat{k}$$

$$\omega_1 \hat{J} + \dot{\theta}\hat{K}$$

$$\vec{\alpha} = \cancel{\dot{\omega}_1 \hat{J} + \omega_1 \dot{\hat{J}} + \ddot{\theta}\hat{k} +} \dot{\theta}\dot{\hat{k}}$$

$$= \dot{\theta}(\vec{\omega} \times \hat{k}) = \dot{\theta}[(\omega_1 \hat{J} + \dot{\theta}\hat{k}) \times \hat{k}]$$

$$= \dot{\theta}\omega_1 (\hat{J} \times \hat{k})$$

$$= \dot{\theta}\omega_1 \hat{I}$$

$$\hat{K} = \hat{k}$$

$$\hat{\imath} = \cos\theta \hat{I} + \sin\theta \hat{J}$$

$$\vec{V}_p = \vec{V}_o + (\vec{v}_{p/o})_{rel} + \vec{\omega} \times \vec{r}_{p/o}$$

$$0 + 0 + (\omega_1 \hat{J} + \dot{\theta}\hat{k}) \times L\hat{\imath} = L(\omega_1 \hat{J} + \dot{\theta}\hat{K}) \times (\cos\theta \hat{I} + \sin\theta \hat{J})$$

$$= -L\omega_1 \cos\theta \hat{K} + L\dot{\theta}\cos\theta \hat{J} - L\dot{\theta}\sin\theta \hat{I}$$

$$\vec{a}_p = \cancel{\vec{a}_o} + \cancel{(\vec{a}_{p/o})_{rel}} + \vec{\alpha}\times\vec{r}_{p/o} \cancel{+ 2\vec{\omega}\times(\vec{v}_{p/o})_{rel}} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{p/o})$$

$$= \dot{\theta}\omega_1\hat{I}\times L\hat{i} + (\omega_1\hat{J}+\dot{\theta}\hat{K})\times[(\omega_1\hat{J}+\dot{\theta}\hat{K})\times L\hat{i}]$$

$$= \dot{\theta}\omega_1\hat{I}\times L(\cos\theta\hat{I}+\sin\theta\hat{J}) + (\omega_1\hat{J}+\dot{\theta}\hat{K})[(\omega_1\hat{J}+\dot{\theta}\hat{K})\times L(\cos\theta\hat{I}+\sin\theta\hat{J})]$$

$$= \dot{\theta}\omega_1 L\sin\theta\hat{K} + (\omega_1\hat{J}+\dot{\theta}\hat{K})(-\omega_1 L\cos\theta\hat{K} + \dot{\theta}L\cos\theta\hat{J} - \dot{\theta}L\sin\theta\hat{I})$$

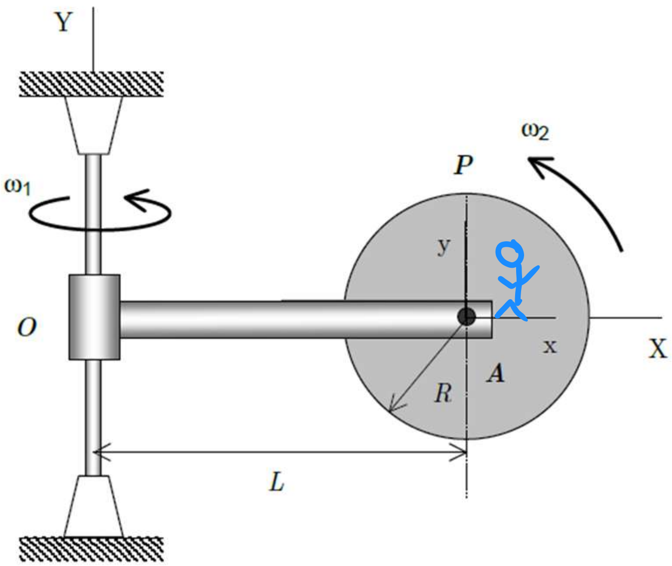
$$= \dot{\theta}\omega_1 L\sin\theta\hat{K} - \omega_1^2 L\cos\theta\hat{I} + \dot{\theta}\omega_1 L\sin\theta\hat{K} - \dot{\theta}^2 L\cos\theta\hat{I} - \dot{\theta}^2 L\sin\theta\hat{J}$$

Example 3.B.8

Find: Determine:

(a) The velocity of point P on the disk at the instant when P is directly above the center A of the disk; and

(b) The acceleration of point P at the same instant.



$\hat{K} = \hat{k}$

$\vec{\omega} = \omega_1 \hat{J} + \omega_2 \hat{k}$

$\omega_1 \hat{j} + \omega_2 \hat{k}$

$\vec{\alpha} = \cancel{\dot{\omega}_1 \hat{J}} + \cancel{\omega_1 \dot{\hat{J}}} + \cancel{\dot{\omega}_2 \hat{k}} + \omega_2 \dot{\hat{k}}$

$= \omega_2 (\vec{\omega} \times \hat{k})$

$= \omega_2 [(\omega_1 \dot{\hat{j}} + \omega_2 \hat{k}) \times \hat{k}] = \omega_1 \omega_2 \hat{i}$

velocity:

$\vec{V}_P = \vec{V}_A + (\vec{V}_{P/A})_{rel} + \vec{\omega} \times \vec{r}_{P/A}$

$= -\omega_1 L \hat{k} + (\omega_1 \hat{j} + \omega_2 \hat{k}) \times R \hat{j}$

$= -\omega_1 L \hat{k} - \omega_2 R \hat{i}$

rigid body eqn for V_A

$V_A = V_O + \omega_{arm} \times r_{A/O}$

$= \omega_1 \hat{J} \times L \hat{I} = -\omega_1 L \hat{K}$

$\hat{K} = \hat{k}$

accel:

$$\vec{a}_p = \vec{a}_A + \cancel{(\vec{a}_{p/A})_{rel}} + \vec{\alpha} \times \vec{r}_{p/A} + 2\vec{\omega} \times \cancel{(\vec{v}_{p/A})_{rel}} + \omega \times (\omega \times r_{p/A})$$

rigid body for a_A

$$a_A = \cancel{a_o} + \cancel{\alpha_{OA}} \times r_{A/O} + \vec{\omega}_{AO} \times (\underbrace{\vec{\omega}_{AO}}_{\omega_1 \hat{j}} \times \underbrace{\vec{r}_{A/O}}_{L\hat{i}}) = -\omega_1^2 L \hat{i}$$

$$\vec{a}_p = -\omega_1^2 L\hat{i} + \omega_1\omega_2 \hat{i} \times R\hat{j} + (\omega_1\hat{j} + \omega_2\hat{k}) \times [(\omega_1\hat{j} + \omega_2\hat{k}) \times R\hat{j}]$$

$$= -\omega_1^2 L\hat{i} + \omega_1\omega_2 R\hat{k} + \omega_1\omega_2 R\hat{k} - \omega_2^2 R\hat{j}$$

ME 274: Basic Mechanics II

Lecture 15: Moving Reference Frame Kinematics -3D

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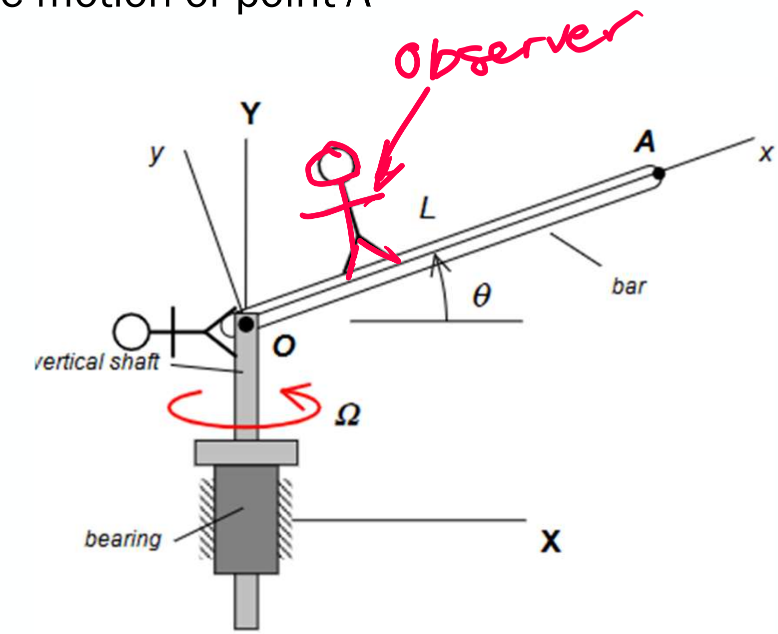
3D Moving Reference Frame Kinematics – Changing observers

Regardless of where you place your observer/ moving reference frame you should arrive at the same final expression for velocity and acceleration.

Problem: A person attached to a moving body is observing the motion of point A

$$\vec{v}_A = \vec{v}_O + \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\vec{r}_{A/O}$$

$$\vec{a}_A = \vec{a}_O + \left(\vec{a}_{A/O}\right)_{rel} + \vec{\alpha}\times\vec{r}_{A/O} + 2\vec{\omega}\times\left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{A/O}\right)$$



Observer on arm OA:

$$\vec{\omega} = \Omega\hat{J} + \dot{\theta}\hat{k}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \dot{\Omega}\hat{J} + \Omega\dot{\hat{J}} + \ddot{\theta}\hat{k} + \dot{\theta}\dot{\hat{k}} = \dot{\theta}\left(\vec{\omega}\times\hat{k}\right)$$

$$\left(\vec{v}_{A/O}\right)_{rel} = \vec{0}$$

$$\left(\vec{a}_{A/O}\right)_{rel} = \vec{0}$$

$\hat{k} = \hat{K}$

$r_{A/O} = L(\cos\theta\,\hat{I} + \sin\theta\,\hat{J})$

$\vec{V}_o = 0$, $\vec{r}_{A/O} = L\hat{i}$

$\vec{V}_A = 0 + 0 + (\Omega\hat{J} + \dot{\theta}\hat{K})\times(L\hat{i})$

$= (\Omega\hat{J} + \dot{\theta}\hat{K})\times L(\cos\theta\,\hat{I} + \sin\theta\,\hat{J})$

$$\boxed{\vec{V}_A = -\Omega L\cos\theta\,\hat{K} + \dot{\theta}L\cos\theta\,\hat{J} - \dot{\theta}L\sin\theta\,\hat{I}}$$

$$\vec{a}_A = \dot{\theta}(\vec{\omega}\times\hat{K})\times(L\hat{\imath}) \rightarrow (\Omega\hat{J}+\dot{\theta}\hat{K})\times[(\Omega\hat{J}+\dot{\theta}\hat{K})\times(L\hat{\imath})]$$

$$(\vec{\omega}\times\hat{K}) = (\Omega\hat{J}+\dot{\theta}\hat{K})\times\hat{K} = \Omega\hat{I}$$

$$= \dot{\theta}\Omega\hat{I}\times L(\cos\theta\hat{I}+\sin\theta\hat{J}) + (\Omega J+\dot{\theta}\hat{K})\times[(\Omega\hat{J}+\dot{\theta}\hat{K})\times L(\cos\theta\hat{I}+\sin\theta\hat{J})]$$

$$= \dot{\theta}\Omega L\sin\theta K + (\Omega J+\dot{\theta}\hat{K})\times(-\Omega L\cos\theta\hat{K} - \dot{\theta}L\cos\theta\hat{J} - \dot{\theta}L\sin\theta\hat{I})$$

$$= \dot{\theta}\Omega L\sin\theta\hat{K} - \Omega^2 L\cos\theta\hat{I} + \dot{\theta}\Omega L\sin\theta\hat{K} - \dot{\theta}^2 L\cos\theta\hat{I} - \dot{\theta}^2 L\sin\theta\hat{J}$$

$$\vec{a}_A = (-\Omega^2 L\cos\theta - \dot{\theta}^2 L\cos\theta)\hat{I} - \dot{\theta}^2 L\sin\theta\hat{J} + 2\dot{\theta}\Omega L\sin\theta\hat{K}$$

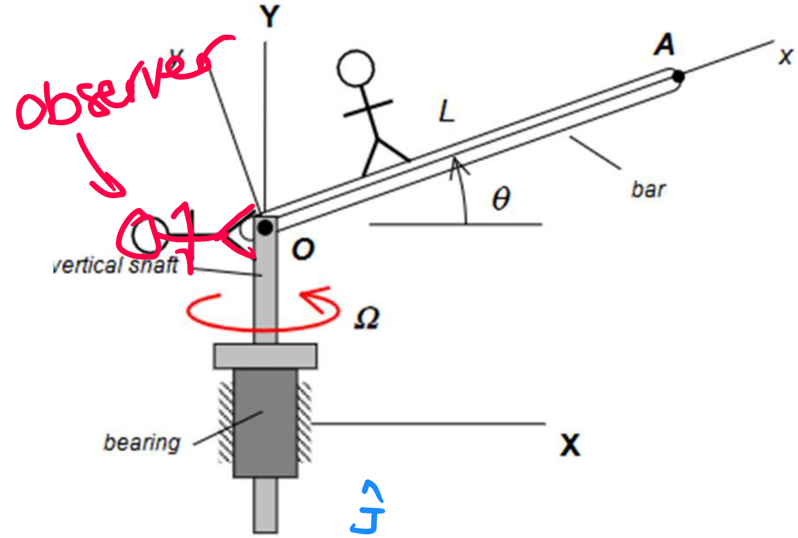
3D Moving Reference Frame Kinematics – Changing observers

Regardless of where you place your observer/ moving reference frame you should arrive at the same final expression for velocity and acceleration.

Problem: A person attached to a moving body is observing the motion of point A

$$\vec{v}_A = \vec{v}_O + \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\vec{r}_{A/O}$$

$$\vec{a}_A = \vec{a}_O + \left(\vec{a}_{A/O}\right)_{rel} + \vec{\alpha}\times\vec{r}_{A/O} + 2\vec{\omega}\times\left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{A/O}\right)$$



Observer on vertical shaft:

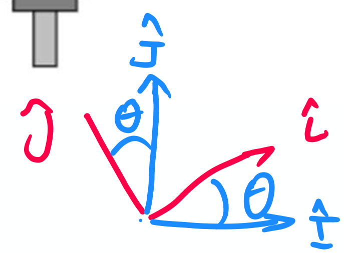
$$\vec{\omega} = \Omega\hat{J}$$

$$r_{A/O} = L(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \vec{0}$$

$$\left(\vec{v}_{A/O}\right)_{rel} = L\dot{\theta}\hat{j}$$

$$\left(\vec{a}_{A/O}\right)_{rel} = -L\dot{\theta}^2\hat{i}$$



$$\vec{V}_A = \vec{0} + L\dot{\theta}\hat{j} + \Omega\hat{J}\times L(\cos\theta\,\hat{I} + \sin\theta\,\hat{J})$$

$$= L\dot{\theta}\hat{j} - \Omega L\cos\theta\,\hat{K} \qquad \hat{j} = -\sin\theta\,\hat{I} + \cos\theta\,\hat{J}$$

$$\vec{V}_A = -L\dot{\theta}\sin\theta\,\hat{I} + L\dot{\theta}\cos\theta\,\hat{J} - \Omega L\cos\theta\,\hat{K}$$

$$\boxed{\vec{V}_A = -\Omega L\cos\theta\,\hat{K} + \dot{\theta}L\cos\theta\,\hat{J} - \dot{\theta}L\sin\theta\,\hat{I}}$$

✓ matches other case

$$\vec{a}_A = 0 - L\dot{\theta}^2\hat{i} + 0 + 2\Omega\hat{J}\times(L\dot{\theta}\hat{j}) + \Omega\hat{J}\times[\Omega\hat{J}\times(L\cos\theta\hat{I} + L\sin\theta\hat{J})]$$

$$= -L\dot{\theta}^2(\cos\theta\hat{I} + \sin\theta\hat{J}) + 2\Omega\hat{J}\times[L\dot{\theta}(-\sin\theta\hat{I} + \cos\theta\hat{J})] + \Omega\hat{J}\times[\Omega\hat{J}\times(L\cos\theta\hat{I} + L\sin\theta\hat{J})]$$

$$= -L\dot{\theta}^2\cos\theta\hat{I} - L\dot{\theta}^2\sin\theta\hat{J} + 2L\Omega\dot{\theta}\sin\theta\hat{K} + \Omega\hat{J}\times(-\Omega L\cos\theta\hat{K})$$

$$= -L\dot{\theta}^2\cos\theta\hat{I} - L\dot{\theta}^2\sin\theta\hat{J} + 2L\Omega\dot{\theta}\sin\theta\hat{K} - \Omega^2 L\cos\theta\hat{I}$$

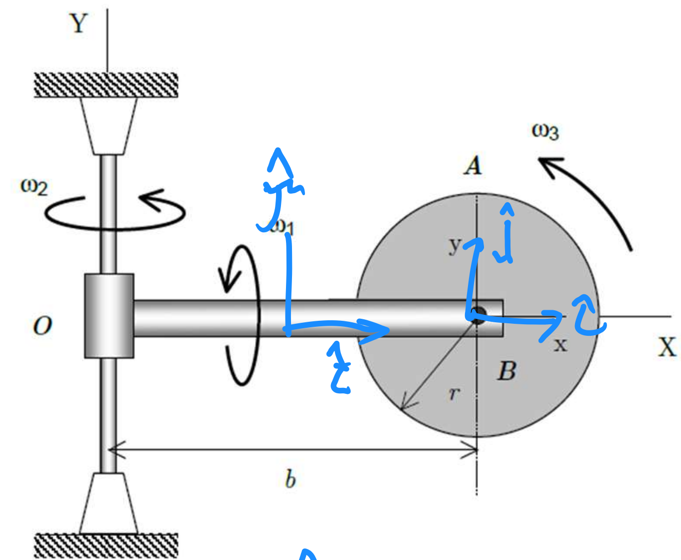
$$\vec{a}_A = (-L\dot{\theta}^2\cos\theta - \Omega^2 L\cos\theta)\hat{I} - L\dot{\theta}^2\sin\theta\hat{J} + 2L\Omega\dot{\theta}\sin\theta\hat{K}$$

$$\vec{a}_A = (-\Omega^2 L\cos\theta - \dot{\theta}^2 L\cos\theta)\hat{I} - \dot{\theta}^2 L\sin\theta\hat{J} + 2\dot{\theta}\Omega L\sin\theta\hat{K}$$

Example 3.B.11

Given: Rotation rates ω_1, ω_2 and ω_3 are all constant. The XYZ axes are fixed, and the xyz axes are attached to the disk. At the instant shown, A is directly above the center B of the disk and the xyz and XYZ axes are aligned.

Find: Determine:
(a) The velocity of point A; and
(b) The acceleration of point A.



$\hat{i}\ \hat{j}\ \hat{k} \rightarrow$ attached to disk

$\hat{i}\ \hat{j}\ \hat{k} \rightarrow$ attached to OB

$\hat{I}\ \hat{J}\ \hat{K} \rightarrow$ fixed

rotation of $\hat{i}\ \hat{j}\ \hat{k}$

$\vec{\omega}_{OB} = \omega_2 \hat{J} - \omega_1 \hat{i}$

$\vec{\alpha}_{OB} = \cancel{\dot{\omega}_2 \hat{J}} + \cancel{\omega_2 \dot{\hat{J}}} - \cancel{\dot{\omega}_1 \hat{i}} - \omega_1 \dot{\hat{i}}$

$\vec{\alpha}_{OB} = -\omega_1 (\vec{\omega}_{OB} \times \hat{i})$

rotation of $\hat{i}\ \hat{j}\ \hat{k}$

$\vec{\omega}_{AB} = \omega_2 \hat{J} - \omega_1 \hat{i} - \omega_3 \hat{k}$

$\alpha_{AB} = \cancel{\dot{\omega}_2 \hat{J} + \omega_2 \dot{\hat{J}} - \dot{\omega}_1 \hat{i}} - \omega_1 \dot{\hat{i}} - \cancel{\dot{\omega}_3 \hat{k}} - \omega_3 \dot{\hat{k}}$

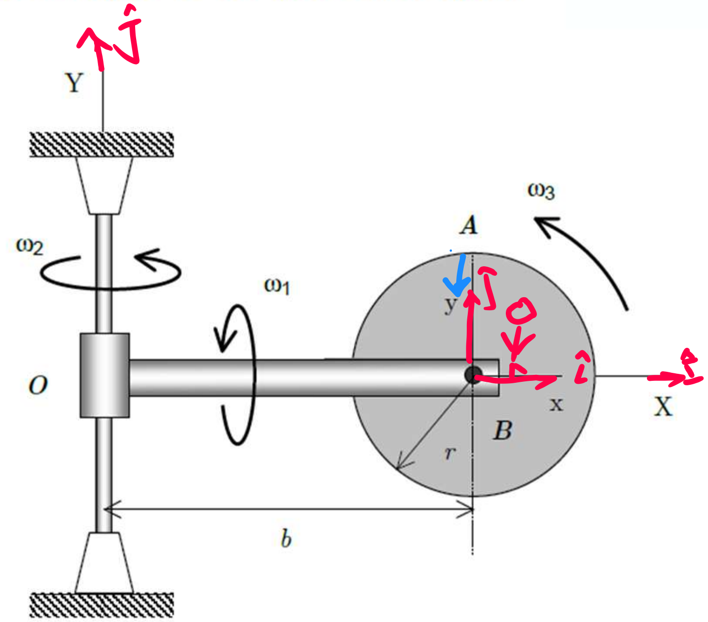
$= \omega_1 (\vec{\omega}_{OB} \times \hat{i}) - \omega_3 (\vec{\omega}_{AB} \times \hat{k})$

$= \omega_1 [(\omega_2 \hat{J} - \omega_1 \hat{i}) \times \hat{i}] - \omega_3 [(\omega_2 \hat{J} + \omega_1 \hat{i} - \omega_3 \hat{k}) \times \hat{k}]$

Example 3.B.11

Given: Rotation rates ω_1, ω_2 and ω_3 are all constant. The XYZ axes are fixed, and the xyz axes are attached to the ~~disk~~. At the instant shown, A is directly above the center B of the disk and the xyz and XYZ axes are aligned.

arm OB



Find: Determine:

(a) The velocity of point A; and

(b) The acceleration of point A.

$\vec\omega$ & $\vec\alpha$

at this instant $\hat i = \hat I$

$$\vec\omega = \omega_2 \hat J - \omega_1 \hat i$$

$$= \omega_2 \hat J - \omega_1 \hat I$$

$$\vec\alpha = \cancel{\dot\omega_2 \hat J + \omega_2 \dot{\hat J} - \dot\omega_1 \hat i} - \omega_1 \dot{\hat i}$$

$$= -\omega_1 (\vec\omega \times \hat i) = -\omega_1 [(\omega_2 \hat J - \omega_1 \hat i) \times \hat i]$$

$$= \omega_1 \omega_2 \hat K$$

a) velocity of A

$$\vec V_A = \vec V_B + (\vec V_{A/B})_{rel} + \vec\omega \times \vec r_{AB}$$

rigid Body for $V_B \rightarrow$ $\vec V_B = \cancel{\vec V_0} + \vec\omega \times \vec r_{B/0}$

$$= \omega_2 \hat J \times b \hat I = -\omega_2 b \hat K$$

$$\vec{V}_A = -\omega_2 b\,\hat{K} + (-r\omega_3\,\hat{i}) + (\omega_2\hat{J} - \omega_1\hat{i}) \times r\hat{j}$$

$$= -\omega_2 b\hat{K} - r\omega_3\hat{I} - \omega_1 r\hat{K}$$

$$\boxed{= -r\omega_3\hat{I} + (-\omega_1 r - \omega_2 b)\hat{K}}$$

b) acceleration of A

$$\vec{a}_A = \vec{a}_B + (\vec{a}_{A/B})_{rel} + \vec{\alpha} \times \vec{r}_{A/B} + 2\vec{\omega} \times (\vec{V}_{A/B})_{rel} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/B})$$

rigid body for $\vec{a}_B$

$$\vec{a}_B = \vec{a}_0 + \vec{\alpha}_{BO} \times \vec{r}_{B/O} + \vec{\omega}_{BO} \times (\vec{\omega}_{BO} \times r_{B/O})$$

$$= \vec{0} + \vec{0} + \omega_2\hat{J} \times (\omega_2\hat{J} \times b\hat{i}) = -\omega_2^2 b\hat{I}$$

$$\vec{a}_A = -\omega_2^2 b\hat{I} - \omega_3^2 r\hat{j} + \omega_1\omega_2\hat{K} \times r\hat{j} + 2(\omega_2\hat{J} - \omega_1\hat{i}) \times (-r\omega_3\hat{i})$$
$$+ (\omega_2\hat{J} - \omega_1\hat{i}) \times [(\omega_2\hat{J} - \omega_1\hat{i}) \times r\hat{j}]$$

$$= -\omega_2^2 b \hat{I} - \omega_3^2 r \hat{J} - \omega_1 \omega_2 r \hat{I} + 2\omega_2 \omega_3 r \hat{K}$$

$$+ (\omega_2 \hat{J} - \omega_1 \hat{I}) \times [-\omega_1 r \hat{K}]$$

$$= -\omega_2^2 b \hat{I} - \omega_3^2 r \hat{J} - \omega_1 \omega_2 r \hat{I} + 2\omega_2 \omega_3 r \hat{K} - \omega_1 \omega_2 r \hat{I} - \omega_1^2 r \hat{J}$$

$$\boxed{\vec{a}_A = (-\omega_2^2 b - 2\omega_1 \omega_2 r)\hat{I} + (-\omega_3^2 r - \omega_1^2 r)\hat{J} + 2\omega_2 \omega_3 r \hat{K}}$$