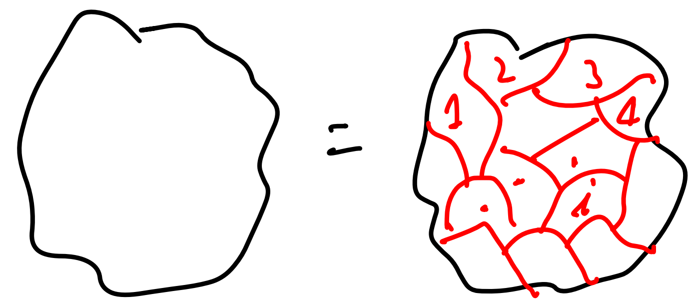
3/27/2026

Newton-Euler Equations

** Chapter 5.A of the Lecturebook*
** Chapter 6.A of Meriam Textbook 7th Edition.*

What is a rigid body?



$m = \sum_i m_i$

Forces and Moments

From Newton's Law and Angular Momentum Equation:

$\vec{F}_i = m_i \vec{a}_i$

$\vec{M}_{Oi} = \dfrac{d}{dt}[\vec{r}_{i/O} \times (m_i \vec{v}_i)]$

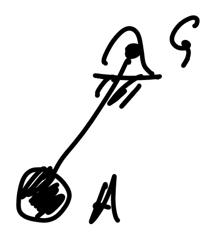
Consider that a body can be recongnized as a combination of particles:

Forces: $\vec{F} = \sum m_i \vec{a}_i \Rightarrow \vec{F} = m \vec{a}_G$

Moments: $\vec{M}_A = \underbrace{\dfrac{d}{dt}\left[\sum_i \vec{r}_{i/A} \times m_i \vec{V}_{i/A}\right]}_{\text{from volume of R.B.}} + \underbrace{\vec{r}_{G/A} \times m \vec{V}_A}_{\text{if RB is a particle @ A}}$

Euler: $\vec{M}_A = \dfrac{d}{dt}[I_A \vec{\omega}] + \underbrace{\vec{r}_{G/A} \times m \vec{V}_A}$

$\Rightarrow \vec{M}_A = I_A \vec{\alpha}$



Newton-Euler Equations

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Forces and Moments

Special Form of Euler's Equation:

$\vec{M}_A = I_A\vec{\alpha} + \vec{r}_{G/A} \times (m\vec{a}_A)$

- When $\vec{r}_{G/A} = 0$:

A : center of gravity

* Always a safe choice since G always exists for any R.B.

- When $\vec{a}_A = 0$:

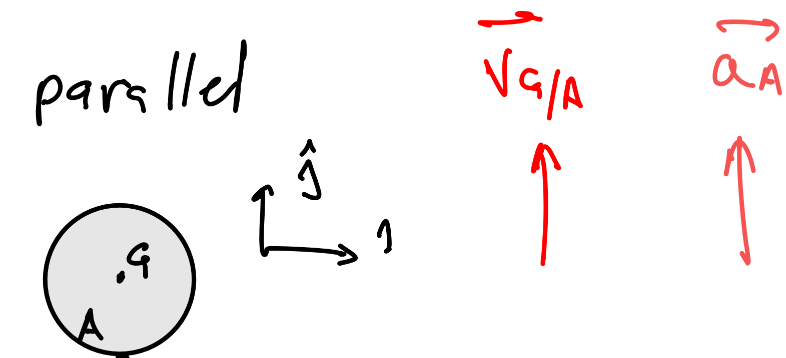
A fixed point

$\vec{a}_A = \vec{0}$

- When $\vec{r}_{G/A} \times (m\vec{a}_A) = 0$ with none zero terms:

$\vec{r}_{G/A}$ and $\vec{a}_A$ are parallel

* Rolling w/o Slipping



$\vec{a}_A$ only in $\hat{j}$ $\qquad$ $\vec{r}_{G/A}$ in $\hat{j}$

Mass Moment of Inertia

When shape is regular, use formula. When shape is not regular, use Radius of Gyration k:

$$I_G = mk_G^2$$

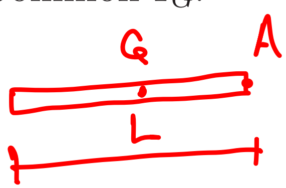
Note: I_G is calculated at the center of gravity.

If I is needed for a different point of the body, use parallel Axis Theorem:

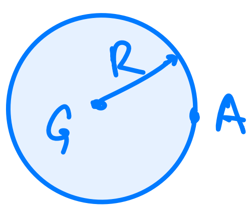
$$I_A = I_G + md_{AG}^2$$

with d_{AG} being the distance between two parallel axis.

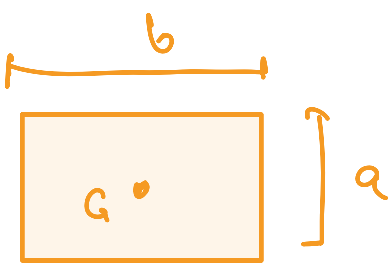
Common I_G:



$I_G = \frac{1}{12} m L^2$, $I_A = \frac{1}{12} m L^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{3} m L^2$



$I_G = \frac{1}{2} m R^2$, $I_A = \frac{1}{2} m R^2 + m R^2 = \frac{3}{2} m R^2$



$I_G = \frac{1}{12}(a^2 + b^2)$

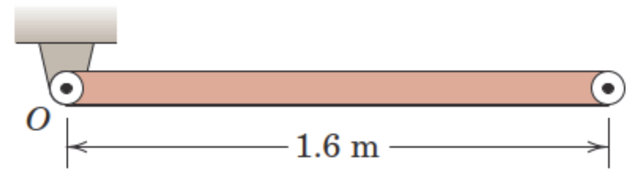
Main Method

1. Consider Coordinate System
2. Draw F.B.D. (and K.D. if using Newton's Law Method)
3. Consider Kinetics Equations:
   - Newton-Euler Equations:
     $\vec{F} = m\vec{a}_G$

     $\vec{M}_A = I_A\vec{\alpha}$ (for applicable points)
4. Set up equations based on Kinematics to solve for all unknowns
5. Solve

Practice Problem

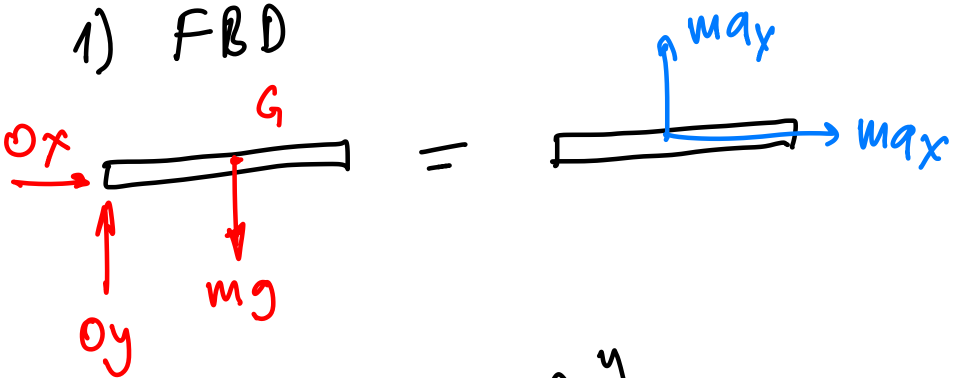
6/33 The uniform 20-kg slender bar is pivoted at O and swings freely in the vertical plane. If the bar is released from rest in the horizontal position, calculate the initial value of the force R exerted by the bearing on the bar an instant after release.



Problem 6/33

Solution attempt

1) FBD



2) Coord

3) Kinetics

x: $\Sigma F_x = m a_x \Rightarrow O_x = m a_x$ ① — 5 unknowns

y: $\Sigma F_y = m a_y \Rightarrow O_y - mg = m a_y$ ②

$(+\circlearrowleft)$ M_O: $\Sigma M_O = I_O \alpha \Rightarrow -\frac{L}{2} mg = I_O \alpha$ ③

4) Kinematics

$$\vec{a}_G = \cancel{\vec{a}_O}^{\,0} + \vec{\alpha} \times \vec{r}_{G/O} - \cancel{\omega^2 \vec{r}_{G/O}}^{\,0}$$

0 (rFr)

Practice Problem

$$\vec{a}_G = \alpha\hat{k} \times \left(\frac{L}{2}\hat{i}\right)$$

$$a_{G_x}\hat{i} + a_{G_y}\hat{j} = \alpha\frac{L}{2}\hat{j}$$

$\hat{i}$ → $\hat{j}$

$$a_{G_x} = 0 \quad ④$$

$$a_{G_y} = \alpha\frac{L}{2} \quad ⑤$$

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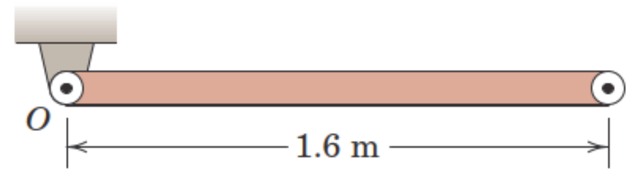


Problem 6/33

5) Solve

④ in ① $O_x = 0$

⑤ in ② $O_y = mg + m\alpha\frac{L}{2} \Rightarrow O_y = mg - \frac{3}{2}\frac{mg}{L}\cdot\frac{L}{2}$

$O_y = \frac{1}{4}mg$

③ $-\frac{1}{2}mg = \frac{1}{3}mL^2\alpha$

$\alpha = -\frac{3}{2}\frac{g}{L}$

$\vec{\alpha} = -\frac{3}{2}\frac{g}{L}\hat{k}$ rad/s^2 , $\vec{R}_O = \frac{1}{4}mg\hat{j}$ N

MASS MOMENT OF INERTIA.

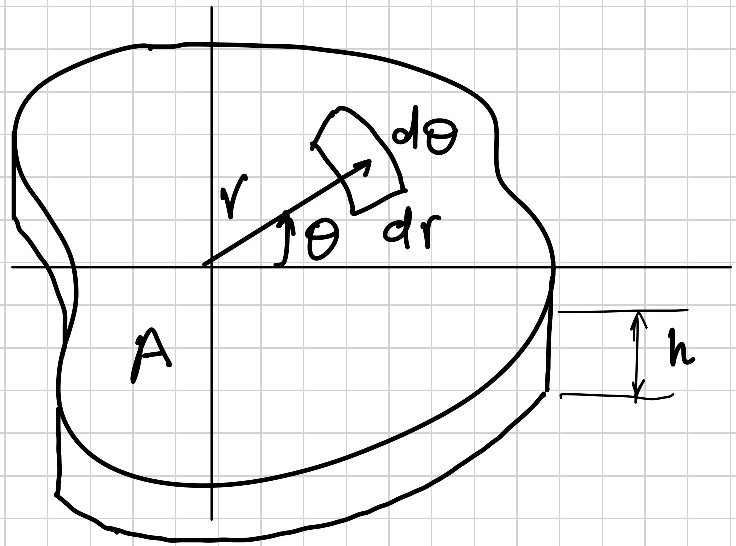
$I_A = \int r^2 dm$, where $dm = \rho dV$

$\Rightarrow \quad I_A = \int \rho r^2 dV$ (similar to polar area of moment in ME270)

POLAR:

$V = h\, r\, dr\, d\theta$

$\Rightarrow \quad I_A = \int\int \rho h r^3 dr d\theta$

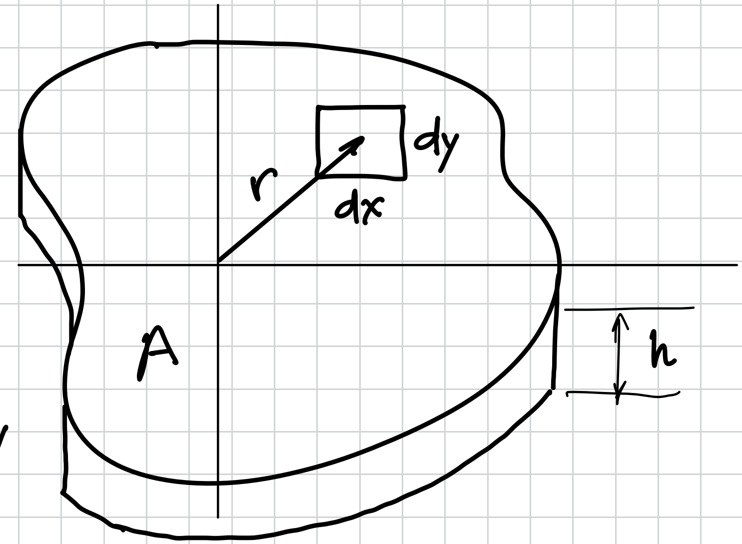


CARTESIAN

$$dV = h\,dx\,dy$$

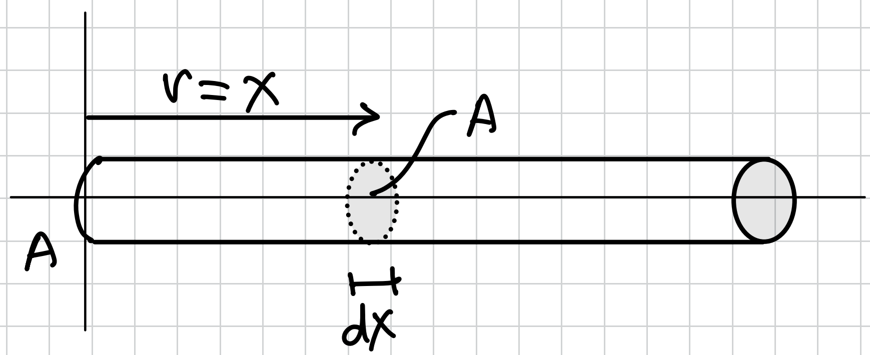
$$r = \sqrt{x^2+y^2}$$

$$I_A = \iint \rho h (x^2+y^2)\,dx\,dy$$



SLENDER BAR

$$I_A = \int \rho A x^2\,dx$$



MOMENT OF INERTIA.

when shape is regular, use table. when shape is irregular, use radius of gyration k_G. ← always applied to center of gravity

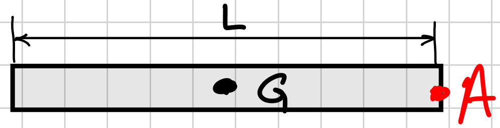
$$I_G = m k_G^2$$

if need to adjust for a point other than G, use parallel axis theorem

$$I_A = I_G + m d_{AG}^2$$

COMMON I_G

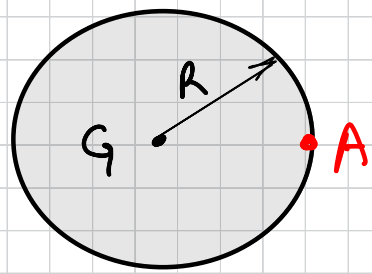
ROD



$$I_G = \frac{1}{12} m L^2$$

$$I_A = \frac{1}{12} m L^2 + m \left(\frac{L}{2}\right)^2$$
$$= \frac{1}{3} m L^2$$

DISK

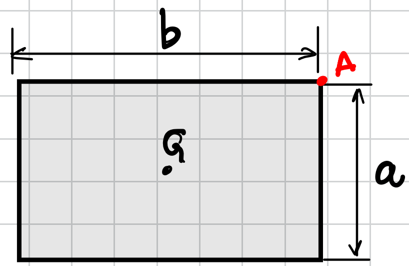


$$I_G = \frac{1}{2} m R^2$$

$$I_A = \frac{1}{2} m R^2 + m R^2$$

$$= \frac{3}{2} m R^2$$

RECTANGULAR PLATE

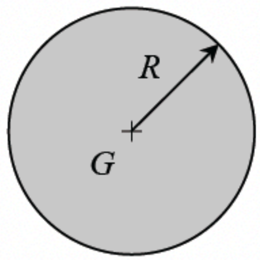


$$I_G = \frac{1}{12} (a^2 + b^2)$$

$$I_A = \frac{1}{12} (a^2 + b^2) + \frac{1}{4} m (a^2 + b^2)$$

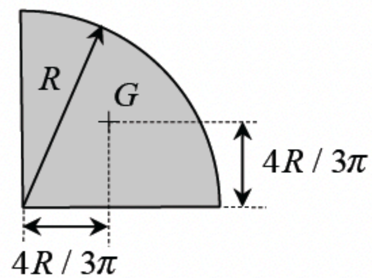
$$I_A = \frac{1}{3} m (a^2 + b^2)$$

Uniform disk of mass m



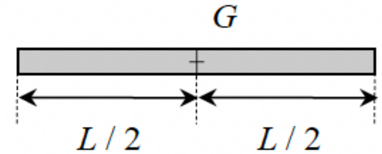
$$I_G = \frac{1}{2} m R^2$$

Quarter circular plate of mass m



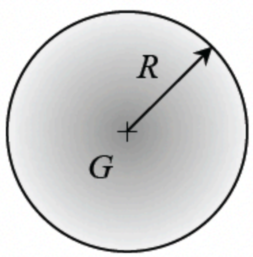
$$I_G = \left(\frac{9\pi^2 - 64}{18\pi^2} \right) m R^2$$

Uniform thin bar of mass m



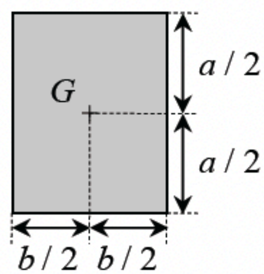
$$I_G = \frac{1}{12} m L^2$$

Uniform sphere of mass m



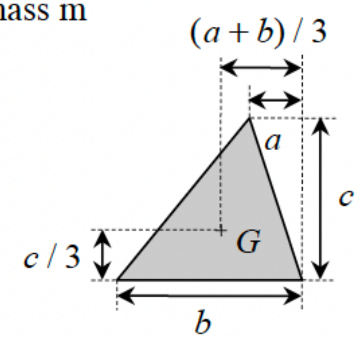
$$I_G = \frac{2}{5} m R^2$$

Uniform rectangular plate of mass m

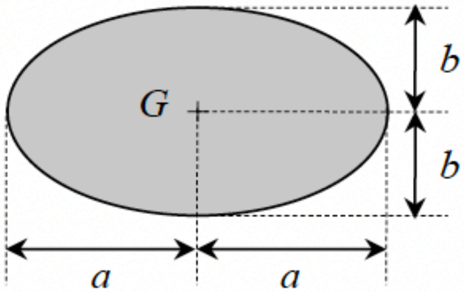
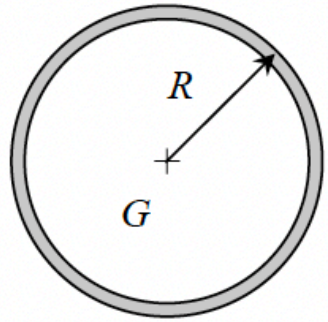
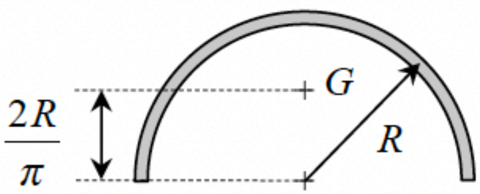
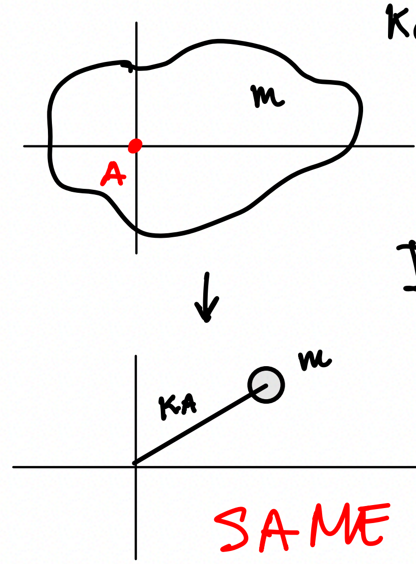


$$I_G = \frac{1}{12} m \left(a^2 + b^2 \right)$$

Uniform triangular plate of mass m



$$I_G = \frac{1}{18} m \left(a^2 + b^2 + c^2 - ab \right)$$

Uniform elliptical plate of mass m  $$I_G = \frac{1}{4}m\left(a^2 + b^2\right)$$	Uniform thin circular ring of mass m  $$I_G = mR^2$$
Uniform thin semi-circular ring of mass m  $$I_G = \left(1 - \frac{4}{\pi^2}\right)mR^2$$	Radius of gyration k_G  $I_G = m k_G^2$ SAME I_A

Back to Newton - Euler:

$$\sum \vec{F} = m\boxed{\vec{a}_G} \leftarrow \text{must use acceleration of G always!}$$

In the Euler equation, one can use any point A, but need to be consistent

$$\sum \vec{M}_{\boxed{A}} = I_{\boxed{A}}\vec{\alpha} + \vec{r}_{G/\boxed{A}} \times \left(m\vec{a}_{\boxed{A}}\right)$$

SAME point A for ALL terms

Also, use same sign convention for both equations.

Ojo: In the parallel axis theorem, A and G are NOT interchangeable!

$$I_A = I_G + m\,d_{AD}^2 \qquad I_G \neq I_A + m d_{GA}^2$$

Practice Problem

Example 5.A.11

Given: A stepped drum (having a mass of M and radius of gyration about its center O of k_O) is attached to a smooth shaft passing through its center O. A cable wrapped around the outer radius of the drum is attached to block A. A second cable is wrapped around the inner radius of the drum with this cable pulled over an ideal pulley and is attached to block B. Assume that the cables do not slip on the drum. The system is released from rest.

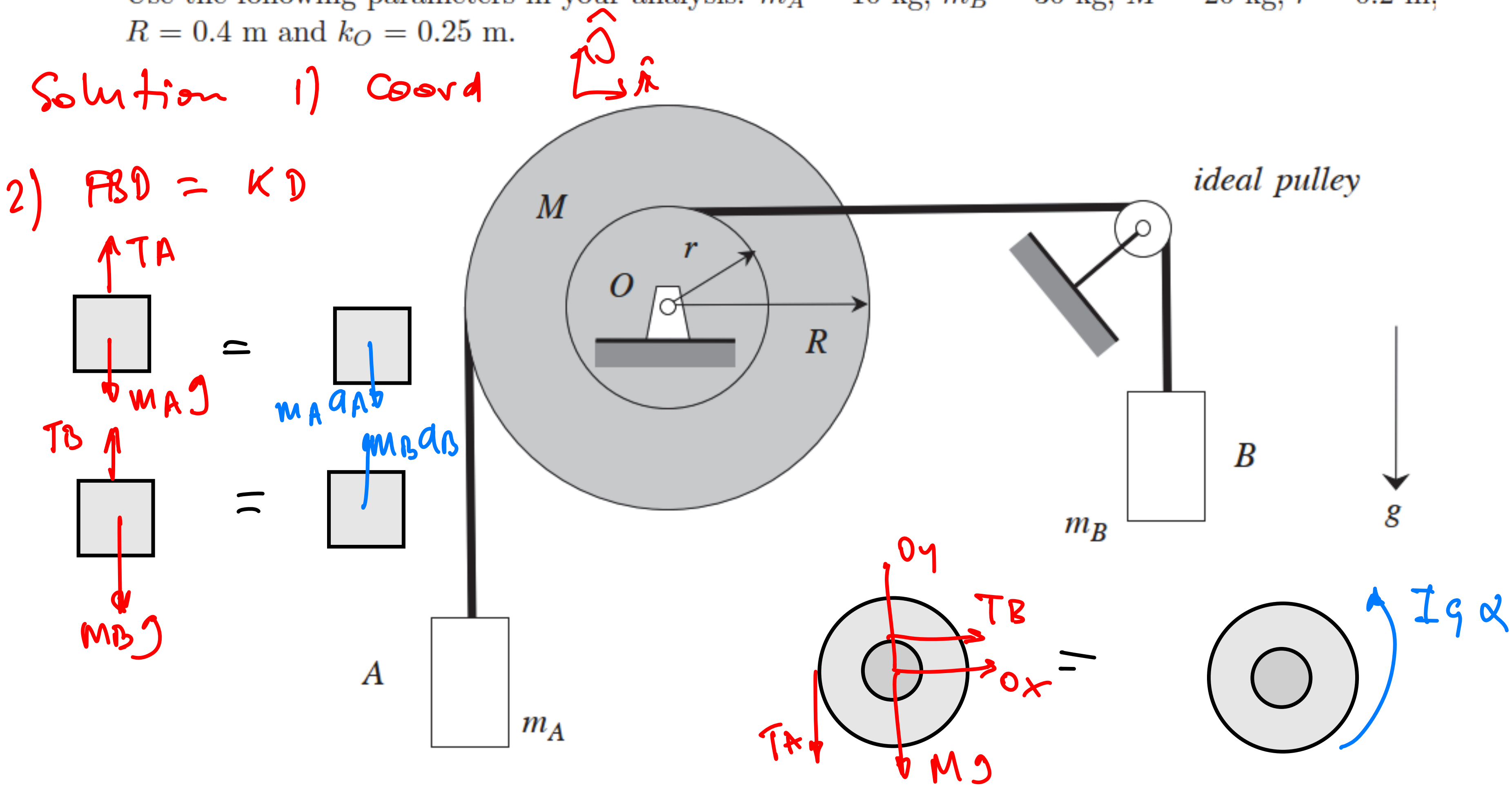
Find: Determine the angular acceleration of the drum on release. Write your answer as a vector.

$\vec{\alpha}$

Use the following parameters in your analysis: $m_A = 10$ kg, $m_B = 30$ kg, $M = 20$ kg, $r = 0.2$ m, $R = 0.4$ m and $k_O = 0.25$ m.

Solution 1) Coord

2) FBD = KD



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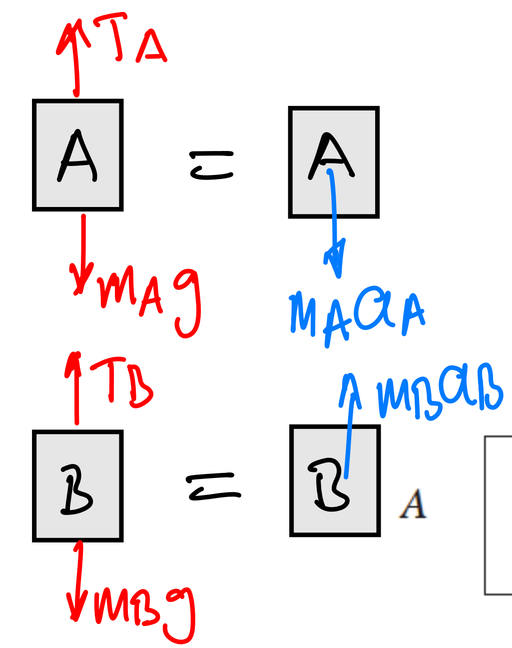
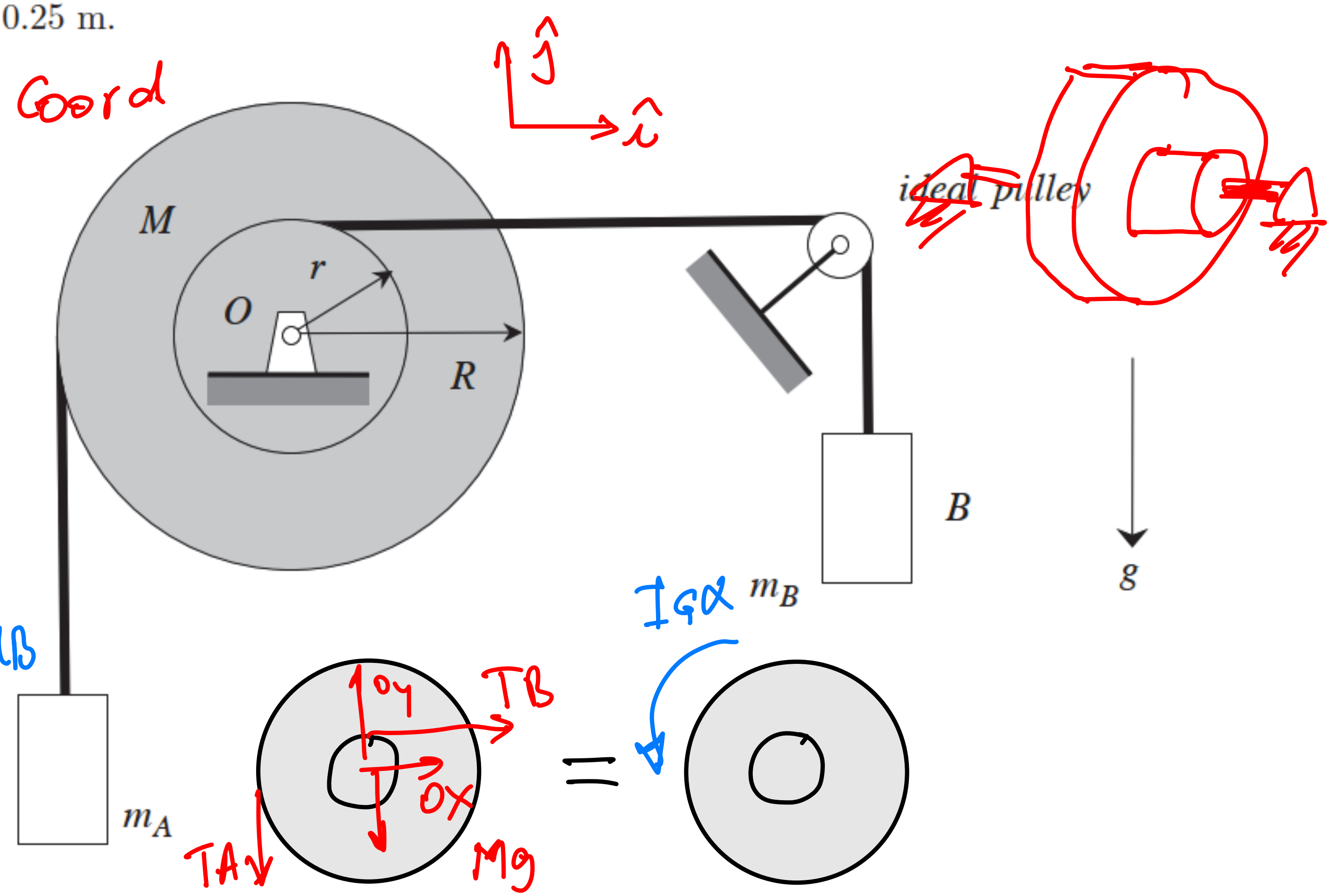
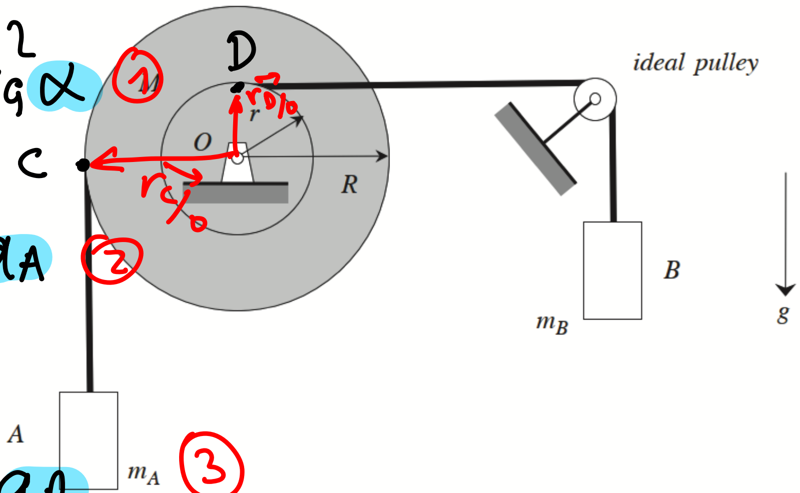




3) Kinetics DRUM

$$+\circlearrowleft \ \sum M_G = I_G \alpha \Rightarrow T_A R - T_B r = M k_G^2 \alpha \quad ①$$



Mass A

$$+\uparrow \ \sum F_y = m_A a_A \Rightarrow T_A - m_A g = -m_A a_A \quad ②$$

Mass B

$$+\uparrow \ \sum F_y = m_B a_B \Rightarrow T_B - m_B g = +m_B a_B \quad ③$$

If we try to do $\sum F_x$ and $\sum F_y$ of the drum, we introduce additional unknowns O_x and O_y. So let's directly try Kinematics.

4) Kinematics

orFr

$$\vec{a}_C = \cancel{\vec{a}_O}^{\,0} + \alpha \times \vec{r}_{C/O} - \cancel{\omega^2 \vec{r}_{C/O}}$$

$$a_{cx}\hat{i} + a_{cy}\hat{j} = \alpha\hat{k} \times (-R)\hat{i} = -\alpha R\hat{j}$$

$$\hat{i}: \quad a_{cx} = 0$$

$$\hat{j}: \quad a_{cy} = -\alpha R \Rightarrow a_A = \alpha R \quad ④$$

$$\vec{a}_D = \cancel{\vec{a}_O}^{\,0} + \alpha \times \vec{r}_{D/O} - \cancel{\omega^2}\,\vec{r}_{D/O}$$

$$a_{Dx}\hat{i} + a_{Dy}\hat{j} = \alpha\hat{k} \times r\hat{j} = -\alpha r\hat{i}$$

$\hat{i}$: $\quad a_{Dx} = -\alpha r \quad \Rightarrow \quad a_B = \alpha r$ (5)

$\hat{j}$: $\quad a_{Dy} = 0$

$$\alpha = -4.94\,\hat{k}\ \frac{\text{rad}}{s^2}$$