3/25/2026

Rigid Body Kinetics

* *Chapter 5 of the Lecturebook*
* *Chapter 6 of Meriam Textbook 7th Edition.*

Where are we?

- Kinetics: the branch of mechanics that is concerned with the relationship between the motion and its causes, specifically, forces and moments.
  - Particle motion with Newton's Law
  - Particle motion with Work/Energy
  - Particle motion with Linear Impulse Momentum
  - Particle motion with Angular Impulse Momentum
  - 2D Rigid Body motion with Newton's Law
  - 2D Rigid Body motion with Work/Energy
  - 2D Rigid Body motion with Linear Impulse Momentum
  - 2D Rigid Body motion with Linear Impulse Momentum
  - 3D Body motion
- Vibration: focused on understanding and mitigating the oscillations or repetitive motions of mechanical systems.

KINETICS OF RIGID BODIES

In our early derivations of kinetics of single particles, we used:

$$\vec{F}_i = m_i \vec{a}_i \quad \text{and}$$

$$\vec{M}_{O_i} = \frac{d}{dt}\underbrace{\left[\vec{r}_{i/O} \times (m_i \vec{v}_i)\right]}_{\vec{H}_{O,i}}, \qquad O: \text{ Fixed point}$$

When we had a system of particles, we used:

EXTERNAL

$$\left(\Sigma \vec{F}\right)_{EXT} = \underset{TOT}{m}\, \vec{a}_G$$

$$\left(\Sigma \vec{M}_A\right)_{EXT} = \frac{d}{dt} \sum_i \left[\vec{r}_{i/A} \times (m_i \vec{v}_{i/A})\right] + \vec{r}_{G/A} \times (m \vec{a}_A),$$

G: Center of mass/gravity/inertia

A: arbitrary point

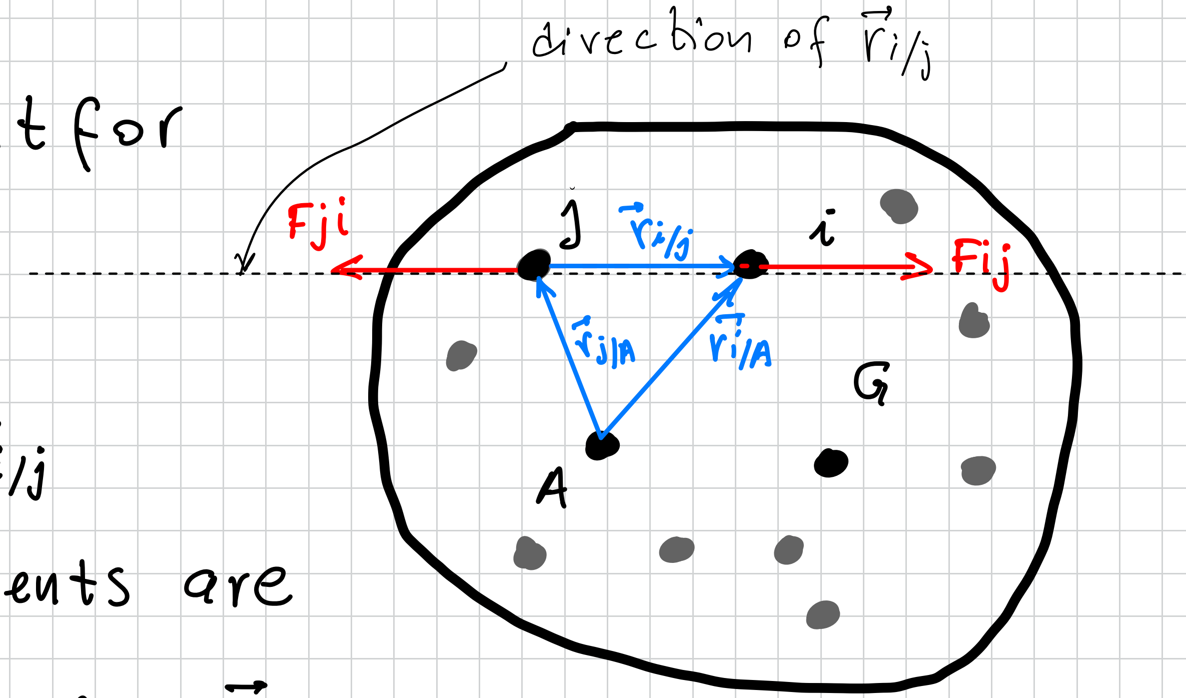
When we think of a continuous system, let's picture a collection of an infinite set of particles of infinitesimal mass:

$$\underbrace{\sum_i (\bullet) m_i}_{\text{idealized discrete}} \longrightarrow \underbrace{\int_{vol} (\bullet)\, dm}_{\text{actual volume}}$$

We need to account for internal forces.

$$\vec{F}_{ij} = -\vec{F}_{ji}$$

along direction of $\vec{r}_{i/j}$



System of N particles

The resulting moments are

$$\Sigma M_{A,int} = \vec{r}_{i/A} \times \vec{F}_{ij} + r_{j/A} \times \vec{F}_{ji}$$

$$= (\vec{r}_{i/A} - \vec{r}_{j/A}) \times \vec{F}_{ij}$$

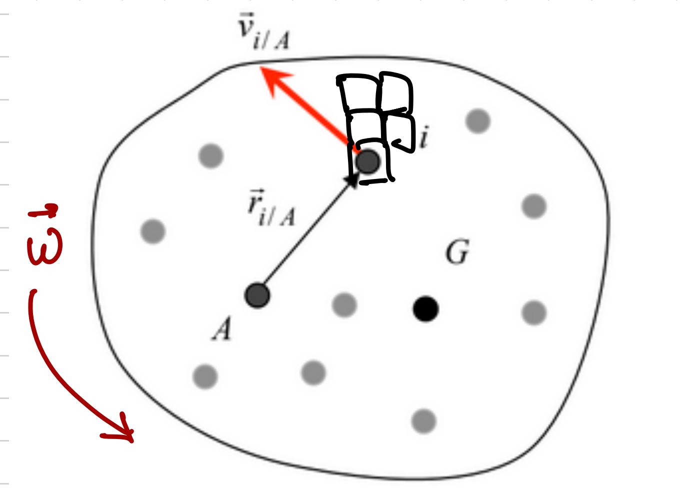
$$= \vec{r}_{ij} \times \vec{F}_{ij} = 0 \quad \text{(colinear)}$$

So, the net effect of internal forces on the system is ZERO!

$$(\Sigma \vec{F})_{EXT} = m\vec{a}_G \qquad (1)$$

$$(\Sigma \vec{M}_A)_{EXT} = \frac{d}{dt}\sum_i \left[\vec{r}_{i/A} \times (m_i \vec{v}_{i/A}\right] + \vec{r}_{G/A} \times (m\vec{a}_A) \qquad (2)$$

We need to produce an equivalent set of equations but for rigid bodies. To do so,



1) Need to enforce rigid connections between all system points

$\Rightarrow$ rigid body velocity eqn.:

$$\vec{v}_{i/A} = \vec{v}_i - \vec{v}_A = \vec{\omega} \times \vec{r}_{i/A} \qquad (3)$$

2) View RB as an infinite set of particles

$$\sum_i (\bullet) m_i \longrightarrow \int_{vol} (\bullet)\, dm = \int (\bullet) \rho dV$$

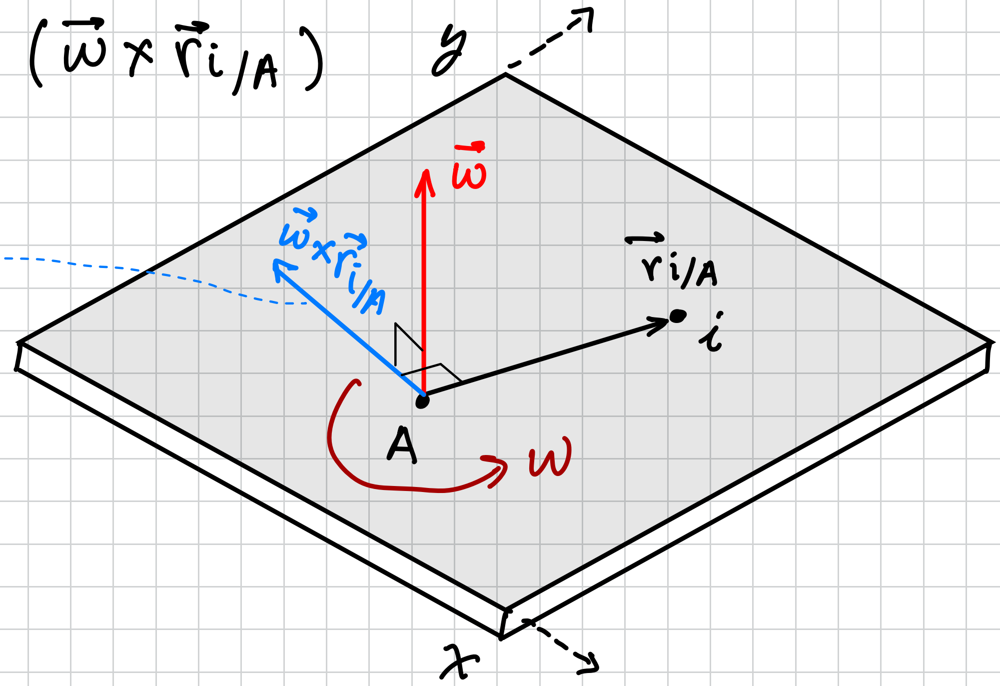
means RB have inertia

$\Rightarrow$ substitute (3) in highlighted part of (2)

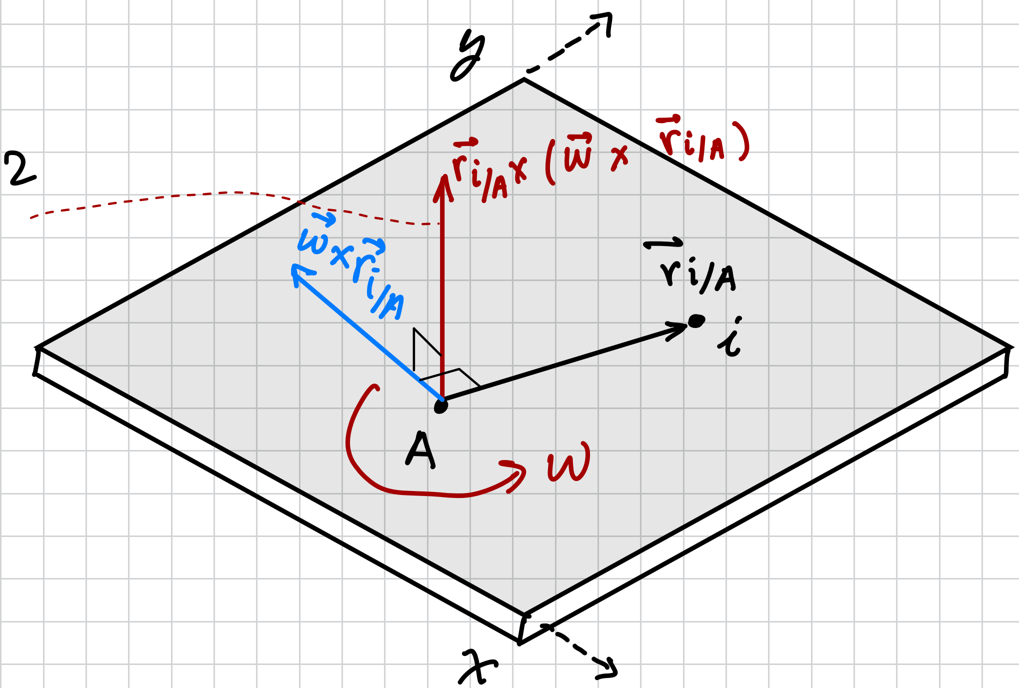
$$\vec{r}_{i/A} \times (m_i \vec{v}_{i/A}) = m_i \vec{r}_{i/A} \times (\vec{\omega} \times \vec{r}_{i/A})$$

$$\vec{r}_{i/A} \times (m_i \vec{v}_{i/A}) = m_i \vec{r}_{i/A} \times (\vec{\omega} \times \vec{r}_{i/A})$$

$$|\vec{\omega} \times \vec{r}_{i/A}| = \omega |\vec{r}_{i/A}|$$



$$|\vec{r}_{i/A} \times (\vec{\omega} \times \vec{r}_{i/A})| = \omega |\vec{r}_{i/A}|^2$$



$$\Rightarrow \vec{r}_{i/A} \times (\vec{\omega} \times \vec{r}_{i/A}) = \omega |\vec{r}_{i/A}|^2 \hat{k} = \vec{\omega} |\vec{r}_{i/A}|^2$$

Therefore, $\vec{r}_{i/A} \times (m_i \vec{v}_{i/A}) = m_i \vec{r}_{i/A} \times (\vec{\omega} \times \vec{r}_{i/A})$

becomes $\vec{r}_{i/A} \times (m_i \vec{v}_{i/A}) = m_i |\vec{r}_{i/A}|^2 \vec{\omega}$

and

$$(\Sigma \vec{M}_A)_{ext} = \frac{d}{dt}\underbrace{\left[\Sigma m_i |\vec{r}_{i/A}|^2 \vec{\omega}\right]}_{I_A \vec{\omega}} + \vec{r}_{G/A} \times (m\vec{a}_A)$$

$$(\Sigma \vec{M}_A)_{ext} = \frac{d}{dt}[I_A \vec{\omega}] + \vec{r}_{G/A} \times (m\vec{a}_A)$$

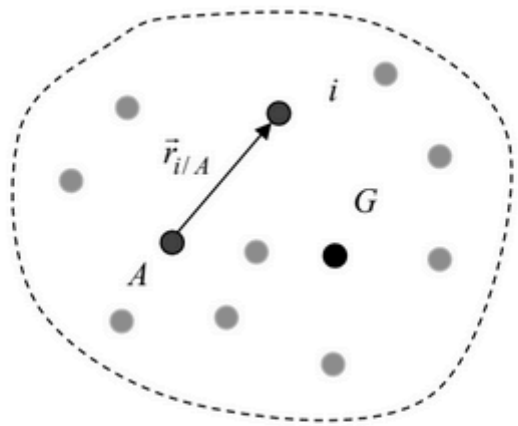
$$\boxed{(\Sigma \vec{M}_A)_{ext} = I_A \vec{\alpha} + \vec{r}_{G/A} \times (m\vec{a}_A)}$$ Euler Equation

where $I_A = \Sigma m_i |\vec{r}_{i/A}|^2$ is the famous mass moment of inertia @ point A of a system of particles

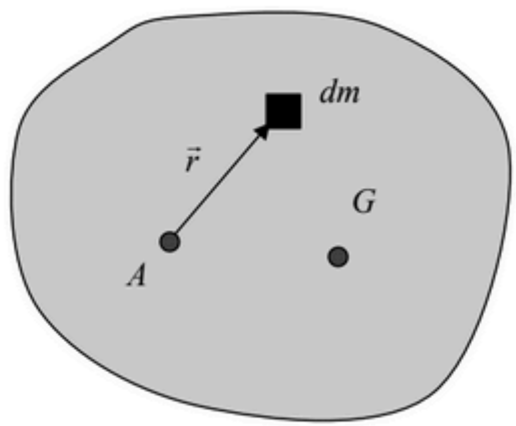
For the case of a R.B.:

$$I_A = \sum m_i \left|\vec{r}_{i/A}\right|^2 \longrightarrow \int r^2 dm$$

(Recall from ME270: $\int r^2 dA$ was the polar area moment)



system of rigidly – connected particles



continuous rigid body

EULER EQUATION – SPECIAL CASES

- If A is G $\Rightarrow \vec{r}_{G/A} = 0$ and $\Sigma \vec{M}_G = I_G \vec{\alpha}$
- If A is fixed $\Rightarrow \vec{a}_o = \vec{0}$ and $\Sigma \vec{M}_o = I_o \vec{\alpha}$
  Basically, A becomes O in this case
- If A is chosen such that its acceleration vector is $\parallel$ to $\vec{r}_{G/A}$, $\Rightarrow \vec{r}_{G/A} \times \vec{a}_A = \vec{0}$, then
  $$\Sigma \vec{M}_A = I_A \vec{\alpha}$$
- All other cases require the full Euler form.
  $$\Sigma \vec{M}_A = I_A \vec{\alpha} + \vec{r}_{G/A} \times m\vec{a}_A$$
- If a body is in pure translation, $\vec{\alpha} = 0$

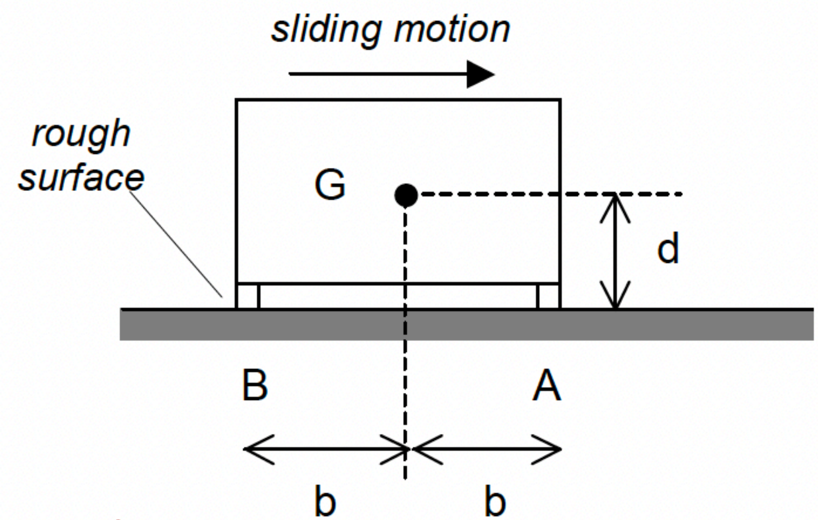
Example 5.A.1

Given: A crate of mass m slides to the right on a rough surface (with a kinetic coefficient of friction of μ_k).

Find: Find the reactions at contact points A and B on the crate.

Given: m, μ_K

Find: N_A, N_B



1) Coordinate $\hat{j}$, $\hat{i}$

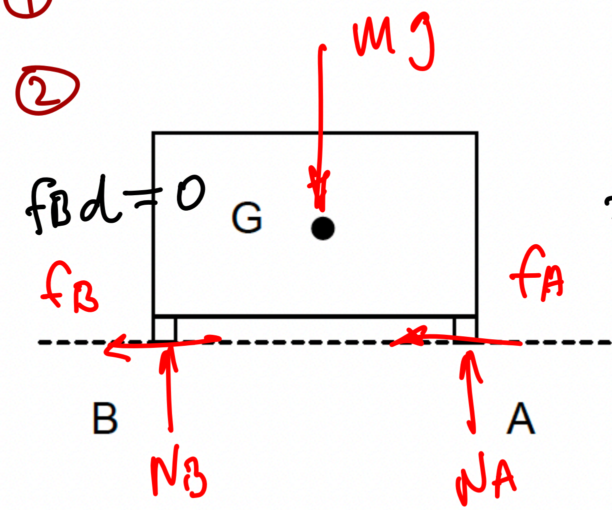
2) FBD = KD

3) Kinetics

$(\xrightarrow{+})\ \sum F_x: \quad -f_A - f_B = m a_x$ ①

$(\uparrow +)\ \sum F_y: \quad N_A + N_B - mg = 0$ ②

③ $(+\circlearrowleft)\ \sum M_G: \quad N_A b - f_A d - N_B b - f_B d = 0$



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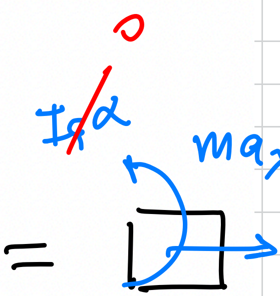


Additional equations:

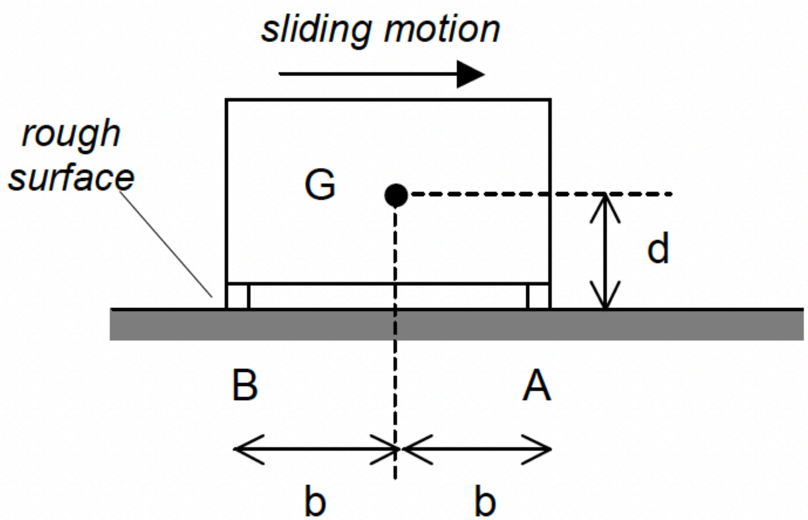
$f_A = \mu_K N_A$ ④

$f_B = \mu_K N_B$ ⑤

Example 5.A.1

Given: A crate of mass m slides to the right on a rough surface (with a kinetic coefficient of friction of μ_k).

Find: Find the reactions at contact points A and B on the crate.

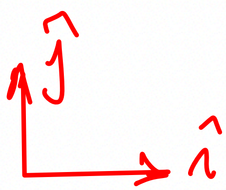


Given: m, μ_k

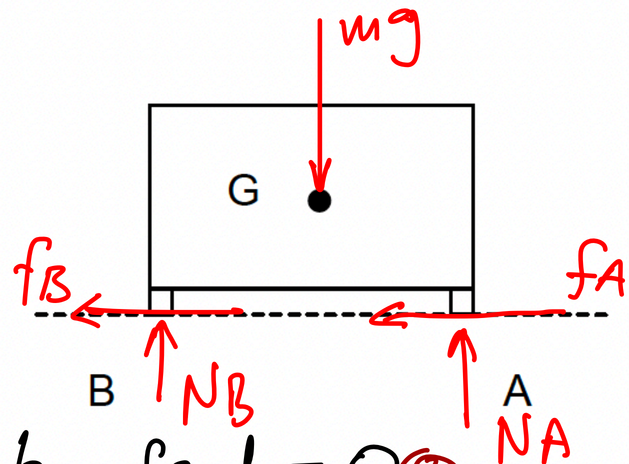
Find : N_A, N_B

solution:

1) coord.



2) FBD = KD



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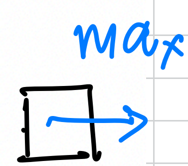


3) Kinetics

$$\Sigma F_x = -f_A - f_B = m a_{Gx} \quad ①$$

$$\Sigma F_y = N_A + N_B - mg = 0 \quad ②$$

$$\circlearrowleft + \quad \Sigma M_G = N_A b - f_A d - N_B b - f_B d = 0 \quad ③$$

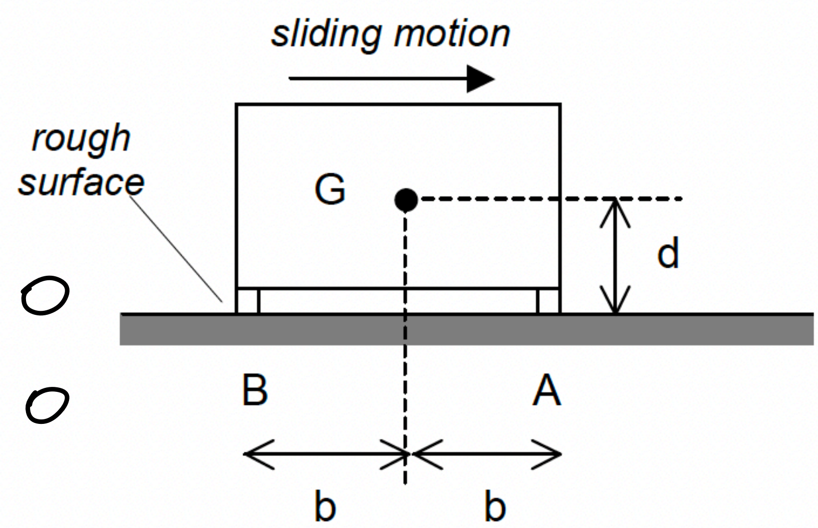
Example 5.A.1

Given: A crate of mass m slides to the right on a rough surface (with a kinetic coefficient of friction of μ_k).

Find: Find the reactions at contact points A and B on the crate.

$$f_A = \mu_K N_A \quad 4$$

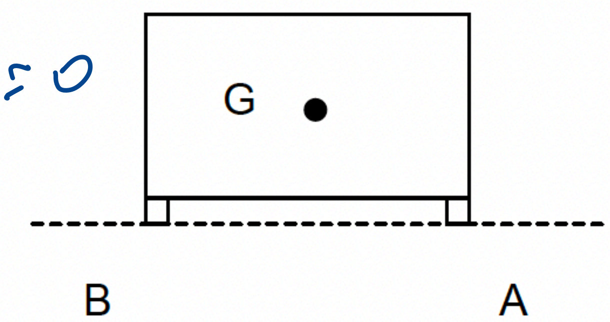
$$f_B = \mu_K N_B \quad 5$$



$$N_A b - \mu_K N_A d - N_B b - \mu_K N_B d = 0$$

$$N_A(b - \mu_K d) + N_B(-b - \mu_K d) = 0$$

$$N_A = mg - N_B$$



$$(mg - N_B)(b - \mu_K d) + N_B(-b - \mu_K d) = 0$$

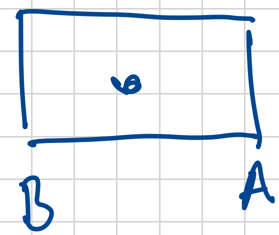
$$mgb - mg\mu_K d - N_B b + \cancel{N_B \mu_K} d$$

$$- N_B b - \cancel{N_B \mu_K} d = 0$$

$$mg\,b - mg\,\mu_K d = 2N_B b$$

$$N_B = \frac{mg(b-\mu_K d)}{2b}, \quad N_A = mg\left[1 - \frac{b-\mu_K d}{2b}\right]$$

$m = 100\,kg$



$b = 0.9\,m$

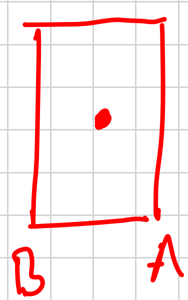
$d = 0.5\,m$

$\mu_K = 0.45$

$N_B = 368\ N$ $\quad$ $N_A = 177\ N$

$b = 0.5$

$d = 0.9$



$N_B = 93\ N$ $\quad$ $N_A = 888\ N$!