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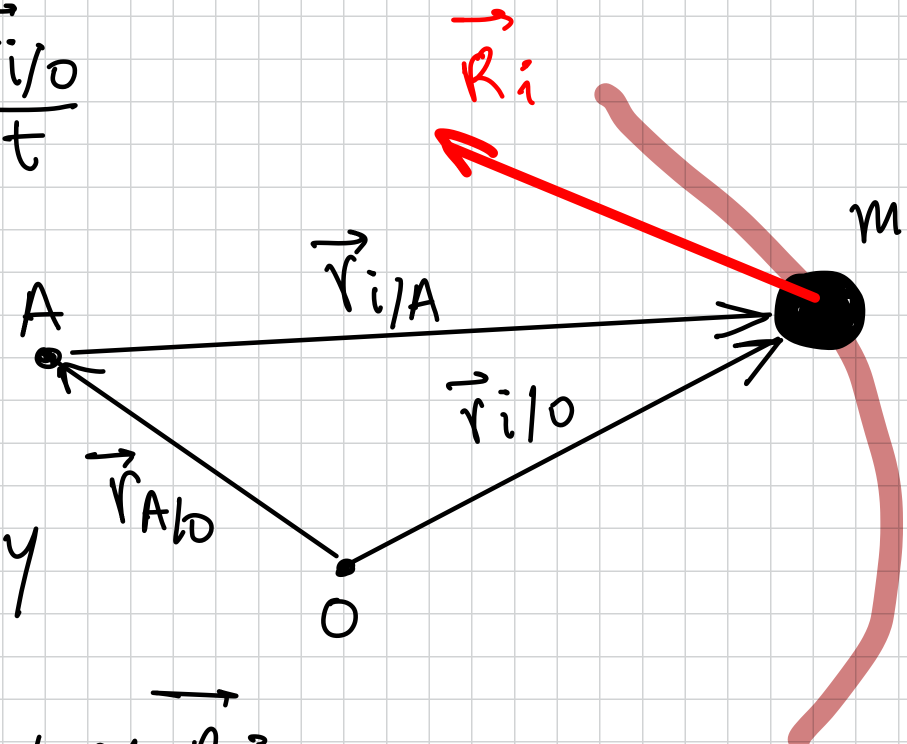
ANGULAR MOMENTUM – SYSTEM OF PARTICLES

In this case, let us consider $\vec{R}_i$ as the resultant force acting on a particle i moving along the path. Using N2L

$$\vec{R}_i = m_i \vec{a}_i = m_i \frac{d\vec{v}_i}{dt},$$

where $\vec{v}_i = \dfrac{d\vec{r}_{i/O}}{dt}$



To find the moment of $\vec{R}_i$ about an arbitrary point A:

$$\vec{M}_A = \vec{r}_{i/A} \times \vec{R}_i$$

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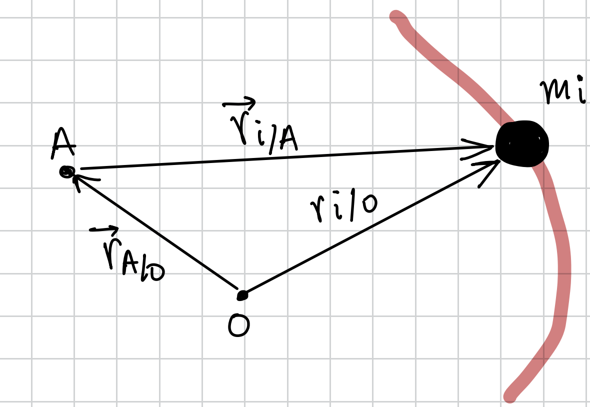
$$\Rightarrow \quad \vec{M}_A = \vec{r}_{i/A} \times m_i \frac{d\vec{v}_i}{dt}$$



but $\frac{d}{dt}\left[\vec{r}_{i/o} = \vec{r}_{A/o} + \vec{r}_{i/A}\right]$

so $\frac{d}{dt}\left[\vec{v}_i = \vec{v}_A + \vec{v}_{i/A}\right]$ (because o is fixed)

$$\frac{d\vec{v}_i}{dt} = \frac{d\vec{v}_A}{dt} + \frac{d\vec{v}_{i/A}}{dt}$$

$$\Rightarrow \quad \frac{d\vec{v}_{i/A}}{dt} = \frac{d\vec{v}_i}{dt} - \frac{d\vec{v}_A}{dt}$$

$$\vec{M}_A = \vec{r}_{i/A} \times m_i\left(\frac{d\vec{v}_A}{dt} + \frac{d\vec{v}_{i/A}}{dt}\right)$$

$$\Rightarrow \quad = \vec{r}_{i/A} \times (m_i \vec{a}_A) + \underbrace{\vec{r}_{i/A}}_{\vec{a}} \times \underbrace{\left(m_i \frac{d\vec{v}_{i/A}}{dt}\right)}_{\left(m_i \frac{d}{dt}\vec{b}\right)}$$

recall $\frac{d}{dt}(\vec a \times m_i \vec b) = \frac{d}{dt}\vec a \times (m_i \vec b) + \vec a \times \frac{d}{dt}(m_i \vec b)$

$$\Rightarrow \vec a \times \frac{d}{dt}(m_i \vec b) = \frac{d}{dt}(\vec a \times m_i \vec b) - \frac{d}{dt}\vec a \times (m_i \vec b)$$

$$\Rightarrow \vec M_A = \vec r_{i/A} \times (m_i \vec a_A) + \frac{d}{dt}\left[\vec r_{i/A} \times m_i \vec v_{i/A}\right] - \cancelto{0}{\underbrace{\frac{d}{dt}\vec r_{i/A}}_{\vec v_{i/A}} \times (m_i \vec v_{i/A})}$$

$$\vec v_{i/A} \times m_i \vec v_{i/A}$$

$$\boxed{\vec M_A = \vec r_{i/A} \times (m_i \vec a_A) + \frac{d}{dt}\left[\vec r_{i/A} \times m_i \vec v_{i/A}\right]}$$

Now, for N particles:

$$\sum_{i=1}^{N} \vec M_A = \sum_{i=1}^{N} (m_i \vec r_{i/A}) \times \vec a_A + \sum_{i=1}^{N} \frac{d}{dt}\left[\vec r_{i/A} \times (m_i \vec v_{i/A})\right]$$

$$= m \vec r_{G/A} \times \vec a_A + \sum_{i=1}^{N} \frac{d}{dt}\left[\vec r_{i/A} \times (m_i \vec v_{i/A})\right]$$

where G is the center of gravity of the system

and m is $\sum_{i=1}^{N} m_i$.

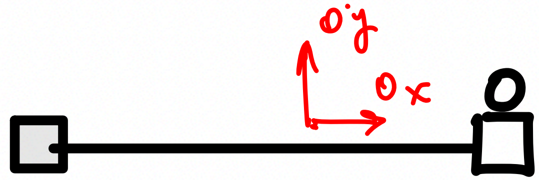
Example 4.D.6

Given: Particles A and B, of masses $4m$ and $2m$, respectively, are attached to the ends a stationary rigid rod of negligible mass. The rod is pinned to ground at O. A third particle C, of mass m, strikes particle B with a speed of v_1. On impact, C sticks to B. The system lies in the horizontal plane.

Find: Determine the angular speed of the bar immediately after the impact occurs.

Solution

1) FBD

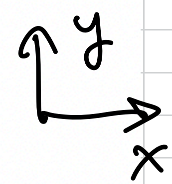


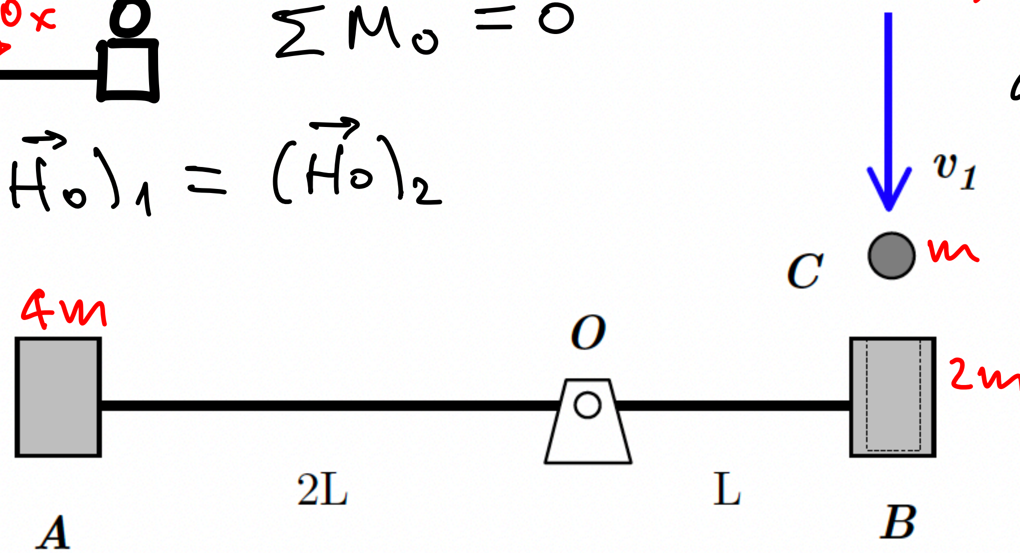
3) Kinetics

$$\sum \vec{M}_O = 0$$

$$(\vec{H}_O)_1 = (\vec{H}_O)_2$$

2) Coord





$$(\vec{H}_O)_1 = \sum_i \vec{r}_{i1} \times m_i \vec{v}_{i1}$$

$$= (-2L\hat{i}) \times \cancel{4m \cdot \vec{0}}^{\,0} + (+L\hat{i}) \times \cancel{2m\vec{0}}^{\,0} + (L\hat{i}) \times m v_1(-\hat{j})$$

$$(\vec{H}_O)_1 = -Lmv_1\hat{k}$$

$$(\vec{H}_o)_2 = (-2L\hat{i}) \times 4m(+2L w_2 \hat{j}) + (L\hat{i}) \times 3m(-L w_2 \hat{j})$$

$$= -16L^2 m w_2 \hat{k} - 3L^2 m w_2 \hat{k}$$

$$(\vec{H}_o)_2 = -19L^2 m w_2 \hat{k}$$

$$(\vec{H}_o)_1 = (\vec{H}_o)_2$$

$$-\cancel{L}\cancel{m} v_1 = 13 L^{\cancel{2}} \cancel{m} w_2$$

$$\boxed{\vec{w}_2 = -\frac{v_1}{19L}\hat{k}}$$

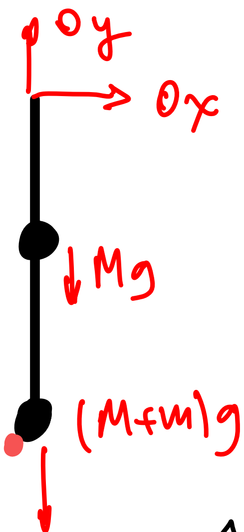
Example 4.D.5

Given: Particles A and B (each having a mass of M) are attached to rigid bar OA (this bar has negligible mass). Bar OA is pinned to ground at end O. This system is at rest when A is struck by a bullet P (having a mass of m) with the bullet traveling in the direction shown with a speed of v_{P1}. Immediately upon impact, the bullet becomes embedded in particle A.

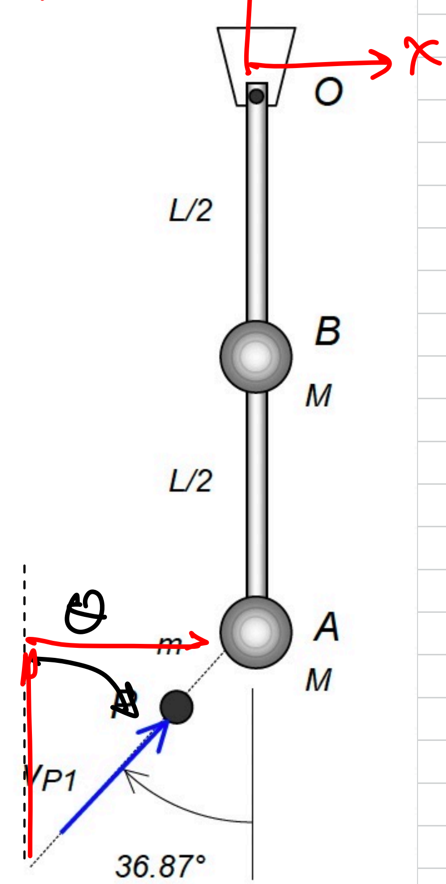
Find: Determine the angular velocity of bar OA immediately after the collision is completed. A.1.M

Use the following parameters in your analysis: $M = 2$ kg, $m = 0.1$ kg, $L = 3$ m and $v_{P1} = 700$ m/s.

Solution 1) FBD



2) Coord



3) Kinetics $\Sigma M_O = 0 \Rightarrow (\vec{H}_O)_1 = (\vec{H}_O)_2$

$$(\vec{H}_O)_1 = \vec{r}_1 \times m\vec{v}_1$$

$$= -L\hat{j} \times m\left(v_{P1}\sin\theta\,\hat{i} + v_{P1}\cos\theta\,\hat{j}\right)$$

$$(\vec{H}_O)_1 = L m v_{P1} \sin\theta \quad \hat{k}$$

$$(\vec{H}_o)_2 = \sum_i \vec{r}_{2i} \times m_i \vec{v}_{2i}$$

$$= -\frac{L}{2}\hat{j} \times M \frac{L}{2}\omega_2 \hat{i} + (-L\hat{j}) \times (M+m)(L\omega_2 \hat{i})$$

$$= M\frac{L^2}{4}\omega_2 \hat{k} + (M+m)L^2\omega_2 \hat{k}$$

$$(\vec{H}_o)_1 = (\vec{H}_o)_2 \Rightarrow \quad \cancel{L} m v_{p1} \sin\theta = M\frac{L^{\cancel{2}}}{4}\omega_2 + (M+m)L^{\cancel{2}}\omega_2$$

$$m v_{p1} \sin\theta = M\frac{L}{4}\omega_2 + ML\omega_2 + mL\omega_2$$

$$m v_{p1} \sin\theta = \omega_2 L\left(\frac{5}{4}M + m\right)$$

$$\vec{\omega}_2 = \frac{m v_{p1} \sin\theta}{L\left(\frac{5}{4}M + m\right)} \hat{k}$$

$$\vec{\omega}_2 = 5.52 \text{ rad/s}$$