ANGULAR IMPULSE & MOMENTUM

3/11/2026

Integration of N2L $\vec{R} = \Sigma \vec{F} = m\vec{a}$ over time resulted in the L-I-M-Eqn:

$$m\vec{v}_2 = m\vec{v}_1 + \int_1^2 \vec{R}\,dt \text{, useful for } \begin{cases} \bullet \text{ relate } \Delta v \text{ with } \Delta t \\ \bullet \text{ conservation of L.M.} \end{cases}$$

In many cases involving motion around Fixed points, L.M. is not conserved! However, angular momentum may be conserved.

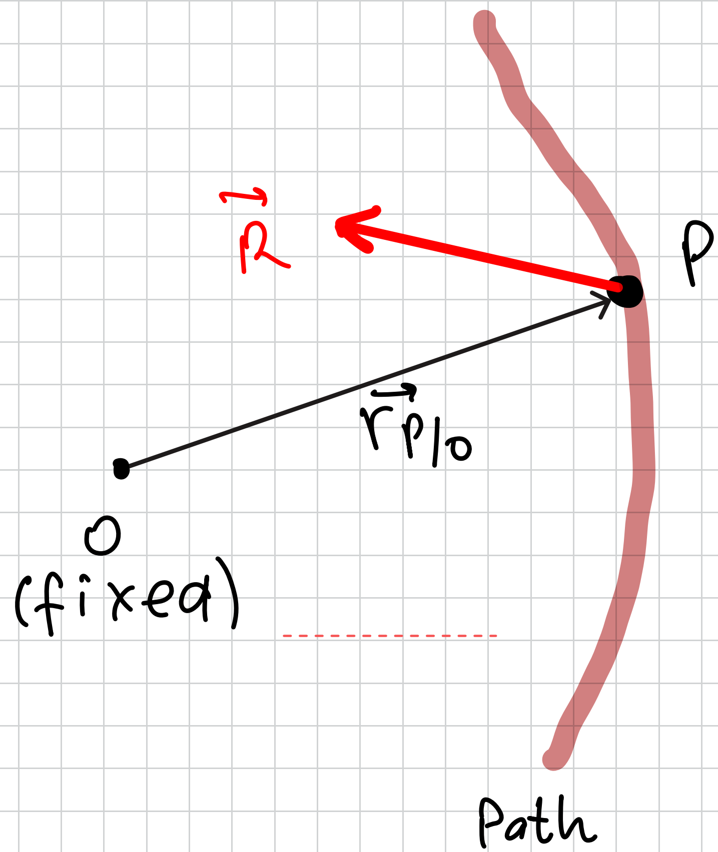
Let $\vec{R}$ be the resultant force acting on P as it moves on a path.

$$\vec{R} = \Sigma \vec{F} = m\vec{a} = m\frac{d(\vec{v}_P)}{dt}$$

Here, $\vec{v}_P = \frac{d}{dt}\vec{r}_{P/O}$ and O is fixed.

Now, the moment around O due to $\vec{R}$ is:

$$\vec{M}_O = \vec{r}_{P/O} \times \vec{R}$$



$\vec{R}$ from N2L into M_O:

$$\vec{M}_O = \underbrace{\vec{r}_{P/O}}_{\vec{a}} \times \frac{d}{dt}\underbrace{(m\vec{v}_P)}_{\vec{b}}$$

$$\frac{d}{dt}(\vec{a}\times\vec{b}) = \frac{d}{dt}(\vec{a})\times\vec{b} + \vec{a}\times\frac{d}{dt}(\vec{b})$$

$$\Rightarrow \quad \vec{a}\times\frac{d}{dt}(\vec{b}) = \frac{d}{dt}(\vec{a}\times\vec{b}) - \frac{d}{dt}(\vec{a})\times\vec{b}$$

So

$$\vec{M}_O = \boxed{\frac{d}{dt}\left[\vec{r}_{P/O}\times m\vec{v}_P\right]} - \boxed{\frac{d}{dt}\vec{r}_{P/O}\times m\vec{v}_P}$$

$$= \frac{d}{dt}\left[\vec{r}_{P/O}\times m\vec{v}_P\right] - \cancelto{0}{\vec{v}_P\times m\vec{v}_P}$$

If we define $\vec{H}_O = \vec{r}_{P/O} \times m\vec{v}_P$, the angular momentum of P about O:

$$\Rightarrow \underbrace{\int_1^2 \vec{M}_O\,dt}_{\text{ANGULAR IMPULSE}} = \int_1^2 \frac{d}{dt}(\vec{H}_O) = \underbrace{\vec{H}_{O_2} - \vec{H}_{O_1}}_{\text{ANGULAR MOMENTUM}}$$

angular impulse is the $\frac{d}{dt}(\vec{H}_O)$

ANGULAR IMPULSE - MOMENTUM EQUATION

Consider the same system at two different states:

$$\underbrace{(\vec{H}_O)_1}_{\substack{\text{angular momentum} \\ \text{@ state 1}}} + \underbrace{\int_1^2 \vec{M}_O\,dt}_{\substack{\text{angular impulse by} \\ \text{external moments}}} = \underbrace{(\vec{H}_O)_2}_{\substack{\text{angular momentum} \\ \text{@ state 2}}}$$

This new equation adds to the energy/momentum toolbox:

W/E: $\frac{1}{2} m v_1^2 + V_1 + \int_1^2 \vec{R}\, ds = \frac{1}{2} m v_2^2 + V_2$

LIM: $m v_1^2 + \int_1^2 \vec{R}\, dt = m v_2^2$

AIM: $\vec{H}_{O1} + \int_1^2 \vec{M}_O\, dt = \vec{H}_{O2}^{2}$

The AIM equation usually applied with polar description

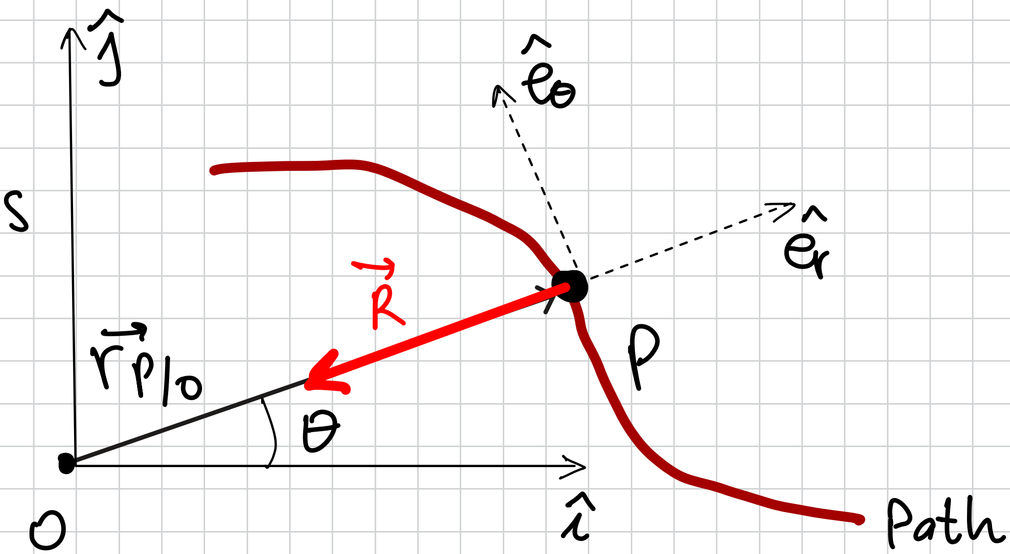
$$\vec{H}_O = m (r^2 \dot{\theta})\, \hat{k}$$

So, when angular momentum is conserved:

$$\vec{H}_{O2} = \vec{H}_{O1} \implies \dot{\theta}_2 = \frac{r_1^2}{r_2^2} \dot{\theta}_1$$

Clearly, this relationship states that the angular velocity must decrease/increase as the radial distance increases/decreases for the angular momentum to be conserved.

CENTRAL FORCE PROBLEMS:



Since the resultant force $\vec{R}$ points towards O, the moment due to $\vec{R}$ is ZERO; so, for central force cases, the angular momentum is conserved.

PROCEDURE

1) FREE BODY DIAGRAMS

2) COORDINATE SYSTEM

3) KINETICS EQUATIONS

- ↳ NEWTON
- ↳ WORK-ENERGY
- ↳ LINEAR IMPULSE MOMENTUM
- ↳ ANGULAR IMPULSE MOMENTUM
- ↳ IMPACT

3) KINEMATICS

4) SOLVE

SOLUTION

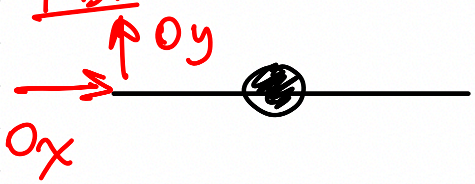
Example 4.D.1

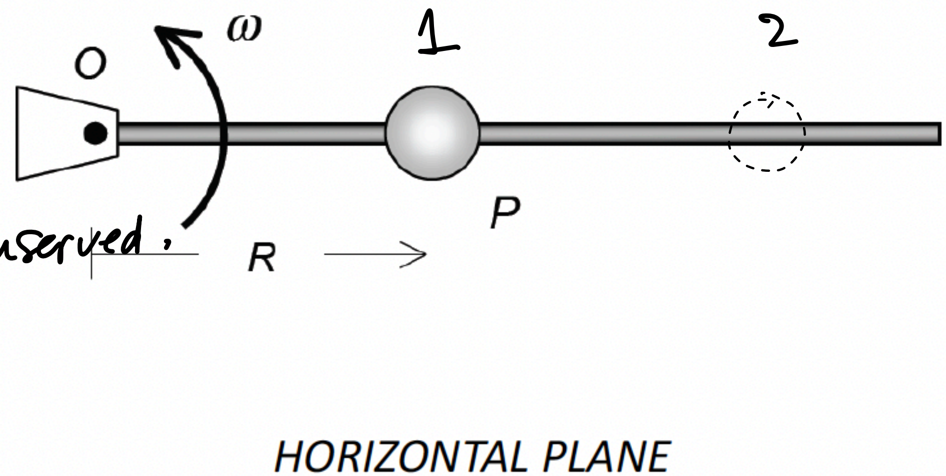
Given: Particle P (weighing 2 lb) is able to slide on a smooth, lightweight horizontal arm that is rotating about a vertical axis. Initially, P is stationary relative to the arm when the arm is rotating at a rate of $\omega_1 = 20$ rad/s and P is $R = 3$ in from the rotation axis of the arm.

Find: Determine the angular speed of the arm when the P has moved outward to a position of $R = 24$ in from the axis of the shaft.

Solution

FBD





$$\Sigma M_O = 0$$

Angular momentum @ O conserved.

Kinetics

$$(\vec{H}_O)_1 = (\vec{H}_O)_2$$

$$(\vec{H}_O)_1 = \vec{r}_1 \times m\vec{v}_1 = R_1 \hat{e}_r \times \frac{W}{g}(\dot{R}_1 \hat{e}_r + R_1 \dot{\theta} \hat{e}_\theta)$$

$$= R_1^2 \frac{W}{g} \omega_1 \hat{k}$$

$$\vec{H}_O)_2 = \vec{r}_2 \times m\vec{v}_2 = r_2\hat{e}_r \times \frac{w}{g}(\dot{r}_2\hat{e}_r + r_2\dot{\theta}_2\hat{e}_\theta)$$

$$= \frac{w}{g} r_2^2 \omega_2 \hat{K}$$

$$r_1^2 \cancel{\frac{w}{g}} \omega_1 = \cancel{\frac{w}{g}} r_2^2 \omega_2$$

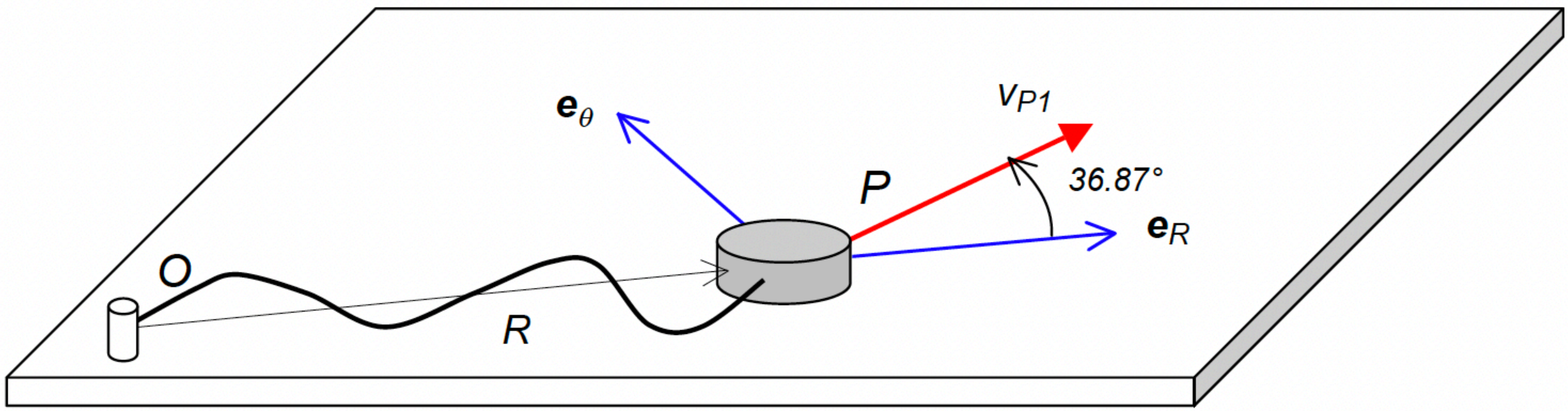
$$\omega_2 = \left(\frac{r_1}{r_2}\right)^2 \omega_1 \longleftarrow \underline{\text{ANSWER}}$$

Example 4.D.3

Given: Disk P having a mass of 0.2 kg is able to slide on a smooth, horizontal surface. A rubber band (having a stiffness of $k = 10$ N/m and unstretched length of 0.6 m) attaches P to a fixed peg at O. Disk P is set into motion with a speed of $v_{P1} = 15$ m/s in the direction shown with $R = 0.4$ m.

Find: Determine the polar coordinates of the velocity of P when P is a distance 1.5 m from O.

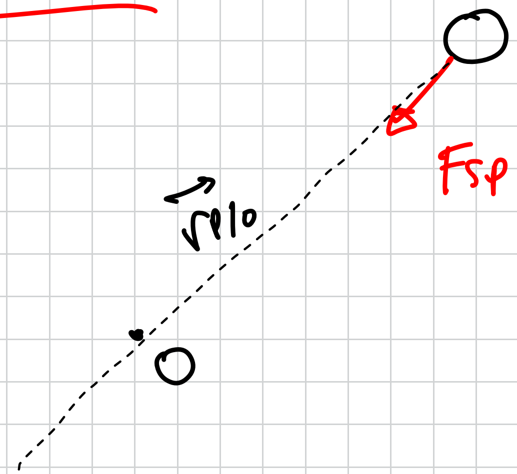
$$\vec{v}_P = v_{Pr}\,\hat{e}_r + v_{P\theta}\,\hat{e}_\theta$$



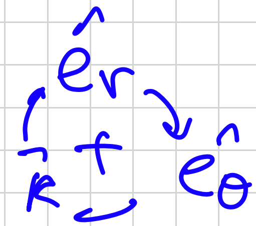
Solution

Hints: asking for Δv after Δx – rotational
- motion is constrained to a fixed pt. O
- Given a spring K, Δx

1 FBD



Since the external forces are 0 and the internal force points @ point O $\Rightarrow$ conservation of angular momentum



2 Coordinates

(given)

3) Kinetics

$$\sum M_O = 0 \Rightarrow (\vec{H}_O)_1 = (\vec{H}_O)_2$$

$$(\vec{H}_O)_1 = \vec{r}_1 \times m\vec{v}_1 = R_1 \hat{e}_r \times m v_{P_1}(\cos\theta\, \hat{e}_r + \sin\theta\, \hat{e}_\theta)$$

$$(\vec{H}_O)_1 = R_1 m v_{P_1} \sin\theta\, \hat{k} \; -$$

$$(\vec{H}_O)_2 = \vec{r}_2 \times m\vec{v}_2 = R_2 \hat{e}_r \times m\underbrace{(V_{2r}\hat{e}_r + V_{2\theta}\hat{e}_\theta)}_{\vec{v}_2}$$

$$(\vec{H}_O)_2 = R_2 V_{2\theta} m\, \hat{k} \; -$$

$$(\vec{H}_o)_1 = (\vec{H}_o)_2 \implies \cancel{m} R_1 v_{p_1} \sin\theta = \cancel{m} R_2 \underline{V_{2\theta}}$$

$$V_{2\theta} = \frac{R_1}{R_2} v_{p_1} \sin\theta$$

$$V_{2\theta} = 2.4 \text{ m/s}$$

Use W/E for V_{2r}

$$T_1 + V_1 + U_{1-2}^{NC} = T_2 + V_2$$

$$T_1 = \frac{1}{2} m v_{p_1}^2 \qquad T_2 = \frac{1}{2} m (V_{2r}^2 + V_{2\theta}^2)$$

$$V_1 = 0 \qquad U_{1-2}^{NC} = 0$$

unstretched $R = 0.6$

$$V_2 = \frac{1}{2} K (R_2 - \overbrace{R_0}^{}) ^2$$

$$\cancel{\tfrac{1}{2}}\cancel{m} v_{p_1}^2 + 0 + 0 = \cancel{\tfrac{1}{2}}\cancel{m} v_{2r}^2 + \cancel{\tfrac{1}{2}}\cancel{m} v_{2\theta}^2 + \cancel{\tfrac{1}{2}} \frac{K}{m} (R_2 - R_0)^2$$

$$V_{r2}^2 = V_{p_1}^2 - V_{\theta 2}^2 - \frac{K}{m} (R_2 - R_0)^2 \implies V_{r2} = 13.36 \text{ m/s}$$

Finally:

$$\boxed{\vec{V} = 13.36\, \hat{e}_r + 2.4\, \hat{e}_\theta \quad \text{m/s.}}$$

ANSWER