WORK & ENERGY EQUATION

3/2/2026

Work and potential energy of spring

Consider a spring K and unstretched Length L_o. If attached to a particle, its force is $F_{sp} = K(L - L_o)$, where L : stretched length

The work done by F_{sp} is

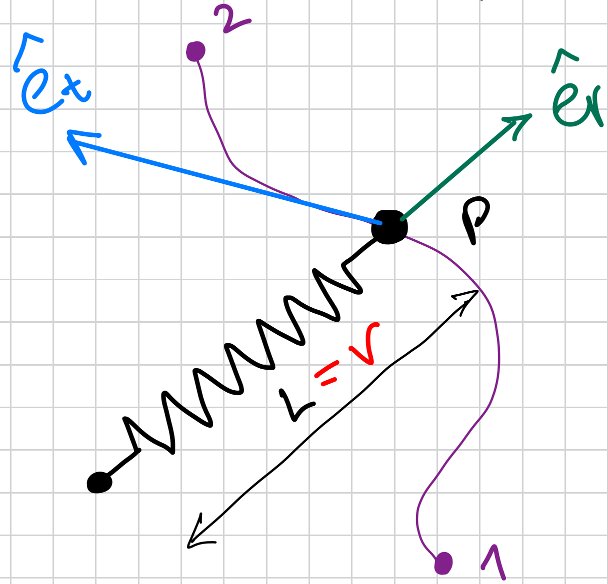
$$U_{1-2} = \int_1^2 (\vec{F}_{sp} \cdot \hat{e}_t)\, ds$$

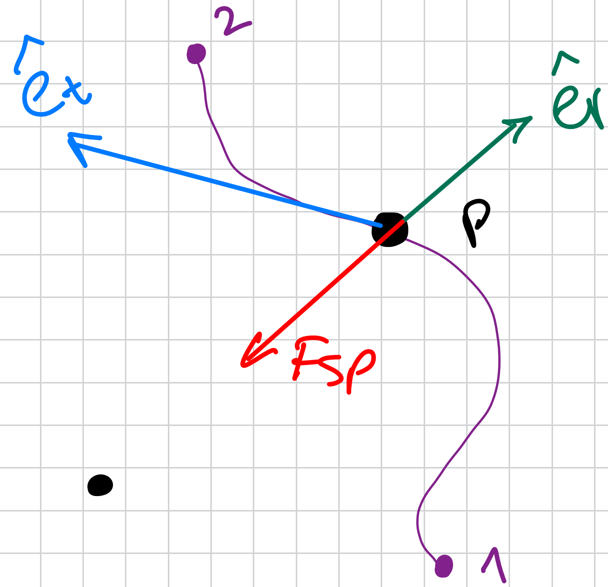
Using a polar description, $\vec{F}_{sp}$ is

$$\vec{F}_{sp} = -K(L - L_o)\,\hat{e}_r$$

Now, we need to relate $\hat{e}_t$ with $\hat{e}_r$ and $\hat{e}_\theta$ in the following diagram:

1





$$\vec v = \dot r\,\hat e_r + r\dot\theta\,\hat e_\theta \qquad \vec v = v\,\hat e_t$$

$$\vec v = \frac{dL}{dt}\hat e_r + L\frac{d\theta}{dt}\hat e_\theta \qquad \vec v = \frac{ds}{dt}\hat e_t$$

$$\Rightarrow \vec v\,dt = dL\,\hat e_r + L\,d\theta\,\hat e_\theta \qquad \vec v\,dt = ds\,\hat e_t$$

$$\hat e_t\,ds = dL\,\hat e_r + L\,d\theta\,\hat e_\theta$$

$$\Rightarrow \quad U_{1-2} = \int_1^2 \left[-K(L-L_0)\hat{e}_r\right] \cdot \left[dL\,\hat{e}_r + L\,d\theta\,\hat{e}_\theta\right]$$

$$\hat{e}_r \cdot \hat{e}_r$$

$$\Rightarrow \quad U_{1-2} = -K\int_1^2 (L-L_0)(1)\,dL$$

$$= -K\left[\frac{1}{2}(L_2-L_0)^2 - \frac{1}{2}(L_1-L_0)^2\right]$$

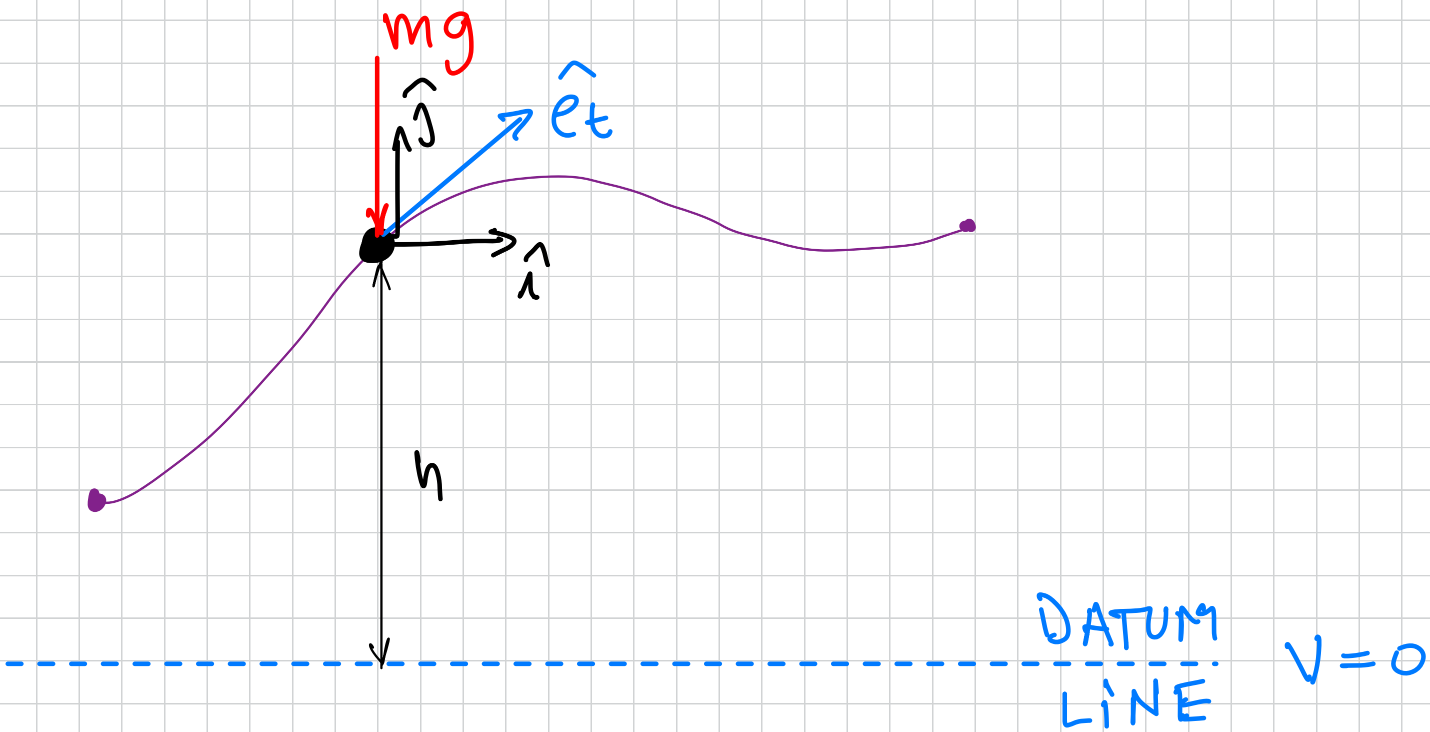
$$\Rightarrow \quad U_{1-2} = -K(V_2 - V_1)$$

The work done by the spring does not depend on the path of the particle, only on the amount of stretch in the spring @ beginning and end of motion.

Work & potential Energy of a weight

Consider a particle with mass m, its force is $\vec{F}_W = -mg\,\hat{j}$

$$\Rightarrow \quad U_{1-2} = \int_1^2 (\vec{F}_W \cdot \hat{e}_t)\, ds$$



$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \qquad \vec{v} = \hat{e}_t \frac{ds}{dt}$$

$$\vec{v}\,dt = dx\,\hat{i} + dy\,\hat{j} \qquad \vec{v}\,dt = \hat{e}_t\,ds$$

$$\Rightarrow \hat{e}_t\,ds = dx\,\hat{i} + dy\,\hat{j}$$

$$\Rightarrow U_{1-2} = \int_1^2 (-mg\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$\hat{j}\cdot\hat{j}$$

$$\Rightarrow U_{1-2} = \int_1^2 -mg\,(1)\,dy = -mg\,[y_2 - y_1]$$

If we define $V = mgh$ (potential energy)

$$\Rightarrow \quad U_{1-2} = -(V_2 - V_1)$$

The work done by the weight does not depend on the path s of the particle, only on the height @ beginning and end of the motion.

$$T_2 + V_2 = T_1 + V_1 + U_{1-2}^{(nc)}$$

→ Only the tangential component of $\vec{R}$ contributes to the work by it

$$U_{1-2} = \int_1^2 (\vec{R} \cdot \hat{e}_t)\, ds$$

→ The perpendicular component of $\vec{R}$ or any force ⊥ to S does NO WORK on P

→ The W-E equation is SCALAR. That is, it cannot be separated into two components (e.g., $\hat{\imath}$ and $\hat{\jmath}$)

→ If there are no non-conservative forces, $U_{1-2} = 0 \Rightarrow T_1 + V_1 = T_2 + V_2$

$\Rightarrow$ energy is conserved!

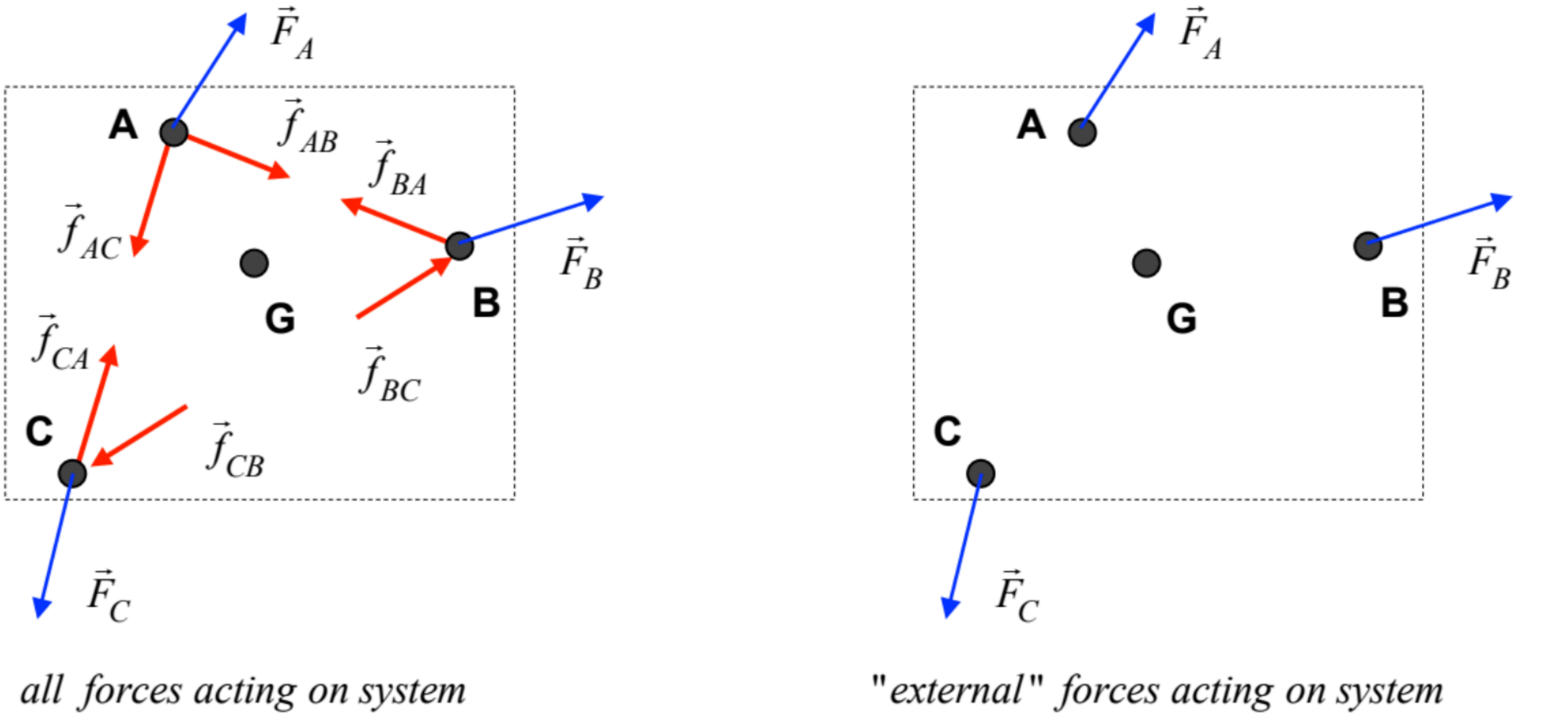
→ Sign criteria of the W-E eqn:

a) $V_{spring} > 0$ always regardless of spring stretching or compressing.

b) V_{weight} sign depends on choice of datum (your design!)

→ W-E eqn can be also applied to systems of particles. Simply the T and V of all particles need to be summed up.

W-E Eqn for systems of particles



Some of the forces acting on the particles result from interactions with other particles.

All external forces act upon the particles by influences from outside of the system.

Regardless of the # of particles:

$$T_2 + V_2 = T_1 + V_1 + U_{1-2}^{(nc)},$$

where

$$T = \sum_{j=1}^{N} T_j = \sum_{j=1}^{N} \frac{1}{2} m_j v_j^2$$

and

$$V = \sum_{j=1}^{N} V_j$$

Work done by external forces is calculated by projecting forces onto path and integrating (the usual way).

Work done by internal forces though:

→ If $\vec{f}_{ij}$ has conservative force components (e.g. spring), NO NEED TO CALCULATE WORK as it is already accounted for in V_{spring}.

→ If $\vec{f}_{ij}$ has reactions due to rigid connections or sliding on smooth surfaces, $W_{net} = 0$

→ If $\vec{f}_{ij}$ has dissipative forces (friction or damping), need to account for these in the W-E eqn.

Try to make system AS LARGE AS POSSIBLE such that workless forces are internal to the system. However, try to leave friction forces external.

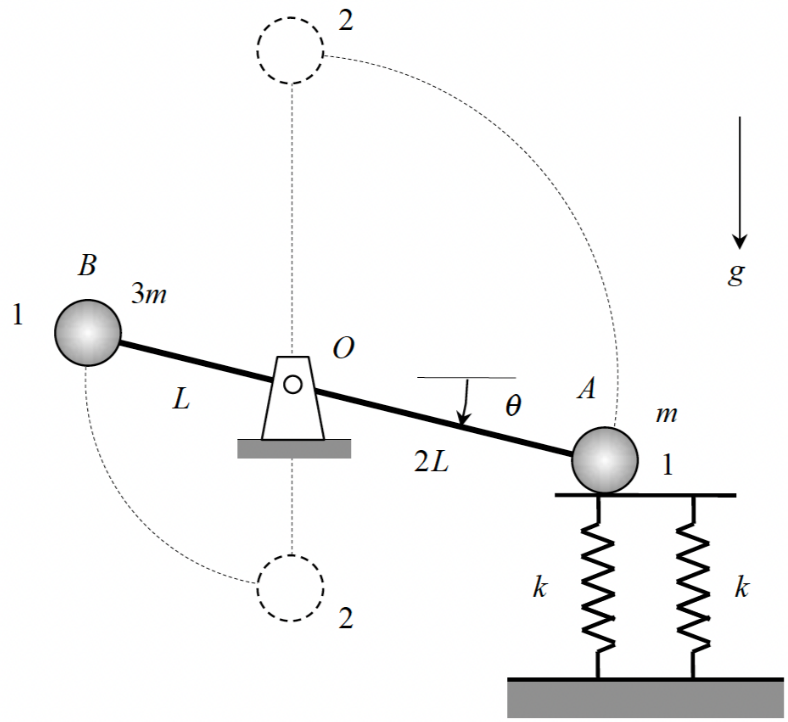
There will be ONLY ONE W-E eqn for the system. If more than one unknown is present, need to look elsewhere. (Kinematics)

Example 4.B.7

Given: Particles A and B, having masses of m and $3m$, respectively, are connected by rigid bar AB, with AB having negligible mass. Bar AB is pinned to ground with a pin joint at O. This system is released from rest at position 1 with $\theta = \theta_1$, with A in contact with a pair of identical springs, as shown in the figure. Each spring has a stiffness of k, and the springs are unstretched when $\theta = 0$. Assume the dimensions of the particles to be negligible.

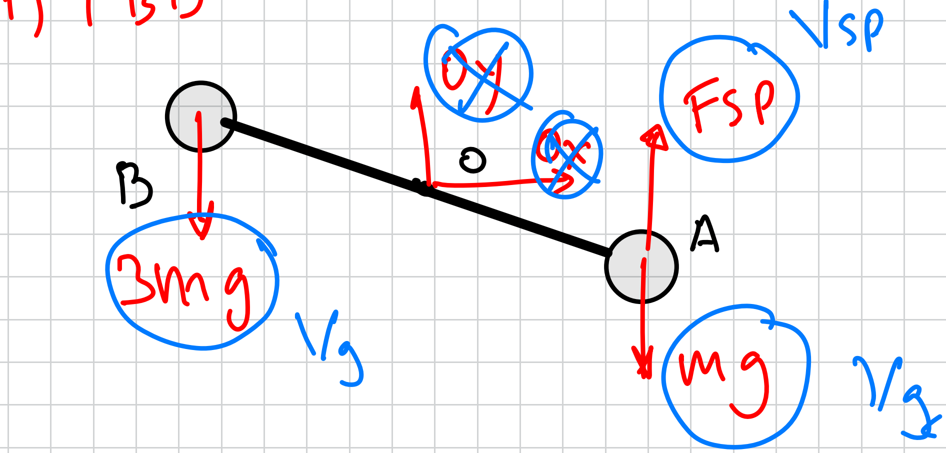
Find: Determine the speeds of particles A and B at position 2, where in position 2 particle A is directly above O.

Use the following parameters in your analysis: $\theta_1 = 36.87°$, $L = 0.1$ m, $m = 10$ kg and $k = 100$ N/m.



Solution

1) FBD

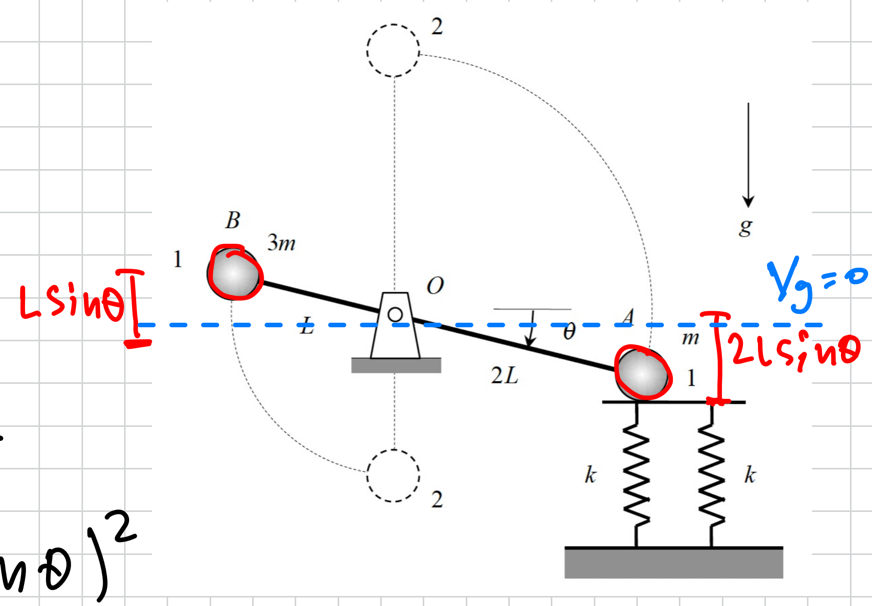


2) Coordinates

3) Kinetics:

$$T_1 + V_1 + \cancel{U^{NC}_{1-2}}^{\,0} = T_2 + V_2$$

$T_1 = 0$ (released from rest)



$$V_1 = -mg\,2L\sin\theta + 3mgL\sin\theta + \frac{1}{2}(2k)(2L\sin\theta)^2$$

$$T_2 = \frac{1}{2} m v_A^2 + \frac{1}{2} 3m v_B^2$$

$$V_2 = mg\,2L - 3mgL = -mgL$$

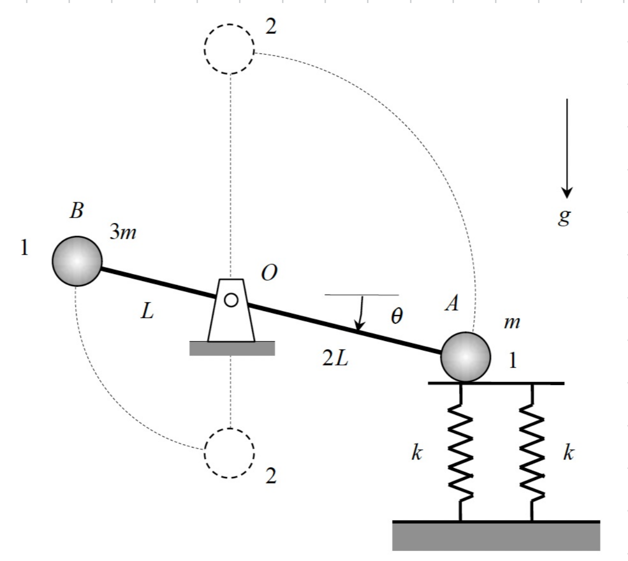
Plug everything into W/E eqn:

$$-mg\,2L\sin\theta + 3mgL\sin\theta + \frac{1}{2}(2k)(2L\sin\theta)^2 = \frac{1}{2} m v_A^2 + \frac{3}{2} m v_B^2 - mgL$$

4) Kinematics:

$$V_A = 2L\dot{\theta} \qquad V_B = L\dot{\theta}$$

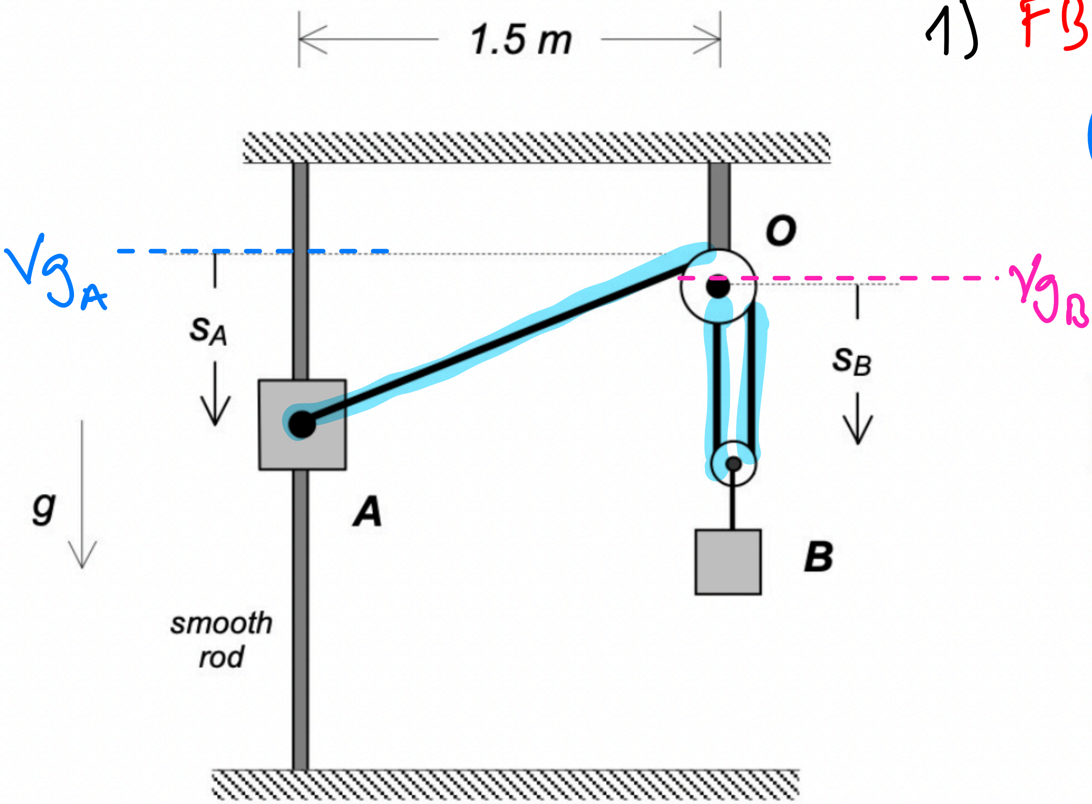
$$V_A = 2V_B$$



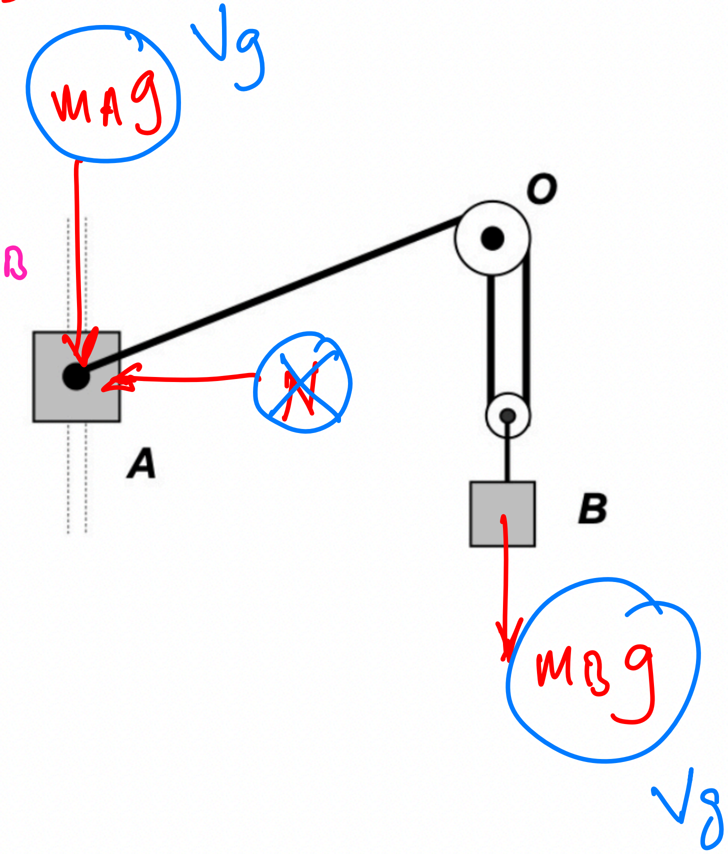
In-class Example 4.2

Given: Particles A and B (having masses of m_A and m_B) are interconnected by the cable-pulley system shown in the figure. Both particles are constrained to vertical motion with particle A able to slide on a smooth vertical rod. The system is released at $s_A = 0$ m with A traveling downward with a speed of v_{A1}. Assume the pulleys to be small, massless and frictionless.

Find: Find the speed of particle A when A has reached the position of s_A.



1) FBD



Use the following parameters in your analysis: $m_A = 10$ kg, $m_B = 10$ kg, $v_{A1} = 5$ m/s and $s_A = 2$ m.

2) Coord

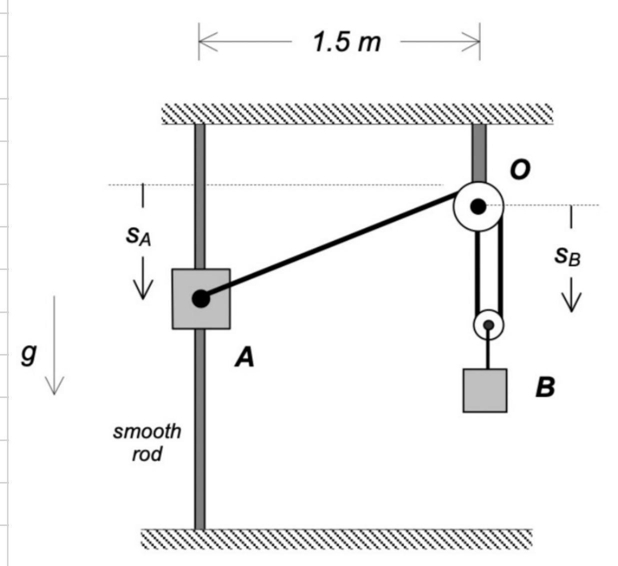
3) $T_1 + V_1 + \cancel{U^{NC}_{1-2}} = T_2 + V_2$ 0

$T_1 = \frac{1}{2} m_A V_{A_1}^2 + \frac{1}{2} m_B V_{B_1}^2$

$V_1 = 0 - m_B g S_{B_1}$

$T_2 = \frac{1}{2} m_A V_{A_2}^2 + \frac{1}{2} m_B V_{B_2}^2$

$V_2 = - m_A g S_A - m_B g S_B$



W/E Eqn:

$$\frac{1}{2} m_A V_{A_1}^2 + \frac{1}{2} m_B \underset{?}{V_{B_1}^2} - m_B g \underset{?}{S_{B_1}} = \frac{1}{2} m_A \underset{?}{V_{A_2}^2} + \frac{1}{2} m_B V_{B_2}^2 - m_A g S_A - m_B g \underset{?}{S_{B_2}} \quad (1)$$

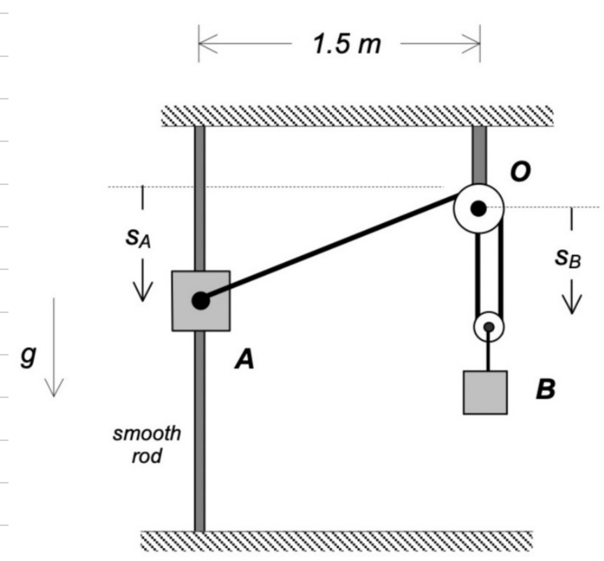
Kinematics

$L = \sqrt{S_A^2 + 1.5^2} + 2 S_{B_2}$; $L = 1.5 + 2 S_{B_1}$

$$L = L$$

$$1.5 + 2S_{B_1} = \sqrt{S_A^2 + 1.5^2} + 2S_{B_2}$$

$$S_{B_2} = S_{B_1} + \frac{1.5 - \sqrt{1.5^2 + S_A^2}}{2}$$



Relate velocities:

$$\dot{L} = 0 = \frac{1}{2}(S_A^2 + 1.5^2)^{-1/2}(2 S_A V_A) + 2 V_B$$

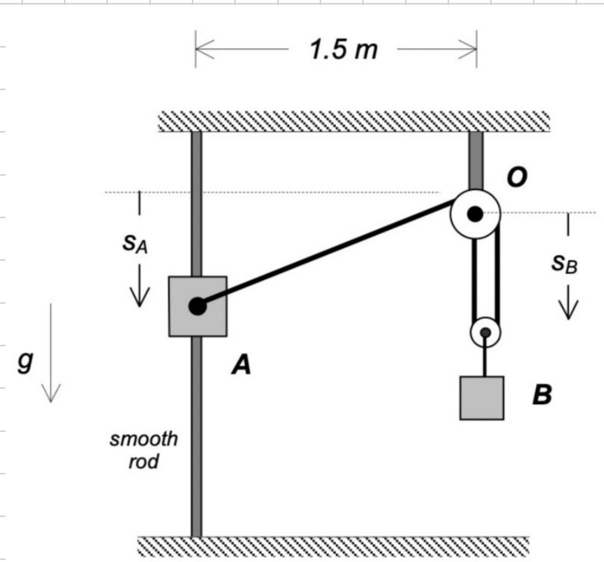
$$V_B = \frac{-S_A V_A}{2\sqrt{1.5^2 + S_A^2}}$$

If A $\downarrow$

B $\uparrow$

$$V_{B_1} = \frac{-(0) V_A}{2\sqrt{1.5^2 + S_A^2}} = 0$$

$$v_{B_2}^2 = \frac{s_A^2 v_A^2}{4(s_A^2 + 1.5^2)}$$



Plug everything into 1

$$v_{A_2} = \left[\frac{m_A v_{A_1}^2 + 2 m_A g s_A + m_B g \left(1.5 - \sqrt{s_A^2 + 1.5^2}\right)}{m_A + \dfrac{m_B s_A^2}{4(s_A^2 + 1.5^2)}} \right]$$