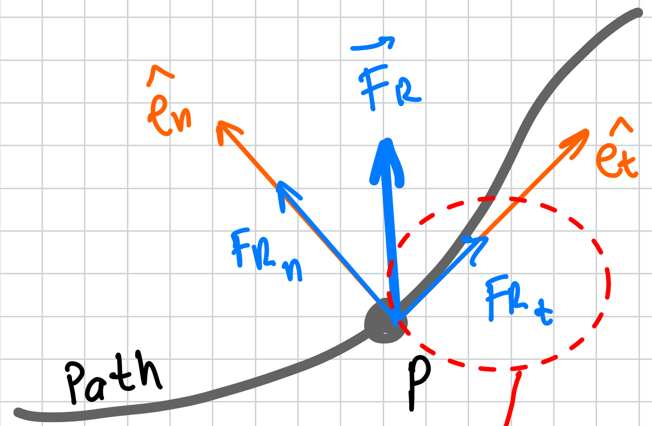
2/27/2026

PARTICLE KINETICS – WORK-ENERGY EQUATION

In the last section, we solved for forces & accelerations separated into normal & tangent components, i.e.:

$$\Sigma F_n = m a_n = m \frac{v^2}{\rho}$$

$$\Sigma F_t = m a_t = m \dot{v}$$



Only the tangential components of F_R are responsible for changing P's speed.

The normal components are only responsible for changing P's direction

⇒ If we want to study the change in speed, we should focus on tangential forces.

Let us develop the work-energy equations

Consider $\vec{R}$ to be the resultant force acting on P as it moves along the path:

$$\Sigma \vec{F} = m\vec{a} \qquad (1)$$

If we want to concentrate in change of speed, then we deal only with the tangential component of eq. (1) => We project (dot product) eq. (1) into $\hat{e}_t$:

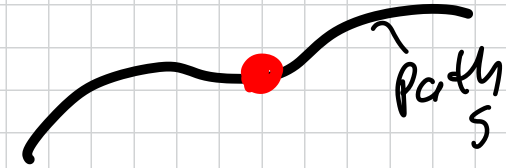
$$\vec{R} \cdot \hat{e}_t = m\vec{a} \cdot \hat{e}_t$$

$$(R_t \hat{e}_t + R_n \hat{e}_n) \cdot \hat{e}_t = m\left[\left(\dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n\right) \cdot \hat{e}_t\right]$$

$$R_t \underbrace{\hat{e}_t \cdot \hat{e}_t}_{1} + R_n \cancel{\hat{e}_n \cdot \hat{e}_t}^{0} = m\dot{v}\,\underbrace{\hat{e}_t \cdot \hat{e}_t}_{1} + m\frac{v^2}{\rho}\cancel{\hat{e}_n \cdot \hat{e}_t}^{0}$$

$$R_t\,(1) = m\dot{v}\,(1)$$

$$\Rightarrow \quad R_t = m\frac{dv}{dt}$$



Now, let's express v as a function of position:

$$v = \frac{ds}{dt} \Rightarrow \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = v\frac{dv}{ds}$$

Such that

$$R_t = m v \frac{dv}{ds}.$$

separating and integrating from states 1 to 2:

$$\int_1^2 R_t\, ds = \int_1^2 m v\, dv$$

$$\boxed{\underbrace{\int_1^2 R_t\, ds}_{U_{1-2}} = \underbrace{\frac{1}{2}m v_2^2 - \frac{1}{2}m v_1^2}_{\Delta T_{2.1}}}$$

Work done by R from 1 to 2

Change in kinetic energy from 1 to 2

$$\boxed{U_{1-2} = T_2 - T_1}$$ Work-Energy Equation

This relationship can also be extended to Cartesian & polar descriptions:

$$U_{1-2} = \int_1^2 (\vec{R} \cdot \hat{e}_t)\, ds = \int_1^2 R_x\, dx + \int_1^2 R_y\, dy ,$$

If $\underbrace{y = f(x)}_{\text{path}}$ is given, $\Rightarrow$ $dy = f'(x)\, dx$

$$\Rightarrow U_{1-2} = \int_1^2 R_x\, dx + \int_1^2 R_y\, f'(x)\, dx ,$$

between x_1 and x_2. This form of the work equation is useful when forces and path are known in Cartesian description.

For the case of polar, we need to relate $\hat{e}_t\, ds$ to $\hat{e}_r$ and $\hat{e}_\theta$

$$\hat{e}_t\, ds = \vec{v}\, dt = \left(\frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta \right) dt$$

$$= dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta$$

and $$\vec{R} = R_r\,\hat{e}_r + R_\theta\,\hat{e}_\theta$$

$$\Rightarrow \quad U_{1-2} = \int_1^2 R_r\,dr + \int_1^2 R_\theta\, r\,d\theta$$

Conservative Forces & Potential Energy

Since $U^{1-2} = \int_1^2 \vec{F} \cdot d\vec{r}$, i.e. the integral of projection of force over path, U is dependent on distance traveled and path geometry. Some forces cause work between 1 and 2 that are path independent. these are called conservative forces. (only dependent on initial and final position)

CONSERVATIVE FORCES
1) forces due to springs
2) forces due to weight

$$U_{1-2} = -(V_2 - V_1)$$

The work done by these forces is the difference between the associated potential energies at the beginning and end of the motion (positions 1 & 2).

$$V = V_{spring} + V_{gravity} \qquad (V\text{: Potential energy})$$

$$V_{spring} = \frac{1}{2}K(L - L_0)^2 \quad \frac{1}{2}Kx^2 \quad > 0 \quad \text{ALWAYS}$$

$$V_{gravity} = mgh \quad \begin{cases} h > 0 & P \text{ above datum} \\ h < 0 & P \text{ below datum} \end{cases}$$

Datum: V_g — 0 — User-defined reference for zero potential due to gravitational effects.

... In Sumary:

V_{sp} and V_g do not require integration!
Just need to evaluate the potential function @ begining and end of motion

If we separate the contributions of conservative and non-conservative work done:

$$U_{1-2} = U_{1-2}^{(c)} + U_{1-2}^{(nc)}$$

$$\Rightarrow U_{1-2} = -(V_2 - V_1) + U_{1-2}^{(nc)}$$

(c) : weights, springs

(NC) : all other non perpendicular forces

The original work-energy eqn is then

$$U_{1-2} = T_2 - T_1$$

$$-(V_2 - V_1) + U_{1-2}^{(nc)} = T_2 - T_1$$

$$\boxed{T_1 + V_1 + U_{1-2}^{(NC)} = T_2 + V_2}$$

Work-Energy EQUATION w/ (C) and (NC) Forces

T_1 : Kinetic energy @ state 1 : $\frac{1}{2} m v_1^2$

V_1 : Potential energy @ state 1 : $\frac{1}{2} K \Delta_1^2 + mgh_1$

U_{1-2} : Work done by non conservative forces 1-2

$\int_1^2 R_t \, ds$ usually $|\vec{F}| \cdot d$ with d in direction of $\vec{F}$

T_2 : Kinetic energy @ state 2 : $\frac{1}{2} m v_2^2$

V_2 : Potential energy @ state 2 : $\frac{1}{2} K \Delta_2^2 + mgh_2$

KINETICS PROBLEMS

2) FREE BODY DIAGRAMS

1) COORDINATE SYSTEM

3) KINETICS EQUATIONS

↳ Energy

3) KINEMATICS

4) SOLVE

SOLUTION

If using energy, Keep system "big" to eliminate external forces.

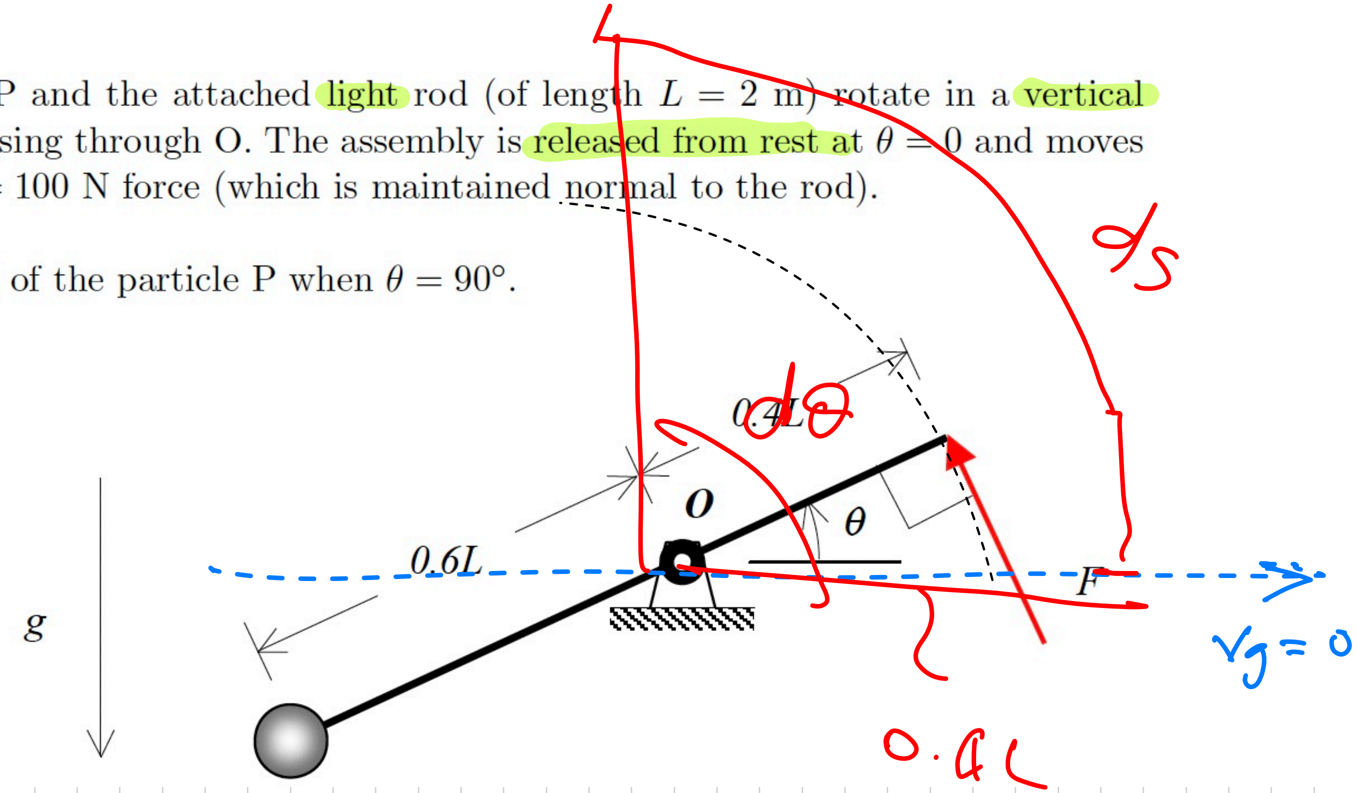
Example 4.B.1

Given: The 6 kg particle P and the attached light rod (of length $L = 2$ m) rotate in a vertical plane about a fixed axis passing through O. The assembly is released from rest at $\theta = 0$ and moves under the action of the $F = 100$ N force (which is maintained normal to the rod).

Find: Determine the speed of the particle P when $\theta = 90°$.

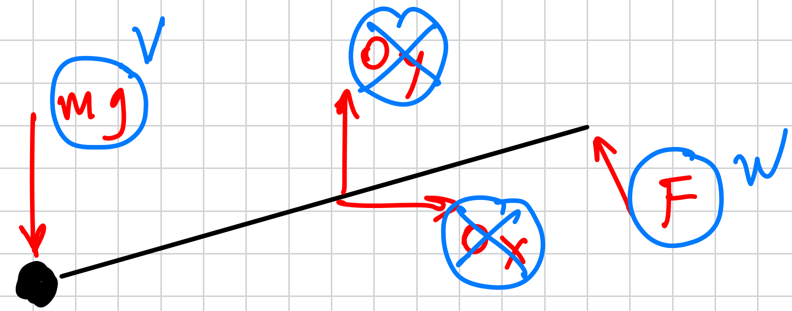
Solution

Problem asks for changes in speed after changes in position



$\Rightarrow$ Clear indicator of use of W/E equation

FBD

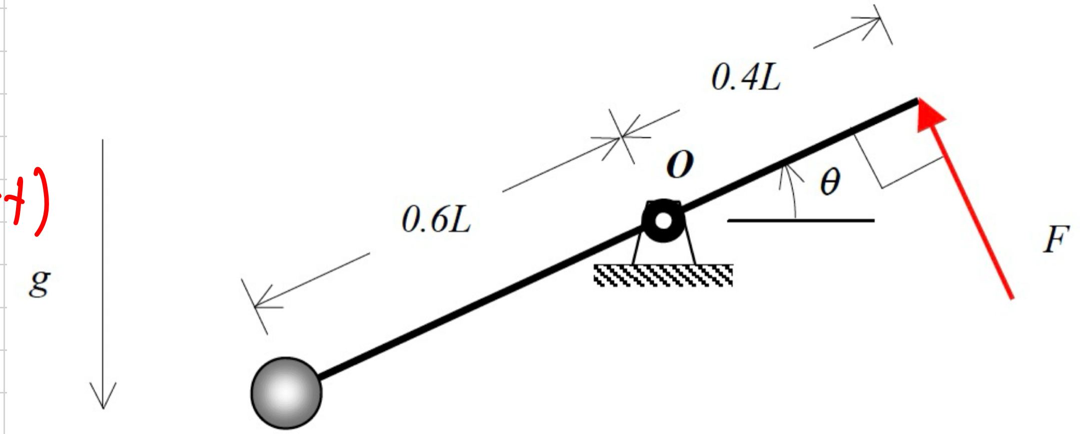


$$T_1 + V_1 + U_{1-2}^{(NC)} = T_2 + V_2$$

Position 1 $(\theta = 0)$

$T_1 = \frac{1}{2} m \cancel{v_1}^{2} = 0$ (starts from rest)

$V_1 = mg\cancel{h_1} = 0$ (datum line)



Position 2 $(\theta = \frac{\pi}{2})$

$T_2 = \frac{1}{2} m v_2^2$; $V_2 = -mgh_2 = -mg(0.6L)$

<u>Work</u>

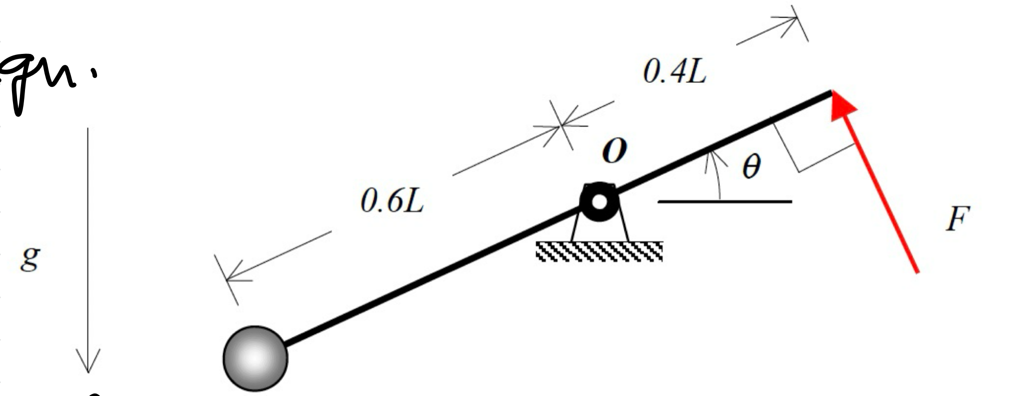
$$U^{NC}_{1-2} = \int_1^2 \underbrace{\vec{F}\cdot\hat{e}_t}\, ds \;, \quad F = Fe_t \quad ds = 0.4L\,d\theta$$

$$= \int_0^{\pi/2} F\underbrace{(e_t \cdot e_t)}_{1} \cdot (0.4L\,d\theta) = \left[0.4FL\theta\right]_0^{\frac{\pi}{2}}$$

$$= 0.2\pi LF$$

Always remember $W = F \times$ distance

Plugging all into W/E eqn.



$$0 + 0 + 0.2\pi L F$$

$$= \frac{1}{2} m v_2^2 - mg(0.6L)$$

$\Rightarrow$ Solve for v_2