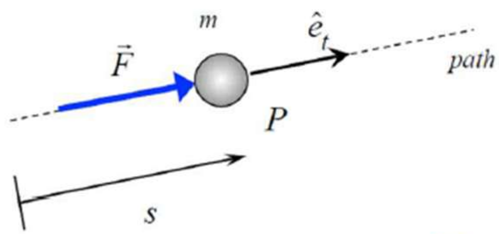
ME 274: Basic Mechanics II

Lecture 19:Particle Kinetics – Work-Energy Equation

PURDUE UNIVERSITY | School of Mechanical Engineering

Kinetics for the rectilinear Motion of Particles – Net force dependent on position

For a particle of mass m traveling along a straight path and experiencing a force $\vec{F} = F\hat{e}_t$



$$\sum F_t = ma_t \quad \Rightarrow \quad F = m\boxed{\frac{dv}{dt}}$$ ← need to write in terms of s

If force is a function of position: $F = F(s)$

$F = m\frac{dv}{dt}$ → using chain rule

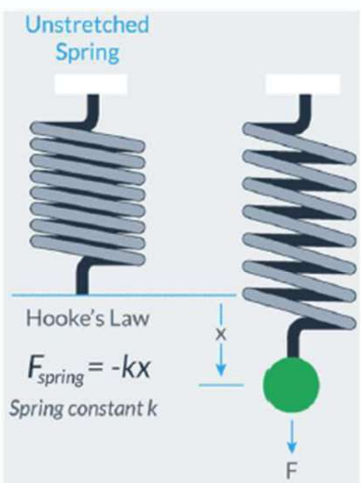
$= m\frac{dv}{ds}\frac{ds}{dt}$ ← velocity!

$F(s) = mv\frac{dv}{ds}$

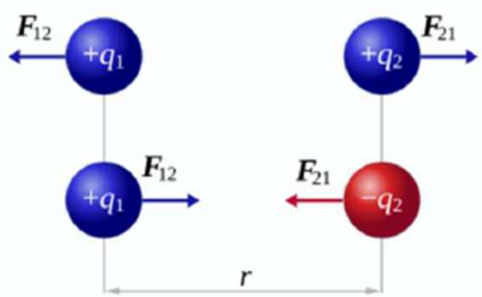
$\int_{s_1}^{s_2} F(s) = \int_{v_1}^{v_2} mv$

$\underbrace{\int_{s_1}^{s_2} F(s)}_{\text{work}} = \frac{mv^2}{2}\Big|_{v_1}^{v_2} = \underbrace{\frac{1}{2}m(v_2^2 - v_1^2)}_{\text{energy}}$

Ex: spring forces



Ex: magnetic forces



$$|F_{12}| = |F_{21}| = k_e \frac{|q_1 \times q_2|}{r^2}$$

Newton's law for the path description of motion

$\vec{R} \rightarrow$ resultant force acting on the particle as it moves along the path

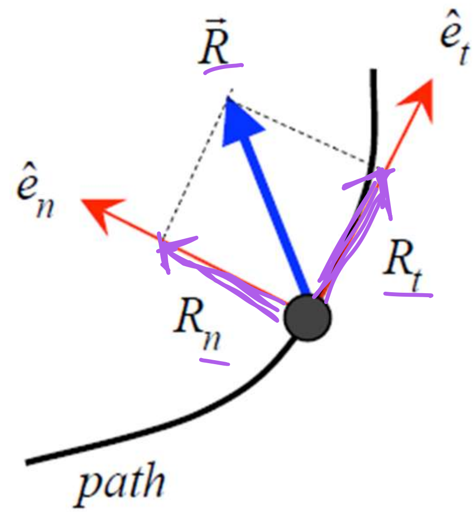
$$\vec{R} = \Sigma \vec{F} = m\vec{a}$$

Remember in the path description $\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$

Breaking $\vec{R}$ into components:

$$R_t = m\dot{v} = m\frac{dv}{dt}$$

$$R_n = m\frac{v^2}{\rho}$$



What is each component responsible for?

$R_t \Rightarrow$ changing speed along path

$R_n \Rightarrow$ turning

If we are interested in studying the **change in speed** of a particle, we need only deal with the **tangential components** of the forces acting of the particle $V(s)$

$$\sum \vec{F} = m\vec{a}$$

projection $\rightarrow$ $\vec{R}\cdot\hat{e}_t = m\vec{a}\cdot\hat{e}_t$ $\swarrow$ $\vec{a}$ path description

$$(R_t\hat{e}_t + R_n\hat{e}_n)\cdot\hat{e}_t = m\left(\frac{dv}{dt}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n\right)\cdot\hat{e}_t$$

$$R_t\hat{e}_t\cdot\hat{e}_t + R_n\hat{e}_n\cdot\hat{e}_t = m\frac{dv}{dt}\hat{e}_t\cdot\hat{e}_t + m\frac{v^2}{\rho}\hat{e}_n\cdot\hat{e}_t$$

$$R_t(1) + \cancel{R_n(0)} = m\frac{dv}{dt}(1) + \cancel{\frac{v^2}{\rho}(0)}$$

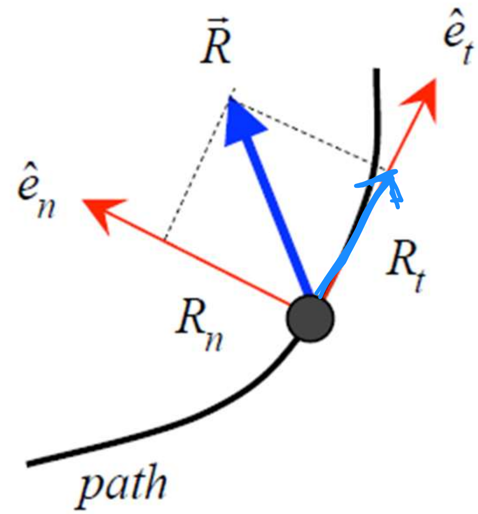
$$R_t = m\frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = v\frac{dv}{ds}$$

$$R_t = mv\frac{dv}{ds}$$

$$\int_1^2 R_t\,ds = m\int_1^2 v\,dv = \frac{1}{2}m{v_2}^2 - \frac{1}{2}m{v_1}^2$$

$$U_{1\to 2} = T_2 - T_1$$

$$U_{1\to 2} = \int_1^2 R_t\,ds$$



Work done by external forces = Change in kinetic energy

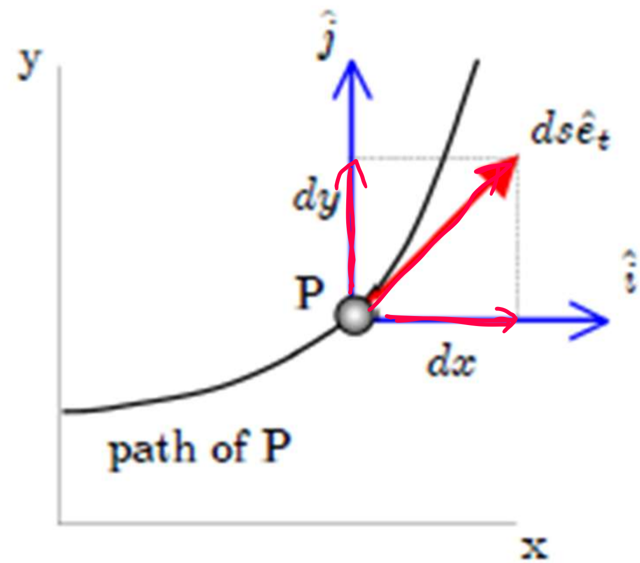
Kinetic energy $\rightarrow$ $T = \frac{1}{2}mv^2$

What if our motion/forces are described with the cartesian description?

From our work-energy equation:

$$U_{1\to 2} = \int_1^2 \left(\vec{R}\cdot\hat{e}_t\right)\, ds$$

To convert to cartesian form, we need to relate $\hat{e}_t$ to $\hat{\imath}, \hat{\jmath}$, and s to x, y



$$U_{1\to 2} = \int_1^2 \left(\vec{R}\cdot\hat{e}_t\right)\, ds$$

$$= \int_1^2 \left(R_x\,\hat{\imath} + R_y\,\hat{\jmath}\right)\cdot\left(dx\,\hat{\imath} + dy\,\hat{\jmath}\right)$$

$$= \int_1^2 R_x\,dx + \int_1^2 R_y\,dy$$

path coords

$\vec{v} = v\,\hat{e}_t = \frac{ds}{dt}\hat{e}_t$

cartesian

$\vec{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}$

$\frac{ds}{\cancel{dt}}\hat{e}_t = \frac{dx}{\cancel{dt}}\hat{\imath} + \frac{dy}{\cancel{dt}}\hat{\jmath}$

$ds\,\hat{e}_t = dx\,\hat{\imath} + dy\,\hat{\jmath}$

$y = f(x)$

Conservative forces and Potential Energy

In general, the work done by a force $\vec{F}$ is dependent on the shape/distance of the path traveled.

Conservative forces are a class of forces where the work is **path independent** and instead only depend on the **start and end positions**.

Examples: spring force, weight

You do not need to integrate conservative forces – instead, take the difference of their potential energy functions between points 1 and 2!

We can rewrite our work energy expression in terms of conservative and non-conservative components:

$$U_{1\to 2} = U_{1\to 2}^{(c)} + U_{1\to 2}^{(nc)} = -(V_2 - V_1) + U_{1\to 2}^{(nc)} \longrightarrow T_2 + V_2 = T_1 + V_1 + U_{1\to 2}^{(nc)}$$

work energy

Spring potential energy

ΔL

$$V_{sp} = \frac{k}{2}(L - L_0)^2$$

- L –current length of spring
- L_0 - unstretched spring length
- k – spring constant

Note: V_{sp} is ALWAYS ≥ 0

Gravitational potential energy

$$V_{gr} = mgh$$

- h – height above/below datum

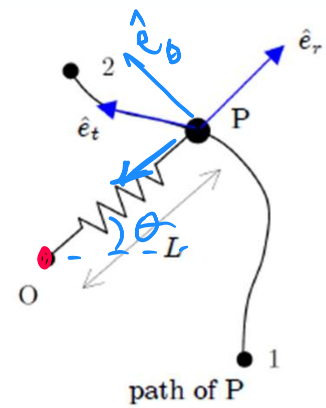
Note: The sign of V_{gr} matches the sign of h, i.e. if the ending position is **above** the datum $V_{gr} > 0$, If the ending position is **below** the datum $V_{gr} < 0$

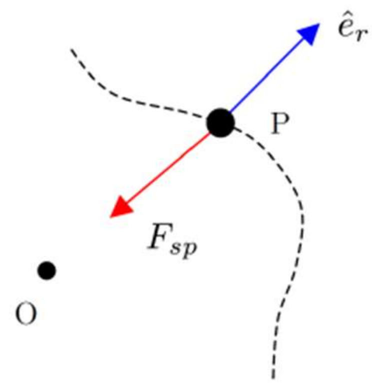
potential energy V

Example – work and potential energy of a spring force

Spring force: $F_{sp} = k(L - L_0)$

polar form $\vec{F}_{sp} = -k(L-L_0)\hat{e}_r$





Work done by spring force:

$$U_{1\to 2} = \int_1^2 (F_{sp} \cdot \hat{e}_t)\, ds$$

Convert between path and polar descriptions:

polar : $\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

path : $\vec{V} = \frac{ds}{dt}\hat{e}_t$

$r = L$

$\dot{r} = \frac{dL}{dt}$

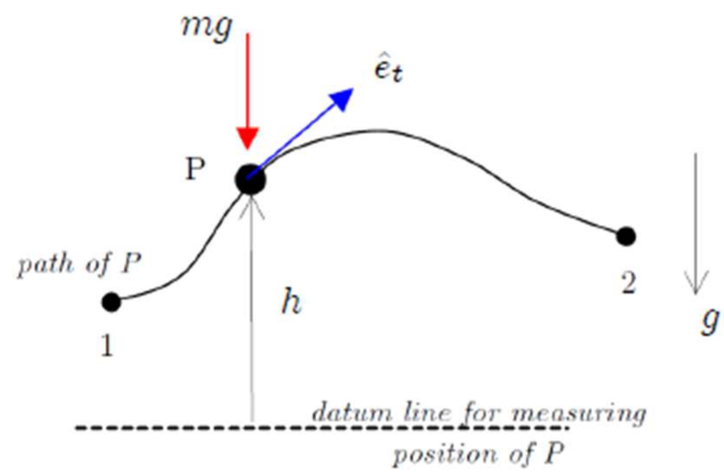
$$\hat{e}_t ds = dL\,\hat{e}_r + L\,d\theta\,\hat{e}_\theta$$

$$U_{1\to 2} = \int_1^2 \left[-k(L-L_0)\hat{e}_r \cdot (dL\,\hat{e}_r + L\,d\theta\,\hat{e}_\theta)\right] = -k\int_1^2 (L-L_0)\,dL$$

$$= -\frac{k}{2}\left[\underbrace{(L_2-L_0)^2}_{V_{sp} = \frac{1}{2}k(\Delta L)^2} - (L_1-L_0)^2\right] = -(V_2 - V_1)$$

Example – work and potential energy of a weight force

Weight force: $\vec{F}_w = -mg\hat{j}$



Work done by weight force: $U_{1\to 2} = \int_1^2 (\vec{F}_w \cdot \hat{e}_t)\,ds$

Convert between path and cartesian descriptions:

$$U_{1\to 2} = \int_1^2 (-mg\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j})$$

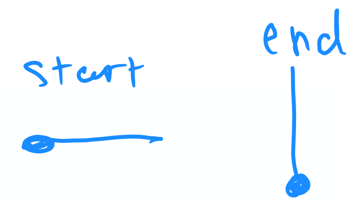
$$= -mg\int_1^2 dy = -mg(h_2 - h_1)$$

$$= -(V_2 - V_1)$$

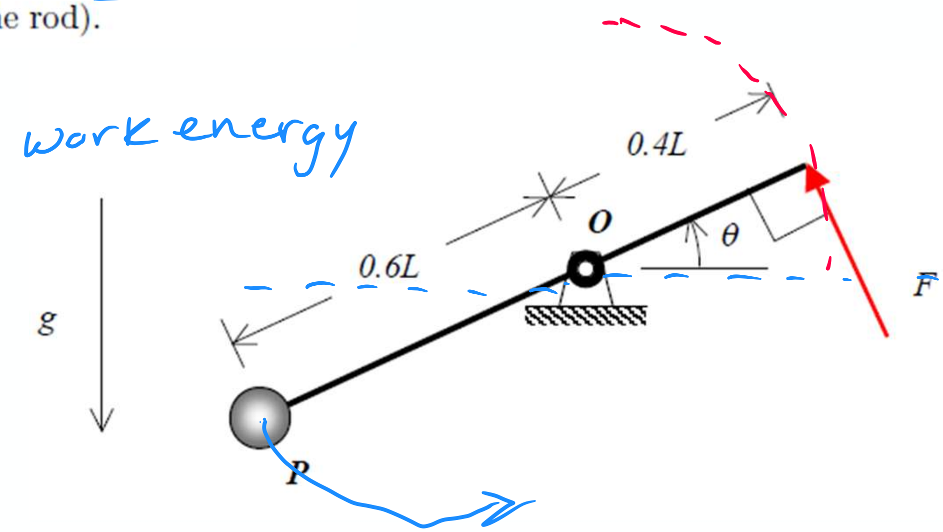
Example 4.B.1

Given: The 6 kg particle P and the attached light rod (of length $L = 2$ m) rotate in a vertical plane about a fixed axis passing through O. The assembly is released from rest at $\theta = 0$ and moves under the action of the $F = 100$ N force (which is maintained normal to the rod).

Find: Determine the speed of the particle P when $\theta = 90°$.



speed after position change → work energy



2) Kinetic

$$T_1 + V_1 + U_{1\to2} = T_2 + V_2$$

$T_1 = 0 \leftarrow$ (RFR)

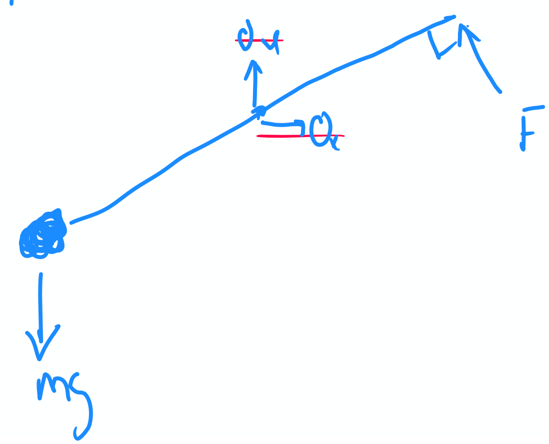
$V_1 = 0 \leftarrow$ at datum

$$U_{1\to2} = \int_1^2 \vec{F}\cdot d\vec{r} = F(.4L)\left(\frac{\pi}{2}\right)$$

$$T_2 = \frac{1}{2} m v_2^2$$

$$V_2 = -mg(0.6L)$$

1) FBD



Solving:

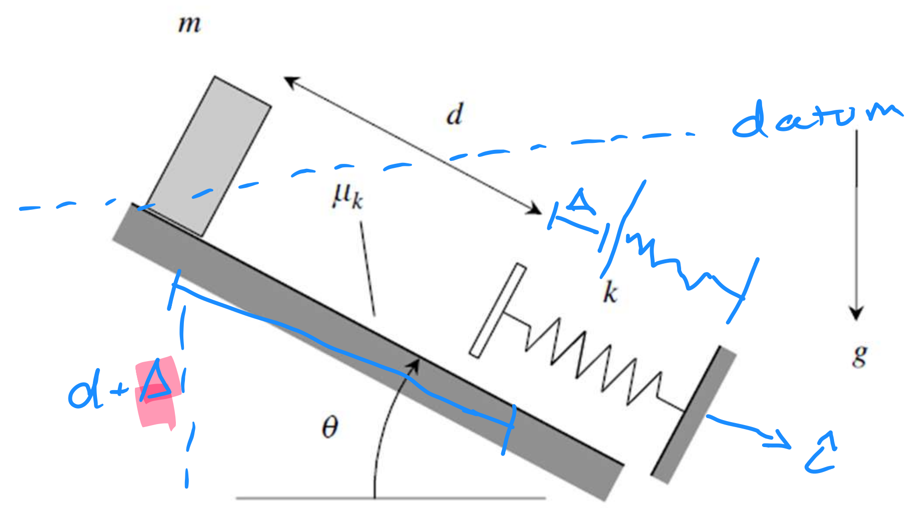
$$0 + 0 + .2\pi L F = \frac{1}{2} m v_2^2 - mg(0.6L)$$

→ solve V_2

Example 4.B.2

Given: A block of mass m is released from rest on a rough, inclined surface (with a coefficient of kinetic friction μ_k) and slides toward an uncompressed spring of stiffness k.

Find: Determine the maximum deflection of the spring.



2) $T_1 + V_1 + U^{NC}_{1\to2} = T_2 + V_2$

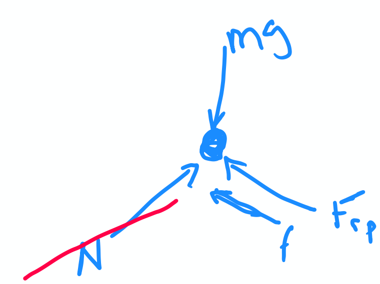
$T_1 = 0$ (RFR)

$V_1 = 0$ at datum

$U^{NC}_{1\to2} = \int_0^{\delta} \vec{F}\cdot\vec{dr} = -f\hat{i}\cdot(d+\Delta)\hat{i}$

$= -f(d+\Delta)$

1) FBD



$T_2 = 0$ ← at max deflection, velocity is instantaneously 0

$V_2 = -mg(d+\Delta)\sin\theta + \frac{1}{2}k\Delta^2$

$0 + 0 - f(d+\Delta) = 0 - mg(d+\Delta)\sin\theta + \frac{1}{2}k\Delta^2$

Solve for Δ