ME 274: Basic Mechanics II

Lecture 21: Particle Kinetics – Angular Impulse Momentum

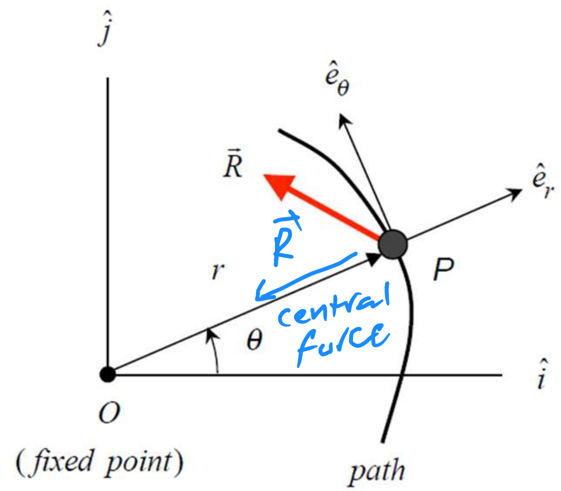
PURDUE UNIVERSITY | School of Mechanical Engineering

Overview: Angular Impulse Momentum

Moment about O | general | polar

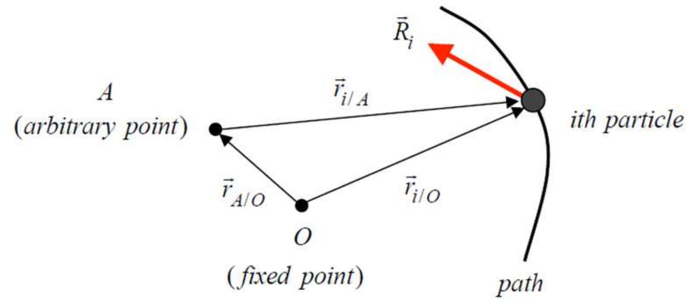
AIM equation: $\int_1^2 \vec{M}_O = \vec{H}_{O2} - \vec{H}_{O1}$, $\quad \vec{M}_O = \vec{r}_{P/O} \times \vec{R}$, $\quad \vec{H}_O = \vec{r}_{P/O} \times m\vec{v}_P$ **or** $\vec{H}_O = m(r^2\dot{\theta})\hat{k}$

- The AIM equation is the rotational analog to the LIM equation. It relates a **change in angular velocity** to a **moment applied over time.**
- Angular momentum, $\vec{H}_O$ is a vector. For 2D motion, it will always be pointed in or out of the page $(\hat{k})$.
- The AIM equation **only gives info about the rotational component** $r\dot{\theta}$ of velocity $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$. If you also need the radial component, use WE.
- In central force problems, $\vec{R}$ points directly towards/away from O. The net moment in these problems $= 0$.
- When $\sum \overrightarrow{\boldsymbol{M}_O} = \mathbf{0}$, angular momentum is conserved! $\Rightarrow \overrightarrow{\boldsymbol{H}_{O2}} = \overrightarrow{\boldsymbol{H}_{O1}}$
- Angular velocity $\vec{\omega}$ must decrease (or increase) as the radial distance r increases (or decreases) when angular momentum is conserved.
  $(r\uparrow, \omega\downarrow \;\; \boldsymbol{or} \;\; r\downarrow, \omega\uparrow)$



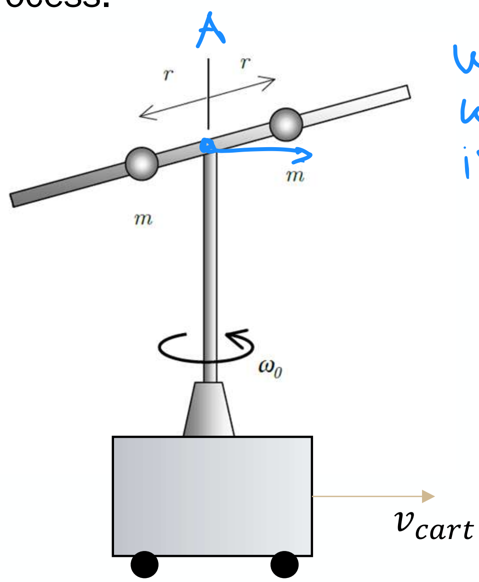
AIM for a system of multiple particles

Note: these equations will be useful in our next chapter on angular momentum of rigid bodies. discrete sum vs. integral

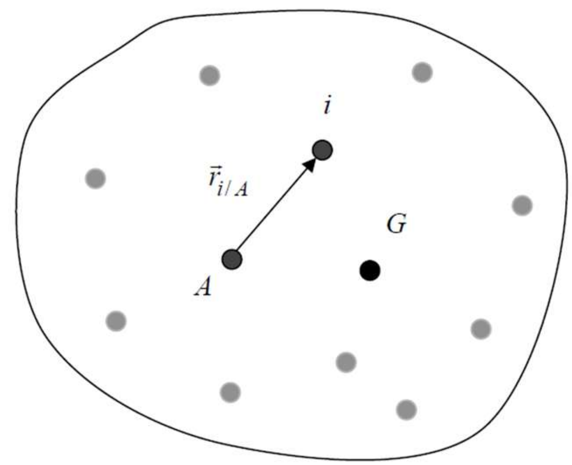


For a system of particles, we sum the contribution of the angular momentum of each point.

Here we use an arbitrary reference point A (can be a moving point) to make these equations more general – later we will see how our selection of this point can simplify our solution process.



useful in cases where our system is rotating + translating



system of N particles

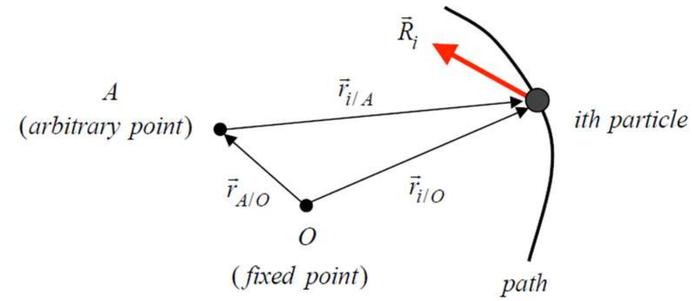
AIM for a system of multiple particles

If we have a system of N particles, looking at the particle $\boldsymbol{i}$

Kinematics:

$$\vec{r}_{i/O} = \vec{r}_{A/O} + \vec{r}_{i/A} \quad , \qquad \vec{v}_i = \vec{v}_A + \vec{v}_{i/A}$$

if O is origin $\quad \vec{r}_i = \vec{r}_A + \vec{r}_{i/A}$



Newton's 2[nd] law:

$$\vec{R}_i = \sum \vec{F}_i = m_i \vec{a}_i = m_i \frac{d\vec{v}_i}{dt}$$

The moment of $\vec{R}_i$ about our reference point A:

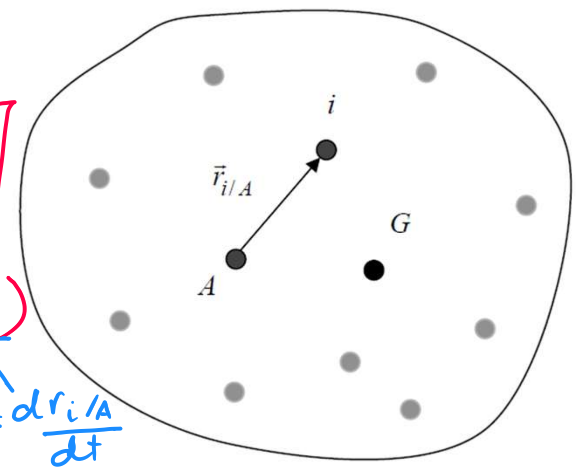
$$\vec{M}_{A,i} = \vec{r}_{i/A} \times \vec{R}_i = \vec{r}_{i/A} \times m_i \frac{d\vec{v}_i}{dt}$$

$$= \vec{r}_{i/A} \times m \frac{d}{dt}\left(\vec{v}_A + \vec{v}_{i/A}\right) = m\left[\left(\vec{r}_{i/A} \times \underbrace{\frac{d}{dt}\vec{v}_A}_{=\vec{a}_A}\right) + \left(\vec{r}_{i/A} \times \frac{d}{dt}\vec{v}_{i/A}\right)\right] \ast$$

$$= \left(r_{i/A} \times m_i \vec{a}_i\right) + \frac{d}{dt}\left(\vec{r}_{i/A} \times m_i \vec{v}_{i/A}\right) - \underbrace{\frac{d\vec{r}_{i/A}}{dt} \times \left(m_i \vec{v}_{i/A}\right)}_{0}$$

$$\vec{v}_{i/A} = \frac{dr_{i/A}}{dt}$$

$$= \left(\vec{r}_{i/A} \times m_i \vec{a}_i\right) + \frac{d}{dt}\underbrace{\left(\vec{r}_{i/A} \times m_i \vec{v}_{i/A}\right)}_{= \vec{H}_{i/A}}$$



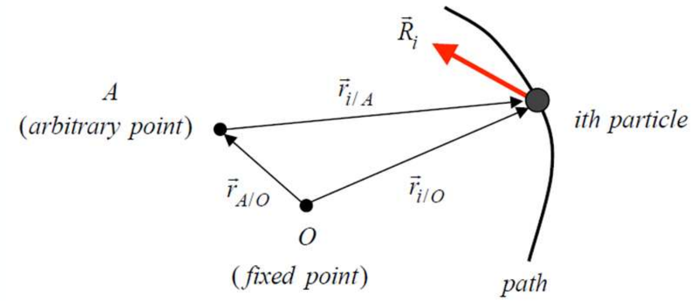
system of N particles

$\ast$ Remember from last lecture: $\frac{d}{dt}\left(\vec{a} \times \vec{b}\right) = \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt} \quad \Rightarrow \quad \vec{a} \times \frac{d\vec{b}}{dt} = \frac{d}{dt}\left(\vec{a} \times \vec{b}\right) - \frac{d\vec{a}}{dt} \times \vec{b}$

AIM for a system of multiple particles

Now looking at the total for all N particles (i.e. for $i = 1, 2, 3, \ldots N$)
Remember the definition of the center of mass for multiple particles:

$$* \quad \vec{r}_G = \frac{1}{m}\sum_{i=1}^{N} m_i \vec{r}_i\,, \qquad \underset{*}{m} = \sum_{i=1}^{N} m_i$$

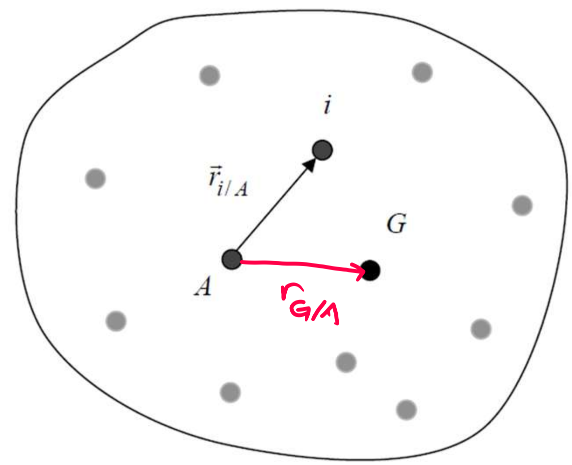


We sum the angular momentum equation of all particles:

$$\vec{M}_{A,i} = \left(\vec{r}_{i/A} \times m_i \vec{a}_A\right) + \frac{d}{dt}\left(\vec{r}_{i/A} \times m_i \vec{V}_{i/A}\right) \leftarrow \text{for 1 particle}$$

$$\sum_{i=1}^{N} \vec{M}_{A,i} = \sum_{i=1}^{N}\left(\overbrace{m_i \vec{r}_{i/A}}^{= m\vec{r}_{G/A}} \times \vec{a}_A\right) + \sum_{i=1}^{N} \frac{d}{dt}\left(\vec{r}_{i/A} \times m_i \vec{V}_{i/A}\right)$$

$$= \underset{\text{total mass}}{\underline{\overset{*}{m}}}\, \vec{r}_{G/A} \times \vec{a}_A + \sum_{i=1}^{N} \frac{d}{dt}\left(\vec{r}_{i/A} \times m_i \vec{V}_{i/A}\right)$$

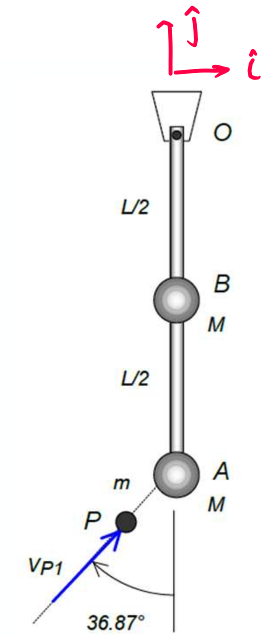


system of N particles

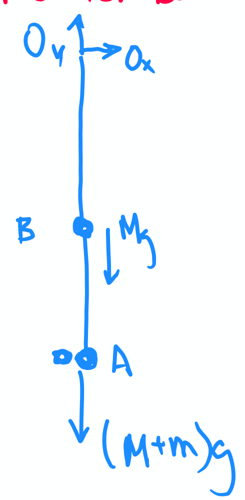
Example 4.D.5

Given: Particles A and B (each having a mass of M) are attached to rigid bar OA (this bar has negligible mass). Bar OA is pinned to ground at end O. This system is at rest when A is struck by a bullet P (having a mass of m) with the bullet traveling in the direction shown with a speed of v_{P1}. Immediately upon impact, the bullet becomes embedded in particle A.

Find: Determine the angular velocity of bar OA immediately after the collision is completed.



remember for AIM, make system as big as possible



$$\Sigma M_O = 0 \rightarrow \text{ang. Mo. cons.}$$

$$\vec{H}_{O1} = \vec{H}_{O2}$$

$\vec{H}_{O1} \rightarrow$ ang. mo. of bullet before impact

$$= \vec{r}_{p/O} \times m_p \vec{V}_{p1}$$

$$= -L\hat{j} \times m(V_{p1}\sin\theta\,\hat{i} + V_{p1}\cos\theta\,\hat{j}) = mLV_{p1}\sin\theta\,\hat{k}$$

$\vec{H}_{O2} \rightarrow$ ang. mo. of bullet, A + B after impact

$$\vec{H}_{O2} = \sum_i^N \vec{r}_{2i} \times m_i V_{2i}$$

O is IC $\rightarrow V_A = r\omega$

$$= -L\hat{j} \times (m+M)(L\omega_2\,\hat{i}) - \frac{L}{2}\hat{j} \times M\frac{L}{2}\omega_2\,\hat{i}$$

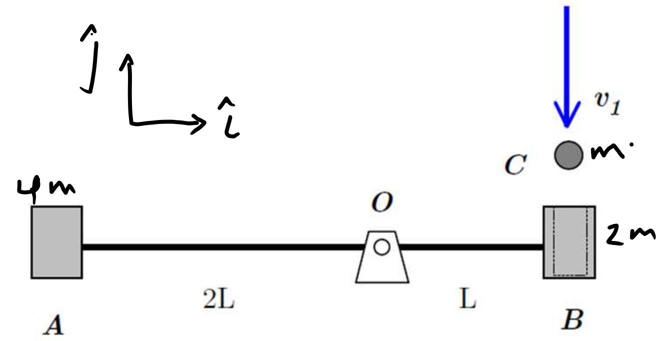
$$= (m+M)L^2\omega_2\,\hat{k} + M\frac{L^2}{4}\omega_2\,\hat{k}$$

$$mLV_{p1}\sin\theta\,\hat{k} = (m+M)L^2\omega_2\,\hat{k} + M\frac{L^2}{4}\omega_2\,\hat{k}$$ $\rightarrow$ solve for ω_2

Example 4.D.6

Given: Particles A and B, of masses $4m$ and $2m$, respectively, are attached to the ends a stationary rigid rod of negligible mass. The rod is pinned to ground at O. A third particle C, of mass m, strikes particle B with a speed of v_1. On impact, C sticks to B. The system lies in the horizontal plane.

Find: Determine the angular speed of the bar immediately after the impact occurs.



$$\sum \vec{M} = 0 \rightarrow \vec{H}_{2O} = \vec{H}_{1O}$$

$\vec{H}_{01}$ → just bullet moving

$$\vec{H}_{01} = \sum_i \vec{r}_{i1} \times m_i v_{i1} = L\hat{i} \times m(-v_1\hat{j}) = -mv_1 L\hat{k}$$

$$\vec{H}_{02} = \sum_i \vec{r}_{i2} \times m_i V_{i2} = L\hat{i} \times 3m(L\omega_2\hat{j}) + (-2L\hat{i}) \times 4m(-2L\omega_2\hat{j})$$

$$= 3m\omega_2 L^2\hat{k} + 16m\omega_2 L^2\hat{k} = 19m\omega_2 L^2\hat{k}$$

$$-mv_1 L = 19m\omega_2 L^2 \longrightarrow \omega_2 = \frac{-v_1}{19L}$$