ME 274: Basic Mechanics II

Lecture 21: Particle Kinetics – Angular Impulse Momentum

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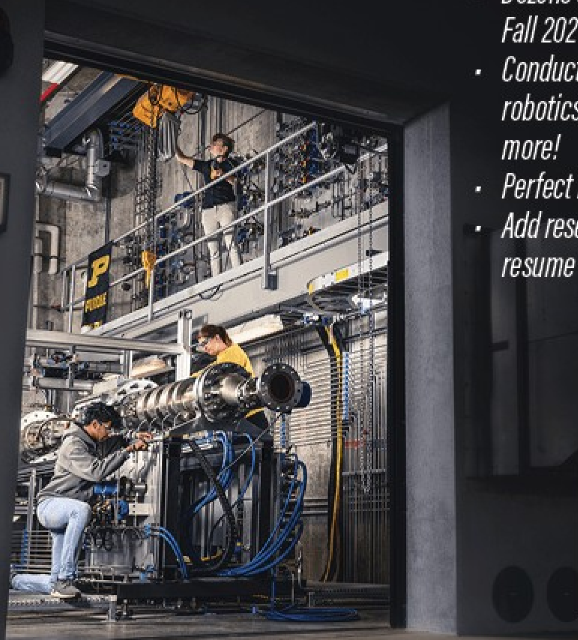

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Purdue BSME Students:
Do you enjoy studying Dynamical Systems?

Purdue BSME Students can pursue a
*Concentration in **Dynamical Systems Modeling & Analysis***

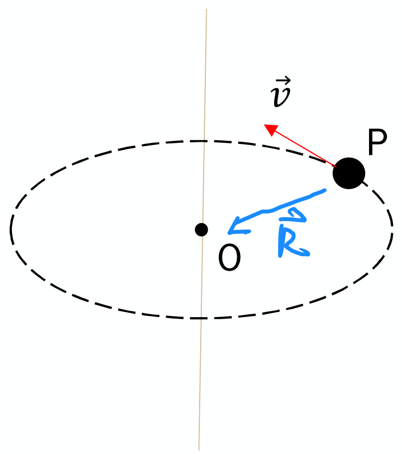
- Develop skills in dynamics, vibrations, controls, and machine learning
- Gain experience in mathematical modeling and analysis of dynamical systems
- Become knowledgeable in the tools and methods required to succeed in advanced study fields and high-demand engineering careers
- Add research coursework and experience to meet your concentration requirements.

Requirements: 10 credit hours

- 1-credit **foundational course** in dynamical systems
- **Two advanced courses** from a designated list focusing on dynamics, vibrations, or approved research, and choose
- **One additional course** from an extended list covering control systems, signal processing, numerical methods, machine learning, nonlinear systems, aeroelasticity, research, or related topics

Talk to your ME undergraduate advisor to determine if this concentration is right for you!

Consider a particle P of mass m constrained to rotate about a fixed point O with a velocity of $\vec{v}$.



Remember from LIM:

$$m\vec{v}_2 = m\vec{v}_1 + \int_1^2 \vec{R}dt$$

If $\vec{R} = 0$, $m\vec{v}_2 = m\vec{v}_1$ → Linear momentum is conserved

In this scenario, the speed v of the particle is constant but the direction is changing.

→ a net force $\vec{R}$ must be acting on the particle. $\Sigma F \neq 0$

→ Linear momentum is not conserved. $m\vec{v}_1 \neq m\vec{v}_2$

→ LIM does not simplify the problem.

need a rotation specific analysis method!

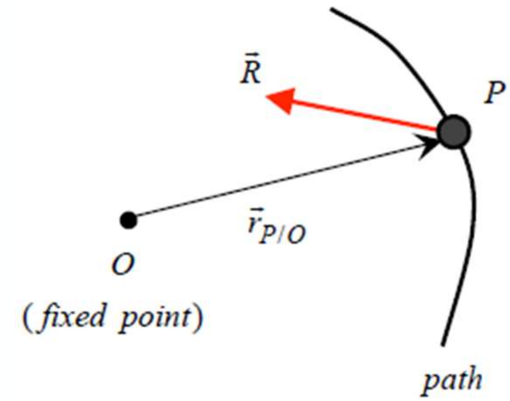
Remember from ME 270 that changes in rotational motion are caused by moments (the effect of a force about a point). here, v = constant → $\Sigma M = 0$

→ How do we adapt this knowledge to find a rotational analog of the LIM equation?

Deriving the Angular Impulse Momentum Equation

Starting with Newton's 2nd Law:

$$\vec{R} = \sum \vec{F} = ma = m\frac{dv_p}{dt}, \qquad v_p = \frac{d\vec{r}_{p/o}}{dt}$$



The moment of the force $\vec{R}$ about point O is found as:

$$\vec{M}_o = \vec{r}_{p/o} \times \vec{R}$$

Substituting in Newton's 2nd Law:

$$\vec{M}_o = \vec{r}_{p/o} \times \left(m\frac{d\vec{v}_p}{dt}\right)$$

Recall the time derivative of the cross product of two vectors is:

$$\frac{d}{dt}(\vec{a}\times\vec{b}) = \frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt} \Rightarrow \vec{a}\times\frac{d\vec{b}}{dt} = \frac{d}{dt}(\vec{a}\times\vec{b}) - \frac{d\vec{a}}{dt}\times\vec{b}$$

Substituting into our moment equation:

$$\vec{M}_o = \frac{d}{dt}\left[\vec{r}_{p/o}\times m\vec{v}\right] - \frac{dr_{p/o}}{dt}\times m\vec{v}$$

$$= \frac{d}{dt}\left[\vec{r}_{p/o}\times m\vec{v}\right] - \cancel{\vec{v}\times m\vec{v}}^{\,0} = \frac{d}{dt}\left[r_{p/o}\times m\vec{v}\right] = \frac{d\vec{H}_o}{dt}$$

H_o

Integrating with respect to time:

$$\underbrace{\int_1^2 M_o\,dt = \int_1^2 d\vec{H}_o = \vec{H}_{o2} - \vec{H}_{o1}}_{\text{Angular impulse momentum}}$$

Key Takeaways

→ $\vec{H}_O$ is the angular momentum about point O, $\vec{H}_O = \vec{r}_{P/O} \times m\vec{v}_P$.

→ Note $\vec{H}_O$ is a vector. For 2D rotation, it will always be pointed in or out of the page ($\hat{k}$).

→ AIM is the rotational equivalent of LIM.

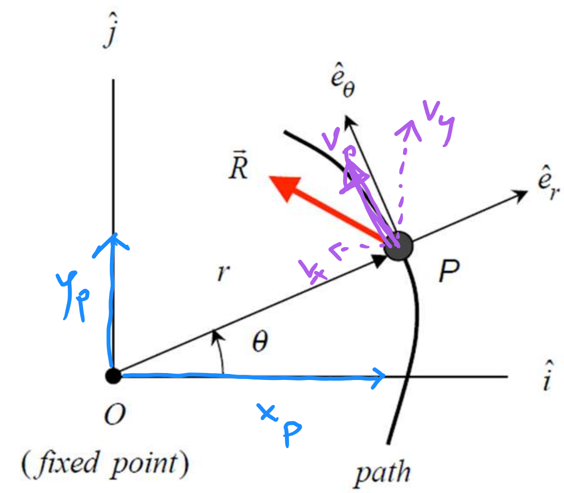
→ AIM relates the time integral of the moment to the change in angular momentum of a particle rotation about a point.

LIM: relates ΔV to Δt, $F(t)$

AIM: relates $\Delta\omega$ to Δt, $M(t)$

Calculating the Angular Momentum of a Particle

For our particle P rotating about the fixed point O, angular momentum is calculated as $\vec{H}_O = \vec{r}_{P/O} \times m\vec{v}_P$. ← general form, all coord. descriptions



To calculate in the cartesian description:

$$\vec{H}_O = \vec{r}_{P/O} \times m\vec{v}_P$$

$$= m\left[(x_P\hat{i} + y_P\hat{j}) \times (V_{Px}\hat{i} + V_{Py}\hat{j})\right]$$

$$= m(x_P V_{Py} - y_P V_{Px})\hat{k}$$

To calculate in the polar description:

$$\vec{H}_O = \vec{r}_{P/O} \times m\vec{v}_P$$

$$= m\left[(r\hat{e}_r) \times (\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)\right]$$

$$= m r^2 \dot{\theta}\hat{k}$$

Note: the polar description is frequently the most useful form for angular momentum.

$\vec{H}_O$ always in $\hat{k}$ for 2D.

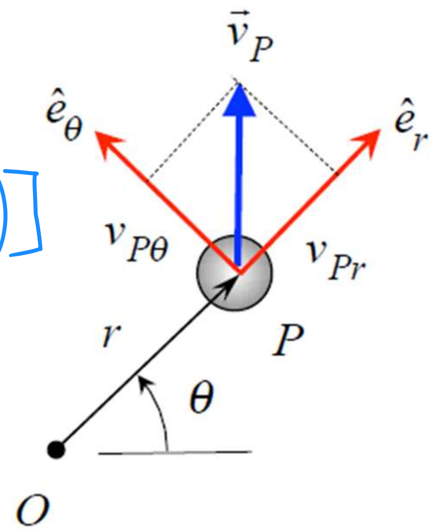
CHALLENGE QUESTION: The radial component of velocity, v_{Pr}, for a particle P cannot make a contribution to the angular momentum of the particle about point O. Why is that?

$$\vec{H}_o = \vec{r}_{p/o} \times m\vec{v}_{p/o}$$

$$= m\left[\underbrace{(r\hat{e}_r)\times(v_{pr}\hat{e}_r}_{\hat{e}_r \times \hat{e}_r = 0} + v_{p\theta}\hat{e}_\theta)\right]$$

$$\vec{H}_o = m\left[(r\hat{e}_r)\times(v_{p\theta}\hat{e}_\theta)\right]$$

$$= m r v_{p\theta}\hat{k}$$

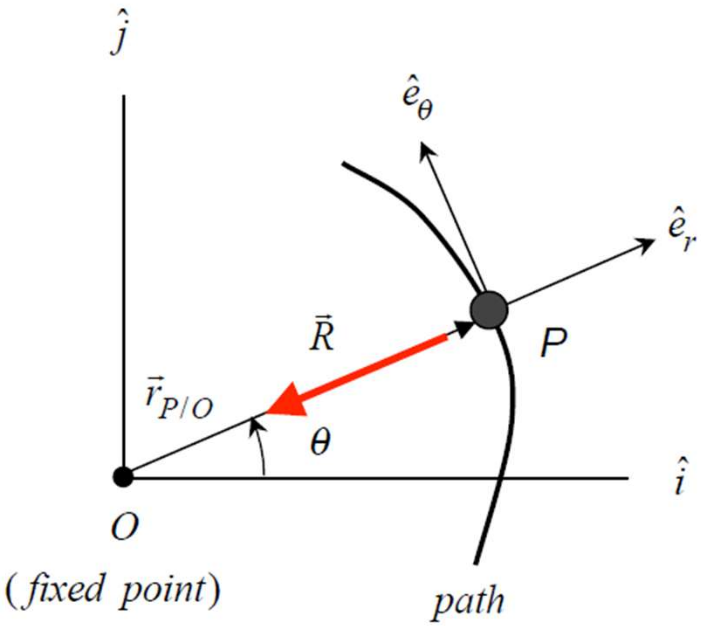


__Key Takeaways__

→ Angular momentum only gives info about the rotational component $r\dot{\theta}$ of velocity $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$.

→ if the total velocity is needed, you will need to also use WE to find $\dot{r}$.

Central Force Problems and Conservation of Angular Momentum



A "central force" problem is a special case where our resultant force $\vec{R}$ is aligned with our $\hat{e}_r$ direction → $\vec{R}$ point directly towards/away from the fixed point O.

The moment of this central force will always be zero!

$$\vec{M}_O = \vec{r}_{P/O} \times \vec{R} = r_{P/O}\hat{e}_r \times R\hat{e}_r = 0$$

From our angular impulse-momentum equation:

$$\int_1^2 \vec{M}_O \, dt = \int_1^2 d\vec{H}_O = \vec{H}_{O2} - \vec{H}_{O1}$$

This means the angular momentum of the particle is constant!

$$\int_1^2 \vec{M}_O dt = 0 \Rightarrow \vec{H}_{O2} - \vec{H}_{O1}$$

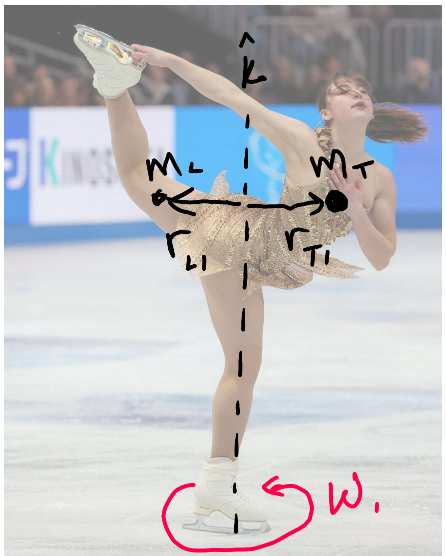
$$\vec{H}_{O2} = \vec{H}_{O1}$$

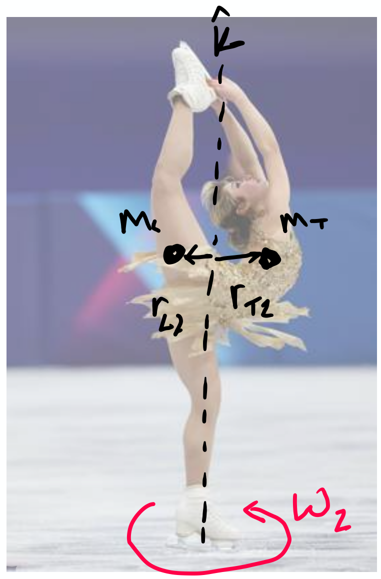
Key Takeaways

→ In central force problems $\vec{H}_{O2} = \vec{H}_{O1}$

→ Angular momentum is conserved!

Olympic gold medalist Alyssa Liu enters a spin rotating slowly. As the spin progresses her rate of rotation increases. How does Alyssa speed up during her spin?





$r_{L1} > r_{L2}$, $r_{T1} > r_{T2}$ $\Rightarrow$ $\omega_1 \quad \omega_2$

By using conservation of angular momentum!

(and years of flexibility and balance training…but mostly angular momentum.)

$$\vec{H}_O = \vec{r}_{P/O} \times m\vec{v}$$

In polar form:

$$\vec{H}_o = m(r^2\dot{\theta})\vec{K}$$

When angular momentum is conserved:

$$\vec{H}_{O2} = \vec{H}_{O1}$$

$$m(r_1^2\dot{\theta}_1) = m(r_2^2\dot{\theta}_2)$$

$$\dot{\theta}_2 = \dot{\theta}\frac{r_1^2}{r_2^2}$$

Key Takeaways

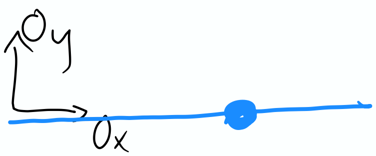
→ Angular velocity $\vec{\omega}$ must decrease (or increase) as the radial distance r increases (or decreases) when angular momentum is conserved.

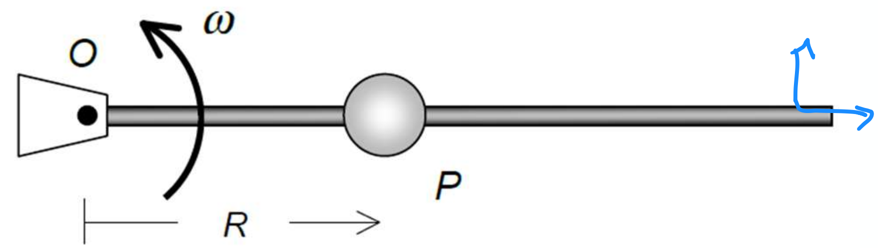
$r\uparrow \ \omega\downarrow \qquad r\downarrow \ \omega\uparrow$

Example 4.D.1

Given: Particle P (weighing 2 lb) is able to slide on a smooth, lightweight horizontal arm that is rotating about a vertical axis. Initially, P is stationary relative to the arm when the arm is rotating at a rate of $\omega_1 = 20$ rad/s and P is $R = 3$ in from the rotation axis of the arm.

Find: Determine the angular speed of the arm when the P has moved outward to a position of $R = 24$ in from the axis of the shaft.





HORIZONAL PLANE

$$\Sigma M_0 = 0$$

$$(\vec{H}_o)_1 = (\vec{H}_o)_2$$

$$\vec{H}_{o1} = \vec{r}_1 \times m\vec{v}_1 = r_1 \hat{e}_r \times \frac{W}{g}(\cancel{\dot{r}_1 \hat{e}_r}^{\,0} + r_1\omega_1 \hat{e}_\theta)$$

$$= \frac{W}{g} r_1^2 \omega_1 \hat{k}$$

$$\vec{H}_{o2} = \vec{r}_2 \times m\vec{v}_2 = r_2 \hat{e}_r \times \frac{W}{g}(\dot{r}_2 \hat{e}_r + r_2 \omega_2 \hat{e}_\theta)$$

$$= \frac{W}{g} r_2^2 \omega_2 \hat{k}$$

$$\Rightarrow r_1^2 \omega_1 = r_2^2 \omega_2$$

$$\omega_2 = \frac{r_1^2}{r_2^2}\omega_1$$

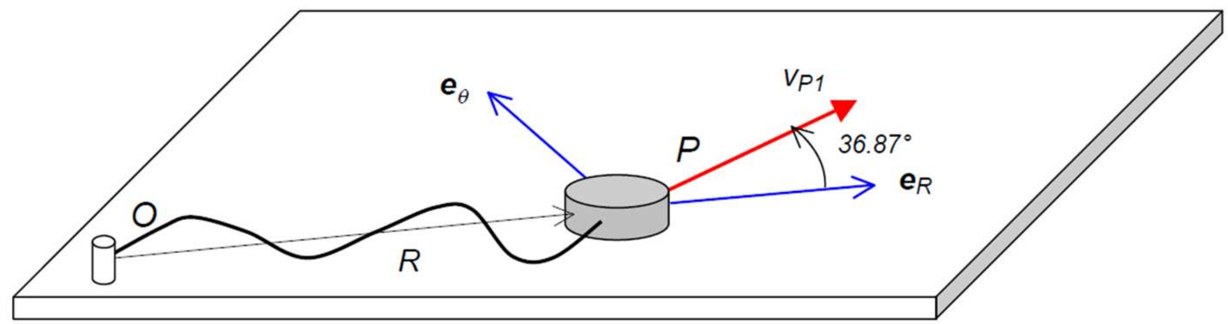
$$= \frac{3^2}{24^2} 20 = 0.3 \text{ rad/s}$$

Example 4.D.3

Given: Disk P having a mass of 0.2 kg is able to slide on a smooth, horizontal surface. A rubber band (having a stiffness of $k = 10$ N/m and unstretched length of 0.6 m) attaches B to a fixed peg at O. Disk P is set into motion with a speed of $v_{P1} = 15$ m/s in the direction shown with $R = 0.4$ m.

Find: Determine the polar coordinates of the velocity of P when P is a distance 1.5 m from O.

FBD



Kinetics

$$\sum M_O = 0 \Rightarrow \vec{H}_{O1} = \vec{H}_{O2}$$

$$\vec{H}_{O1} = \vec{r}_1 \times m\vec{v}_1 = r_1\hat{e}_r \times m(v_{P1}\cos\theta\,\hat{e}_r + v_{P1}\sin\theta\,\hat{e}_\theta)$$

$$= m r_1 v_{P1}\sin\theta\,\hat{k}$$

$$\vec{H}_{O2} = \vec{r}_2 \times m\vec{v}_2 = r_2\hat{e}_r \times m(v_{2r}\hat{e}_r + v_{2\theta}\hat{e}_\theta)$$

$$= m r_2 v_{2\theta}\,\hat{k}$$

angular component: $m r_1 v_{P1}\sin\theta = m r_2 v_{2\theta} \rightarrow v_{2\theta} = \frac{r_1}{r_2} v_{P1}\sin\theta$

To find radial component, use WE!

$$T_1 + V_1 + U^{NC}_{1\to 2} = T_2 + V_2$$

$T_1 = \frac{1}{2} m v_{p1}^2$

$V_1 = 0$ (elastic band cannot be in compression)

$U^{NC}_{1\to 2} = 0$

$T_2 = \frac{1}{2} m (v_{2r}^2 + v_{2\theta}^2)$

$V_2 = \frac{1}{2} k (R_2 - R_0)^2$

$$\Rightarrow \frac{1}{2} m v_{p1}^2 = \frac{1}{2} m v_{2r}^2 + \frac{1}{2} m v_{2\theta}^2 + \frac{1}{2} k (R_2 - R_0)^2$$

Solve for v_{2r}