ME 274: Basic Mechanics II

Lecture 20: Particle Kinetics – Work-Energy

PURDUE UNIVERSITY | School of Mechanical Engineering

Overview: Work-Energy Equation

Q: When do we know to use the work-energy equation in a kinematic analysis?

A: - relating speed to a change in position
- know energy is conserved

Components of the work-energy equation: $\underbrace{T_1 + V_1}_{\text{initial position}} + \overbrace{U_{1\to 2}^{(nc)}}^{\text{work to get from } 1 \to 2} = \underbrace{T_2 + V_2}_{\text{final position}}$

$T = \frac{1}{2}mv^2$ = particle kinetic energy

$V = V_{sp} + V_{gr}$ = potential energy resulting from conservative (path independent) forces

- $V_{sp} = \frac{1}{2}k(L - L_0)^2$ → $V_{sp} > 0$
- $V_{gr} = mgh$ → $V_{gr} > 0$ or < 0, depending on datum

$U_{1\to 2}^{(nc)} = \int_1^2 (R_t)ds$ = work done by $\vec{R}$ from 1 to 2 R → resultant force

- Path description: $U_{1\to 2} = \int_1^2 (\vec{R} \cdot \hat{e}_t)ds$
- Cartesian description: $U_{1\to 2} = \int_1^2 (R_x\hat{\imath} + R_y\hat{\jmath}) \cdot (\underline{dx\,\hat{\imath} + dy\,\hat{\jmath}})$
- Polar description: $U_{1\to 2} = \int_1^2 (R_r\hat{e}_r + R_\theta\hat{e}_\theta) \cdot (\underline{dr\hat{e}_r + r\,d\theta\hat{e}_\theta})$

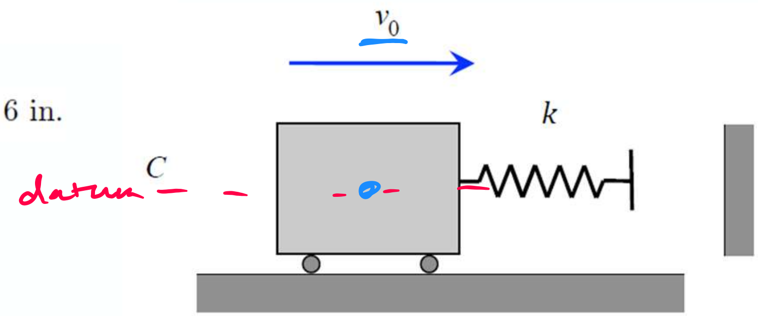
non-conservive → path dependent

Example 4.B.4

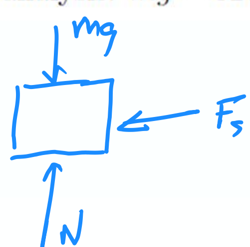
Given: Block C having a weight mg is traveling with a speed of v_0 when it strikes a fixed wall. A spring of stiffness k is attached to the front of the block. The spring eventually contacts the wall, resulting in a maximum spring compression of Δ_{max}.

Find: Determine the stiffness k of the spring.

Use the following parameters in your analysis: $mg = 3200$ lb, $v_0 = 10$ ft/sec and $\Delta_{max} = 6$ in.

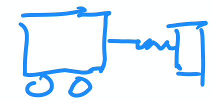


1) FBD



initial position:

$$T_1 + V_1 + U_{1\to2}^{nc} = T_2 + V_2$$



$$T_1 = \frac{1}{2} m v_0^2 \qquad V_1 = 0$$

Non-conservative work: $U_{1\to2}^{nc} = 0 \leftarrow$ mechanical energy conserved

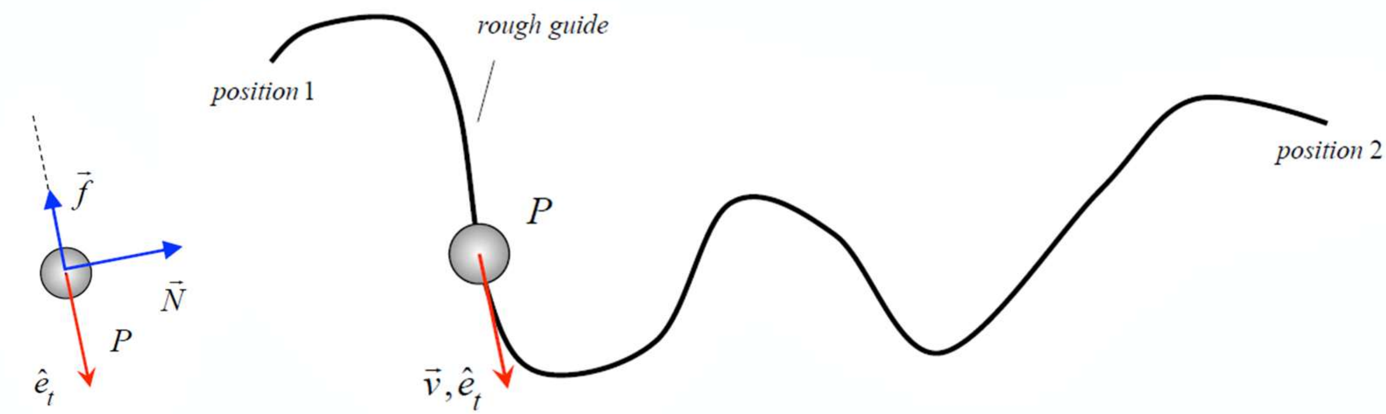
final position:

$$T_2 = 0 \qquad V_2 = \frac{1}{2} k \Delta_{max}^2$$

Work-energy:

$$\frac{1}{2} m v_0^2 = \frac{1}{2} k \Delta_{max}^2 \quad \rightarrow \quad k = \frac{m v_0^2}{\Delta_{max}^2}$$

CHALLENGE QUESTION: A particle slides on a rough, arbitrarily-shaped guide. What can we say about the sign of the work done by the friction force acting on the particle as it moves from position 1 to position 2? Similarly, what is the work done by the normal force acting on the particle?

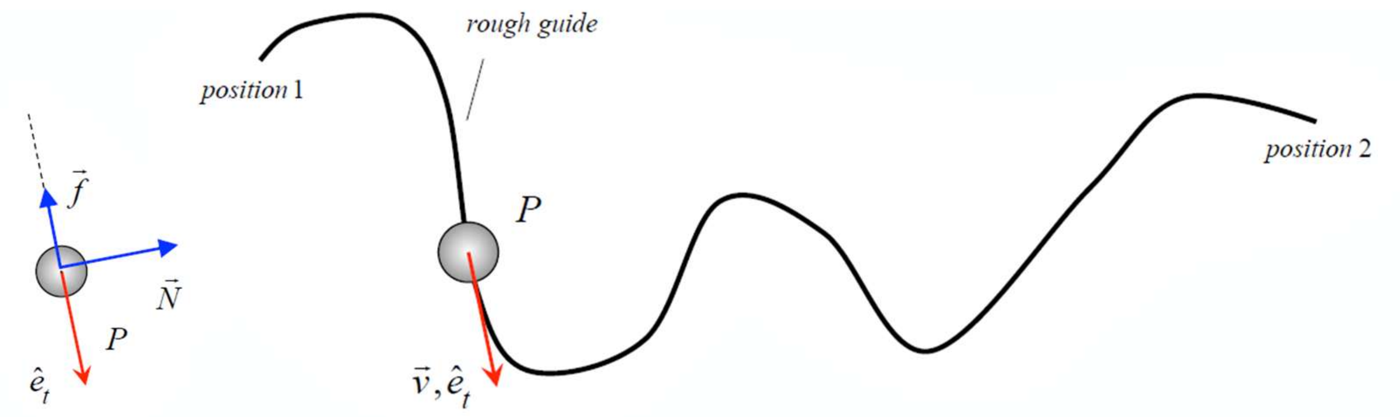


remember : $U^{NC}_{1\to 2} = \int_1^2 (\vec{R}\cdot\hat{e}_t)\,ds$

$\underbrace{\vec{f} = -f\hat{e}_t}$, $\quad -f\underbrace{\hat{e}_t\cdot\hat{e}_t}_{1} = -f < 0$

friction opposes motion
∴ opposite direction of $\vec{v}$

CHALLENGE QUESTION: A particle slides on a rough, arbitrarily-shaped guide. What can we say about the sign of the work done by the friction force acting on the particle as it moves from position 1 to position 2? Similarly, what is the work done by the normal force acting on the particle?



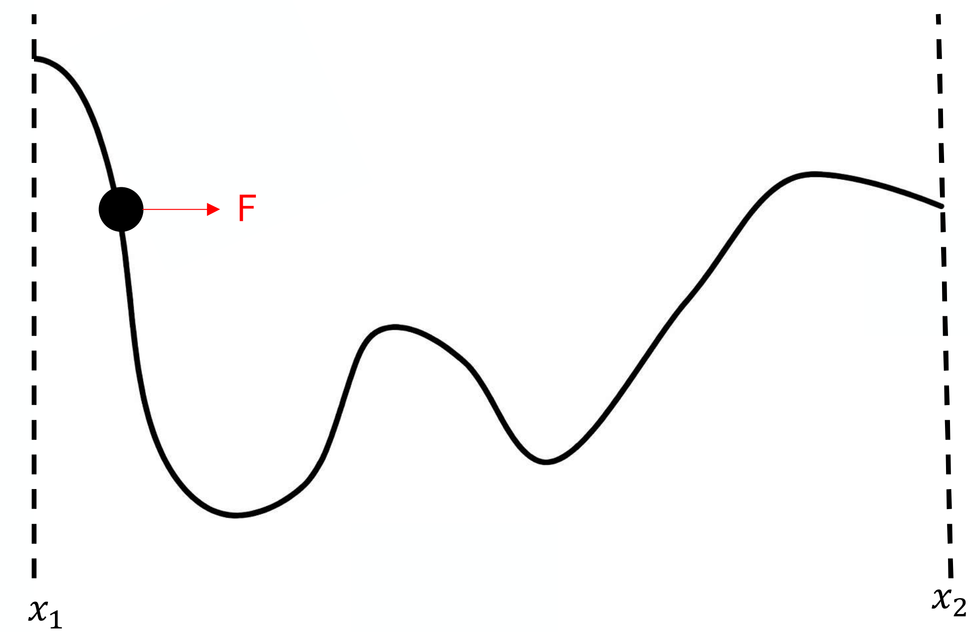
Only the tangential component of the resultant force $\vec{R}$ contributes to the work done by $\vec{F}$:

$$U_{1\to 2} = \int_1^2 \left(\vec{R} \cdot \hat{e}_t\right)\, ds$$

If a force is perpendicular to the path of the particle on which it acts, then the force does NO work on the particle.

Normal force always perpendicular → will never contribute to work

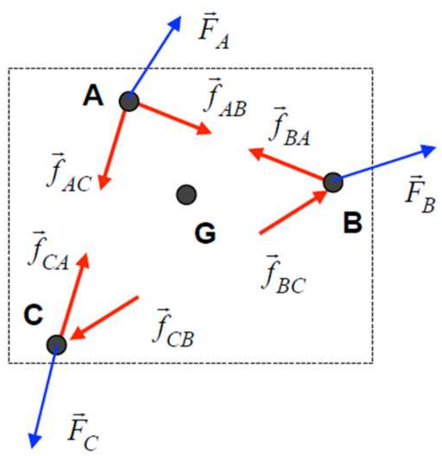
Challenge question: The particle is being pulled along a smooth guide by a force $\vec{F} = F\hat{\imath}$. What is the magnitude of $U_{1\to 2}^{(nc)}$?



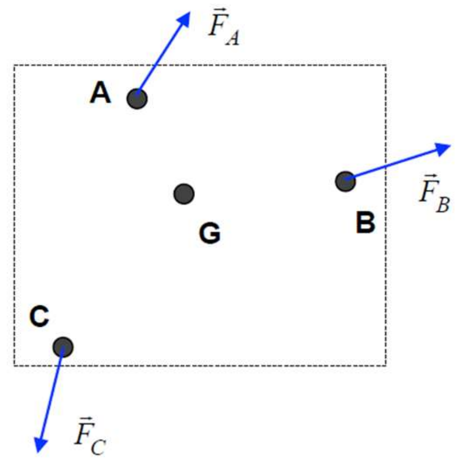
$$U_{1\to 2}^{Nc} = \int_1^2 (\vec{F}\cdot\hat{e}_t)\,ds = \int_1^2 F\hat{\imath}\cdot(dx\,\hat{\imath} + dy\,\hat{\jmath})$$

$$= \int_{x_1}^{x_2} F\,dx = F(x_2 - x_1)$$

Work – Energy Equation for Systems of Particles



all forces acting on system



"external" forces acting on system

There will only be a single work-energy equation, regardless of the number of particles in the system!

If you have more than 1 unknown, use kinematics

$$T_1 + V_1 + U_{1\to 2}^{(nc)} = T_2 + V_2$$

Each term will be the sum for all particles in the system:

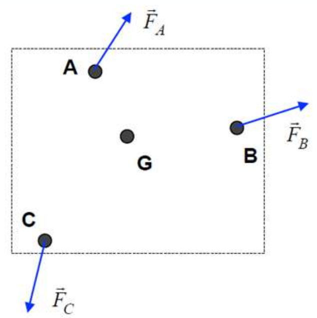
$$T = \sum_{j=1}^{N} T_j = \frac{1}{2}\sum_{j=1}^{N} m_j v_j^2$$

$$V = \sum_{j=1}^{N} V_j$$

$U_{1\to 2}^{(nc)}$ - Total nonconservative work done on the system

Calculating $U_{1\to 2}^{(nc)}$ for systems of particles

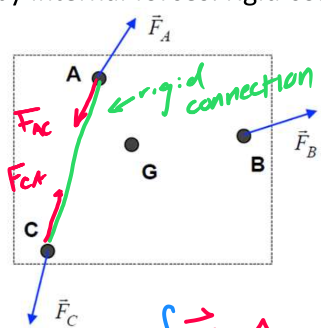
Work done by external forces



"external" forces acting on system

$$U_{1\to 2}^{(nc)} = \int_1^2 \left(\left(\vec{F}_A + \vec{F}_B + \vec{F}_C \right) \cdot \hat{e}_t \right) ds$$

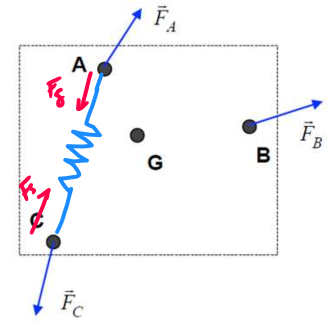
Work done by internal forces: rigid connections



$$\int \vec{F}_{AC} \cdot \hat{e}_t = -\int \vec{F}_{CA} \cdot \hat{e}_t$$

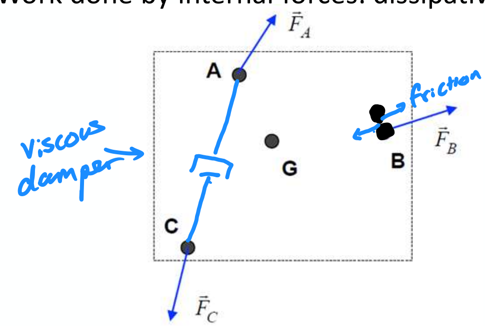
integrals cancel → Net work = 0

Work done by internal forces: conservative forces



include in V term, not in integral

Work done by internal forces: dissipative forces

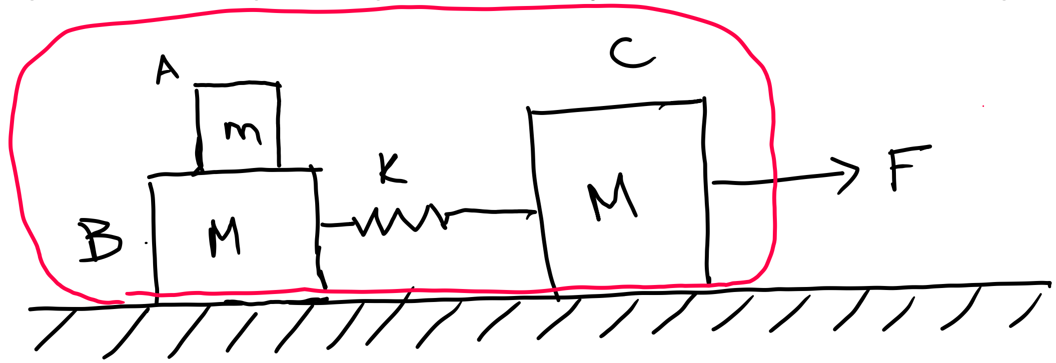


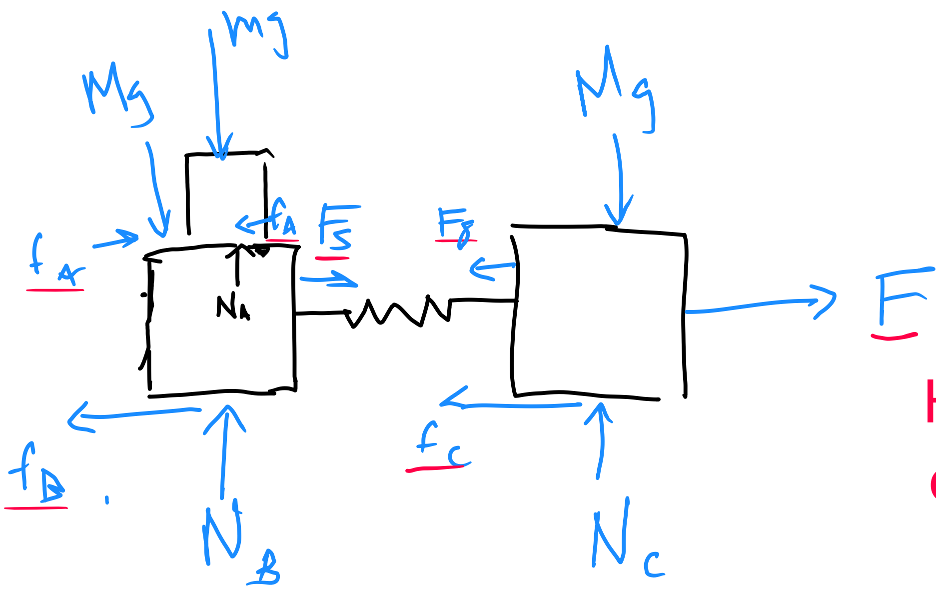
Energy leaves system
(mechanical energy not conserved)
→ must include in work integral

Rule of thumb: make your system as large as possible to make workless forces internal. Make friction/dissipative forces external by your choice of system. (if possible)

Example: two blocks connected by a spring pulled along a rough surface by force F, block A sliding on top of B

System : blocks A, B, C + spring





How do we include each force in our Work-energy equation?

$F \rightarrow$ external $\rightarrow$ in $U_{1\rightarrow 2}$

$F_s \rightarrow$ internal, conservative $\rightarrow$ in V

$mg, Mg, N_A, N_B, N_c \rightarrow$ perpendicular to path $\Rightarrow$ 0 work

$f_A \rightarrow$ internal, disipative $\rightarrow$ in $U_{1\rightarrow 2}$

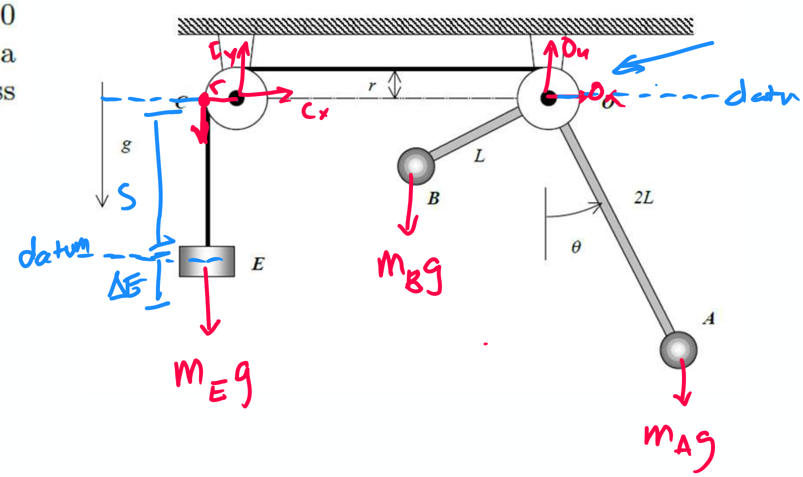
$f_B, f_c \rightarrow$ external disipative $\rightarrow$ in $U_{1\rightarrow 2}$

Example 4.B.6

Given: The system shown below is made up of particles A, B and E (having masses of 5 kg, 20 kg and 80 kg, respectively). Lightweight bars OB and OA are welded together such that there is a right angle between the two bars, and these bars are welded to the pulley at O. Consider the mass of the pulleys to be negligible. The system is released from rest when $\theta = 0$.

Find: Determine the speed of particle E when $\theta = 90°$.

step 1) FBD



1) Kinetics

$$T_1 + V_1 + U^{NC}_{1\to 2} = T_2 + V_2$$

position 1

$T_1 = 0$ (RFR)

$V_1 = -2m_A g L$

$U^{NC}_{1\to 2} = 0$

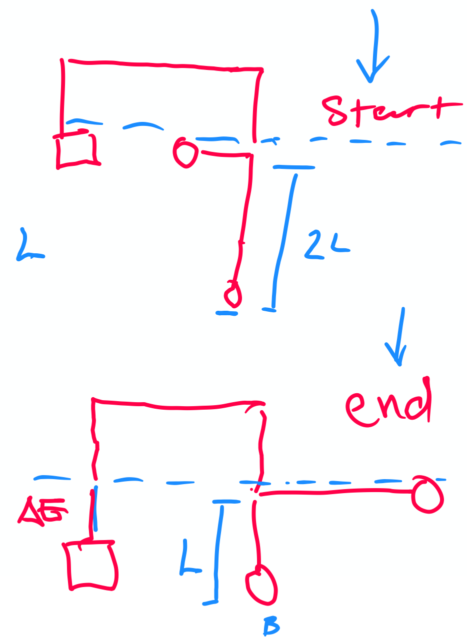
position 2

$T_2 = \frac{1}{2} m_E v_E^2 + \frac{1}{2} m_B v_B^2 + \frac{1}{2} m_A v_A^2$

$V_2 = -m_E g \Delta E - m_B g L$

Work - Energy

$$-2m_A g L = \frac{1}{2} m_E v_E^2 + \frac{1}{2} m_B v_B^2 + \frac{1}{2} m_A v_A^2 - m_E g \Delta E - m_B g L$$



3) kinetics

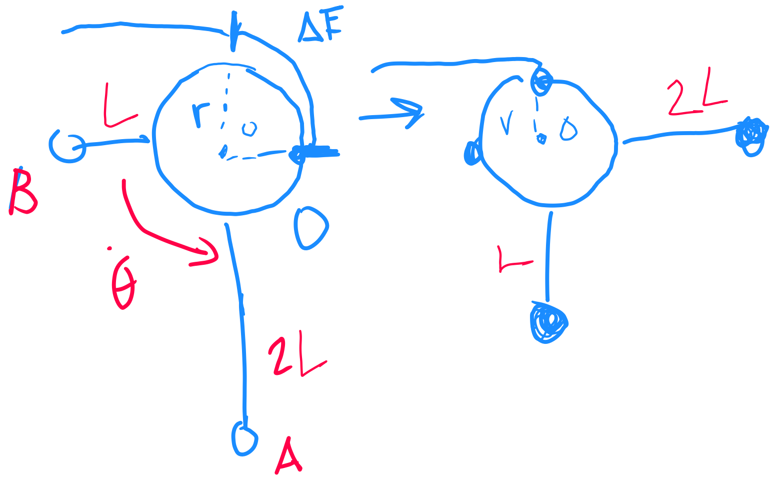
Work - Energy

$$-2m_A g L = \frac{1}{2} m_E V_E^2 + \frac{1}{2} m_B V_B^2 + \frac{1}{2} m_A V_A^2 - mg\Delta E - m_B g L$$

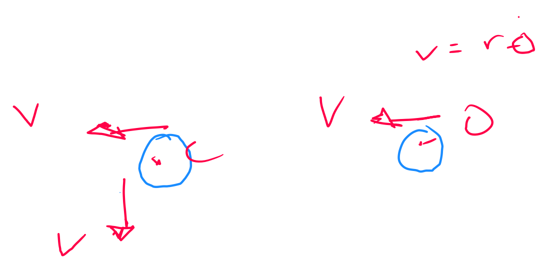
$r\dot{\theta}$ $L\dot{\theta}$ $2L\dot{\theta}$

1 eqn, 4 unknowns → use kinematics!

Solve for ΔE → the change in height of E is a result of the cable unspooling as pully O rotates through 1/4 turn



$$\Delta E = \frac{\pi r}{2}$$



relate V_A, V_B, V_E → pulleys O + C are connected at top by inextensible cable → have the same velocity + ω because of matching radii

$V_B = L\dot{\theta}$ $V_A = 2L\dot{\theta}$ $V_E = r\dot{\theta}$

Substitute into Work-Energy eqn, solve for $\dot{\theta}$, then solve for V_E

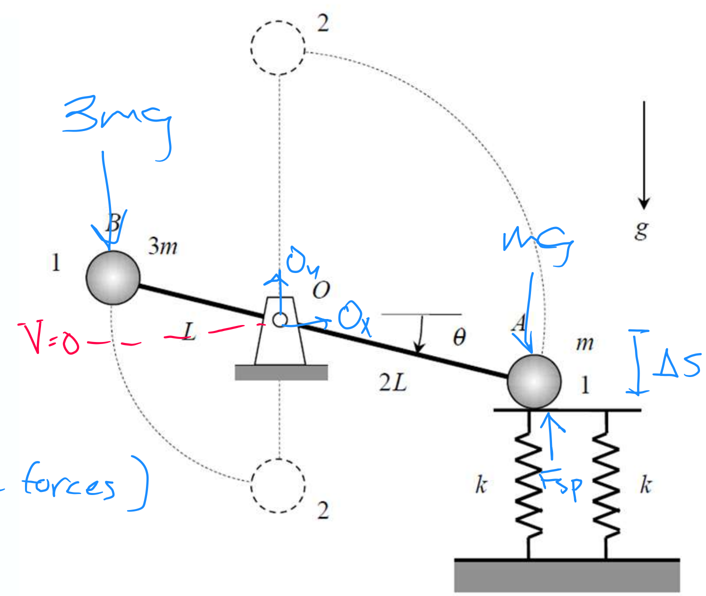
use $|V| = r\omega$ IC rule

Example 4.B.7

Given: Particles A and B, having masses of m and $2m$, respectively, are connected by rigid bar AB, with AB having negligible mass. Bar AB is pinned to ground with a pin joint at O. This system is released from rest at position 1 with $\theta = \theta_1$, with A in contact with a pair of identical springs, as shown in the figure. Each spring has a stiffness of k, and the springs are unstretched when $\theta = 0$. Assume the dimensions of the particles to be negligible.

1) FBD

Find: Determine the speeds of particles A and B at position 2, where in position 2 particle A is directly above O.



2) kinetics

$$T_1 + V_1 + U_{1\to2} = T_2 + V_2$$

$$T_1 = 0 \text{ (RFR)}$$

$$V_1 = \underbrace{3mgL\sin\theta - 2mgL\sin\theta}_{V_{gr}} + 2\Big(\underbrace{\tfrac{1}{2}k(2L\sin\theta)^2}_{V_{sp}}\Big)$$

$U^{NC}_{1\to2} = 0$ (mechanical energy conserved, no external NC forces)

$$T_2 = \tfrac{1}{2}mV_A^2 + \tfrac{3}{2}mV_B^2$$

$$V_2 = 2mg - 3mgL$$

Final work - Energy

$$\Longrightarrow 3mgL\sin\theta - 2mgL\sin\theta + k(2L\sin\theta)^2 = \tfrac{1}{2}mV_A^2 + \tfrac{3}{2}mV_B^2 + 2mgL - 3mgL$$

1 eqn, 2 unknown ⇒ kinematics

kinematics: $V_A = 2L\dot\theta \quad V_B = L\dot\theta \rightarrow V_A = 2V_B$

→ Substitute into work energy & solve!