ME 274: Basic Mechanics II

Week 6 *– Friday, February 20*

Particle kinematics: Particle Kinetics (Intro)

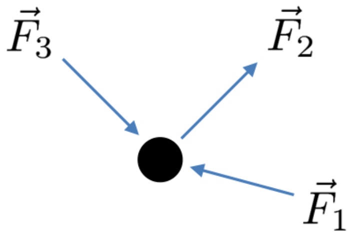
Instructor: Manuel Salmerón

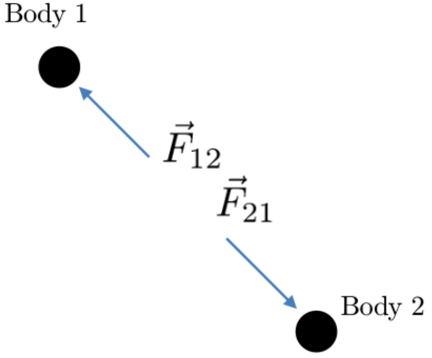
Today's Agenda

1. Summary of Steps for Kinetic Problems
2. Different Coordinate Systems
3. Examples

Summary of Steps for Kinetic Problems

1. Free Body Diagram (FBD)





2. Kinetics

$$\sum \vec{F} = \frac{d(m\vec{v})}{dt} = m\vec{a} \qquad \vec{F}_{12} = -\vec{F}_{21}$$

3. Kinematics

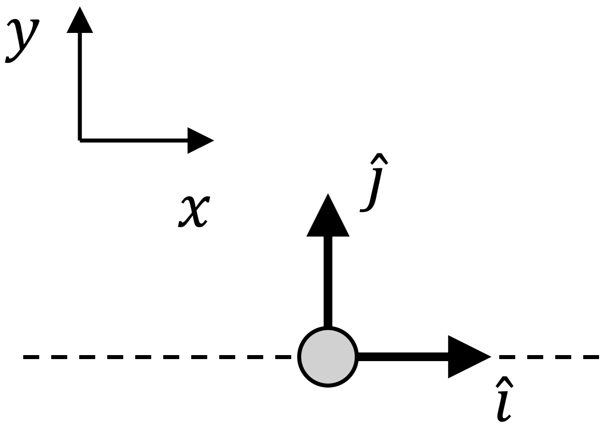
$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds} = \frac{d^2s}{dt^2}$$

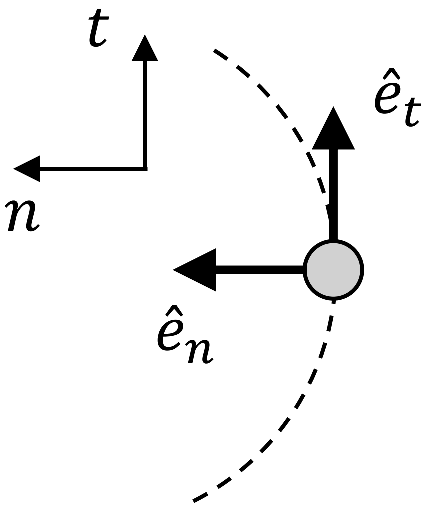
TODAY!

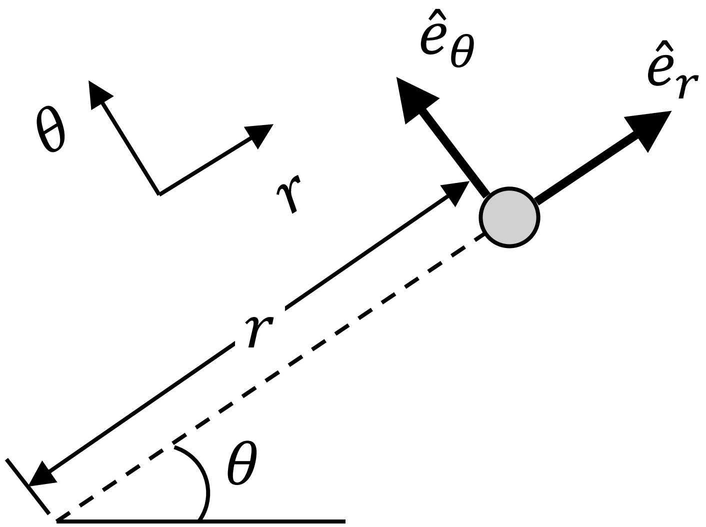
$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega}$$

4. Solve – for whatever is asked!

Different Coordinate Systems







Cartesian Coordinates

$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$

$$\sum F_x = ma_x = m\ddot{x}$$

$$\sum F_y = ma_y = m\ddot{y}$$

Path Coordinates

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

$$\sum F_t = ma_t = m\dot{v}$$

$$\sum F_n = ma_n = m\frac{v^2}{\rho}$$

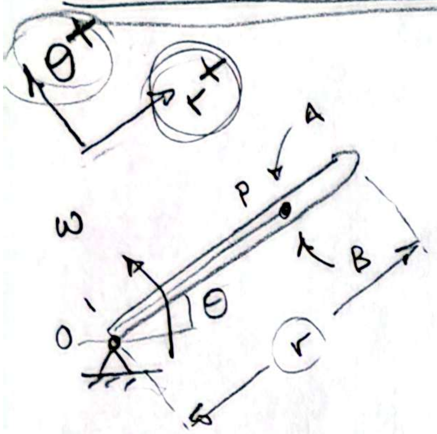
Polar Coordinates

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(\ddot{r} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

$$\sum F_r = ma_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

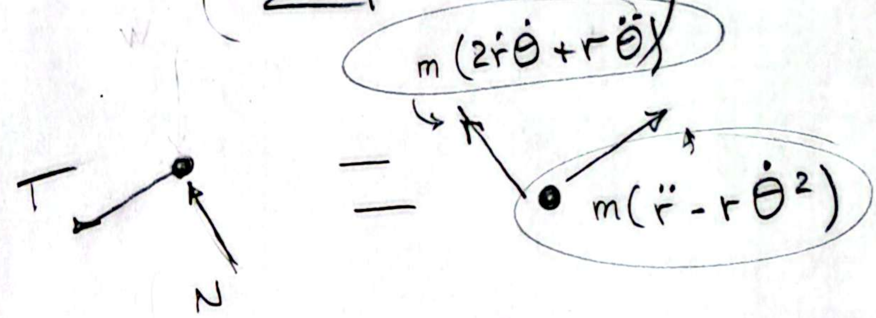
$$\sum F_\theta = ma_\theta = m\left(\ddot{r} + 2\dot{r}\dot{\theta}\right)$$

Example 4.A.10.



1. FBD

$$\left(\Sigma F = m\vec{a}\right)$$



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2. Kinetics

I defined a polar system: $\dot{\theta} = \omega$

$$\Sigma F_r = -T = m\left(\ddot{r} - r\dot{\theta}^2\right) \quad \dots (1)$$

$$\Sigma F_\theta = N = m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) \quad \dots (2)$$

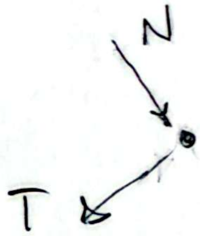
($\ddot{r} \to 0$, $\ddot{\theta} \to 0$)

3. Kinematics ; not here!

4. Solve:

from (1): $T = mr\omega^2$

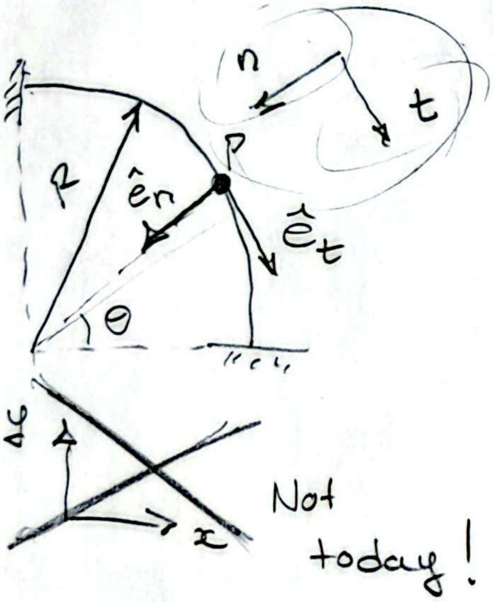
from (2): $N = 2m\dot{r}\omega < 0$ ($\dot{r} > 0$, $\omega > 0$)



True orientation of N. The force N is being exerted by the "A" wall

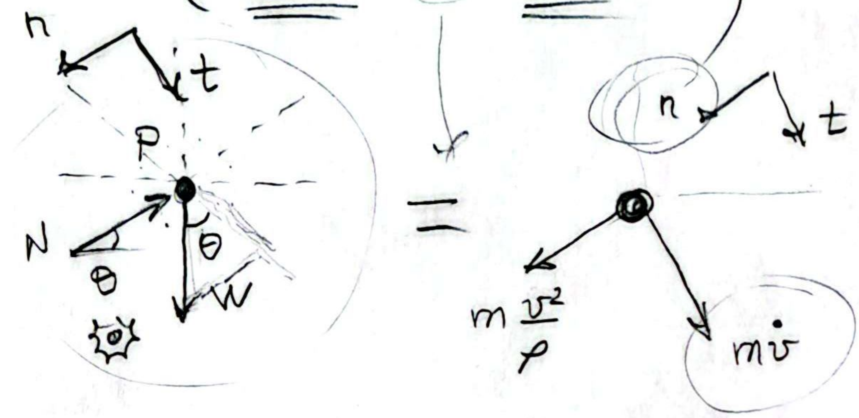
$\therefore$ P is in contact with A

Example 4.A.7



1. Free Body Diagram

$$\left(\sum F = m\vec{a} \right)$$



2. Kinetics

$$\underbrace{\sum F_t = W\cos\theta}_{\text{External}} = \underbrace{m\dot{v}}_{\text{inertial}} \quad \ldots (1)$$

$$\sum F_n = -N + W\sin\theta = m\frac{v^2}{R} \quad \ldots (2)$$

3. Kinematics ; not asked here

4. Solve : from (1) and (2)

$$\dot{v} = \frac{W}{m}\cos\theta \quad \Rightarrow \quad m = \frac{W}{g}$$

$$N = W\sin\theta - m\frac{v^2}{R}$$

Scan

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ME 274: Basic Mechanics II

Week 7 *– Wednesday, February 25*

Particle kinematics: Particle Kinetics

Instructor: Manuel Salmerón

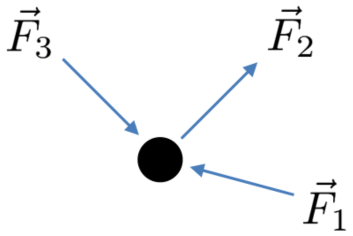
Today’s Agenda

1. Summary of Steps for Kinetic Problems
2. Examples

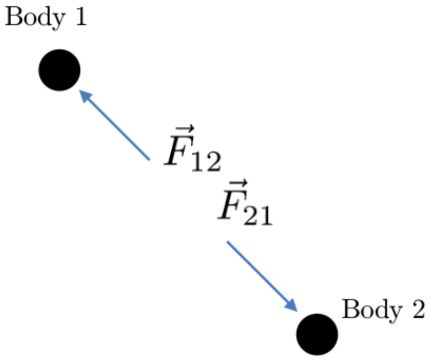
Summary of Steps for Kinetic Problems

1. Free Body Diagram (FBD)

Newton's Second Law



Newton's Third Law



2. Kinetics

$$\sum \vec{F} = \frac{d(m\vec{v})}{dt} = m\vec{a} \qquad \vec{F}_{12} = -\vec{F}_{21}$$

3. Kinematics

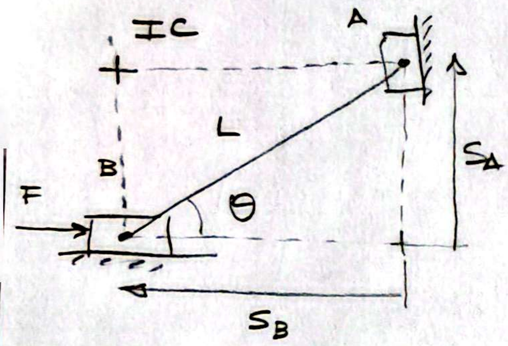
$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds} = \frac{d^2s}{dt^2}$$

TODAY!

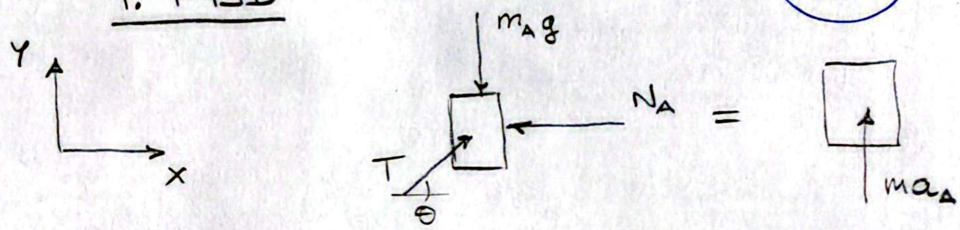
$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}_{A/B} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{A/B}\right)$$

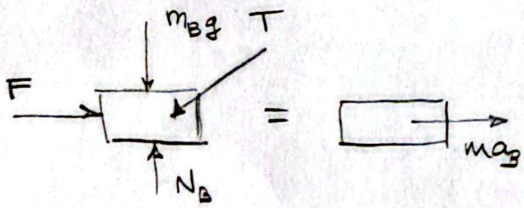
4. Solve – for whatever is asked!

Example 4.A.8



1. FBD





Note that $\sin\theta = \frac{S_A}{L}$ and $S_B = L\cos\theta$ ✓

GOOD

Given: $m_A, m_B, L,$ F, S_A, v_A

Find: $a_A, a_B, F_{AB} \equiv T$

2. Kinetics

A: $\Sigma F_x = -N_A + T\cos\theta = 0 \quad \dots (1)$

$\Sigma F_y = -m_A g + T\sin\theta = m a_A \quad \dots (2)$

B: $\Sigma F_x = F - T\cos\theta = m a_B \quad \dots (3)$

$\Sigma F_y = -m_B g - T\sin\theta + N_B = 0 \quad \dots (4)$

5 Unknowns: N_A, T, a_A, a_B, N_B

4 eqs.

3. Kinematics

Method #1: rigid body equations

$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}_{A/B} - \omega^2 \vec{r}_{A/B}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \alpha \\ S_B & S_A & 0 \end{vmatrix} = -\alpha S_A \hat{i} + \alpha S_B \hat{j}$$

$$a_A \hat{j} = a_B \hat{i} + (\alpha \hat{k}) \times (S_B \hat{i} + S_A \hat{j}) - \omega^2 (S_B \hat{i} + S_A \hat{j})$$

$$a_A \hat{j} = (a_B - \alpha S_A - \omega^2 S_B)\hat{i} + (\alpha S_B - \omega^2 S_A)\hat{j}$$

In $\hat{i}$: $a_B - \alpha S_A - \omega^2 S_B = 0 \quad \dots (5)$ +2 eqs, +2 unknowns α, ω

In $\hat{j}$: $a_A - \alpha S_B + \omega^2 S_A = 0 \quad \dots (6)$

But from the ICR: $\omega = \frac{|v_A|}{|\vec{r}_{A/IC}|} = \frac{|v_A|}{S_B}$ +1 eq., +0 unknowns

Method #2: length constraint

$$L^2 = s_A^2 + s_B^2$$

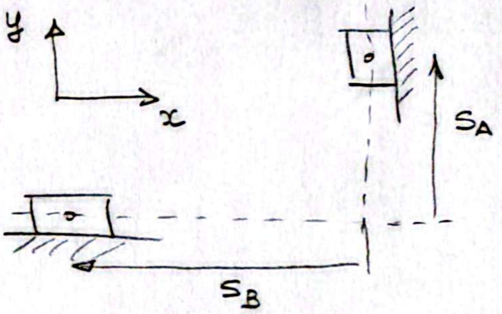
$\frac{d}{dt}$ ↓

$$0 = 2 s_A \dot{s}_A + 2 s_B \dot{s}_B$$

$\frac{d}{dt}$ ↓

$$0 = s_A \ddot{s}_A + \dot{s}_A^2 + s_B \ddot{s}_B + \dot{s}_B^2$$

⚠ Note on the signs:



$\dot{s}_A = +v_A$ (defined positive ↑)

$\ddot{s}_A = +a_A$ (defined positive ↑)

$\dot{s}_B = -v_B$ (defined positive ←)

$\ddot{s}_B = -a_B$ (" " ←)

Considering the signs:

$$+ s_A v_A - s_B v_B = 0 \quad \dots (5')$$

$$+ s_A a_A + v_A^2 - s_B a_B + v_B^2 = 0 \quad \dots (6')$$

The sign convention I chose is independent from the sign of v_A. (?)

If $v_A < 0$; $v_B < 0$: s_A shrinks, s_B grows
($\dot{s}_A < 0$) ($\dot{s}_B > 0$):

If $v_A > 0$, $v_B > 0$: s_A grows, s_B shrinks
($\dot{s}_A > 0$) ($\dot{s}_B < 0$)

ME 274: Basic Mechanics II

Week 7 *– Friday, February 27*

Work and Energy

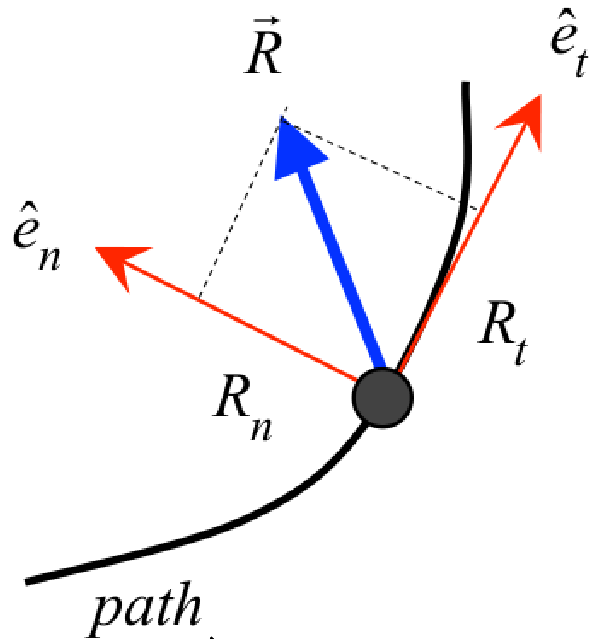
Instructor: Manuel Salmerón

Today's Agenda

1. Work and Energy
2. Example 4.B.1
3. Conservative Forces and Potential Energy
4. Summary

Attendance

Consider the particle P in the figure loaded with a force $\vec{R}$ and the given path coordinate system.



$$\sum F_t = \vec{R} \cdot \hat{e}_t = m\dot{v}$$

$$\sum F_n = \vec{R} \cdot \hat{e}_n = mv^2/\rho$$

Q1. The tangential component of $\vec{R}$ is equal to:

(a) R_n (b) mg

(c) $m\dot{v}$ (d) $m\frac{v^2}{\rho}$

Q2. The normal component of $\vec{R}$ is equal to:

(a) R_t (b) mg

(c) $m\dot{v}$ (d) $m\frac{v^2}{\rho}$

Q3. At a given instant, $\dot{v} = 0$ and $v \neq 0$. Choose the true statement:

(a) P is not moving (b) P is changing direction

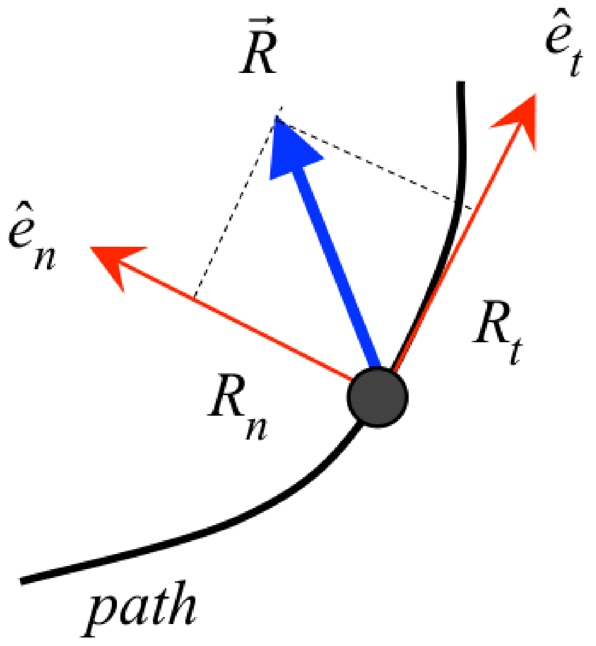
(c) P is moving in a straight path (d) the magnitude of $\vec{a}$ is 0

Q4. At a given instant, $R_n = 0, R_t \neq 0$, and $v \neq 0$. Choose the true statement:

(a) P is not moving (b) P is changing direction

(c) P is moving in a straight path (d) the magnitude of $\vec{a}$ is 0

Work and Energy



Consider the particle P in the figure loaded with a force $\vec{R}$ and the given path coordinate system.

Newton's 2nd Law:

$$\underbrace{\sum F_t = \vec{R}\cdot\hat{e}_t = m\frac{dv}{dt}}$$

Information about the "future": how is v changing?

$$\underbrace{\sum F_n = \vec{R}\cdot\hat{e}_n = m\frac{v^2}{\rho}}$$

Only cares about "now": current v drives direction change

If we want information about a **time interval**:

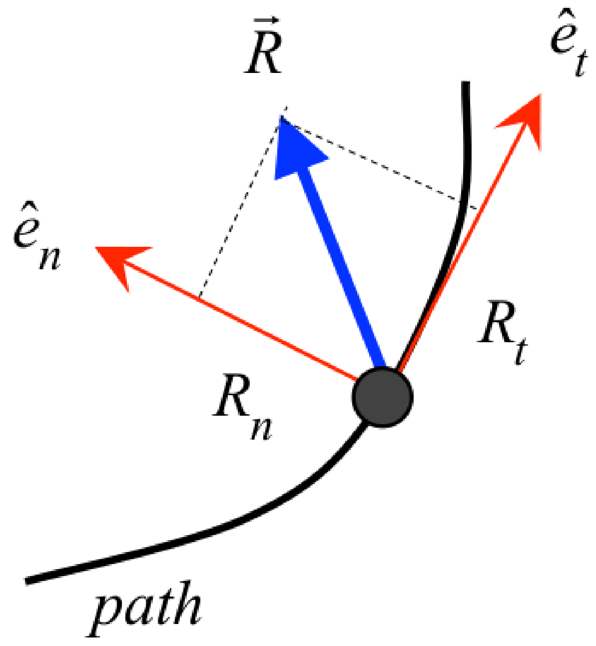
$$\underbrace{\int_{t_1}^{t_2}(\vec{R}\cdot\hat{e}_t)dt} = m\int_{v_1}^{v_2} dv$$

quantity of motion: momentum

total force exerted between t_1 and t_2: impulse

(NOT TODAY)

Work and Energy



Today, we want information about **displacement**:

$$\vec{R} \cdot \hat{e}_t = m\frac{dv}{dt} = m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = m\dot{s}\frac{dv}{ds} = mv\frac{dv}{ds}$$

Thus, the integral is:

$$\int_{s_1}^{s_2}(\vec{R} \cdot \hat{e}_t)ds = m\int_{v_1}^{v_2} vdv$$ **(separating the differentials)**

$$\int_{s_1}^{s_2}\vec{R} \cdot (\hat{e}_t ds) = m\int_{v_1}^{v_2} vdv$$ **(dot product property)**

$$\int_{s_1}^{s_2}\vec{R} \cdot d\vec{s} = m\int_{v_1}^{v_2} vdv$$ **(define $d\vec{s} = \hat{e}_t ds$)**

$$\int_{s_1}^{s_2}\vec{R} \cdot d\vec{s} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$ **(solve r.h.s. integral)**

$$U_{1\rightarrow 2} = T_2 - T_1$$

Work = Change in Kinetic Energy

Conservative Forces and Potential Energy

From Example 4.B.1, the work done by the weight was:

$$U_{1\to 2}^{(gr)} = -\left(V_2^{(gr)} - V_1^{(gr)}\right)$$

For a spring:

$$U_{1\to 2}^{(sp)} = -\left(V_2^{(sp)} - V_1^{(sp)}\right)$$

The work done by springs or the weight of a particle is **independent** of the path of which the forces act.

Such forces are called **conservative**, and their work can be computed using their potential energy functions:

$$V^{(sp)} = \frac{k}{2}(L - L_0)^2 \qquad V^{(gr)} = mgh$$

Conservative Forces and Potential Energy

The complete work-energy equation is thus:

$$T_2 + V_2 = T_1 + V_1 + U_{1\to 2}^{(nc)}$$

where

$T = \frac{1}{2}mv^2$: kinetic energy of the particle

$V = V^{(gr)} + V^{(sp)} = mgh + \frac{k}{2}(L - L_0)^2$: potential energy done by the conservative forces

$U_{1\to 2}^{(nc)} = \int_{s_1}^{s_2} \vec{F} \cdot d\vec{s}$: work done by non-conservative forces, $\vec{F}$

Example 4.B.1

Given: $m, L, F, \theta_1 = 0°, \theta_2 = 90° = \frac{\pi}{2}, v_1 = 0$

Find: $v_2 = ?$: what is the speed at position ②?

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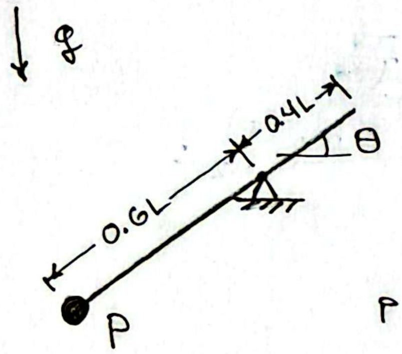
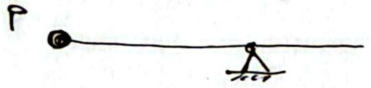


Two positions:

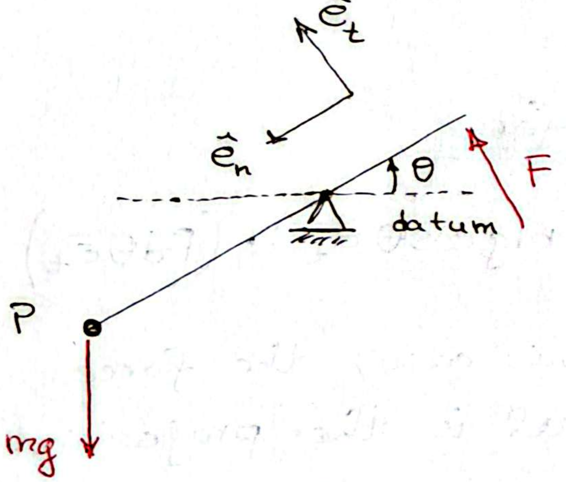
①: $\theta_1 = 0$



②: $\theta_2 = \frac{\pi}{2}$

1. Free Body Diagram

for an arbitrary ("any") position:



- We are told that "F" is <u>always</u> perpendicular to the rod.
- We define a path coordinate system $\hat{e}_t - \hat{e}_n$.
- The angle θ is measured from the "datum" line we drew.

2. Kinetics

: instead of Newton's 2^{nd} law, we use the work energy equation:

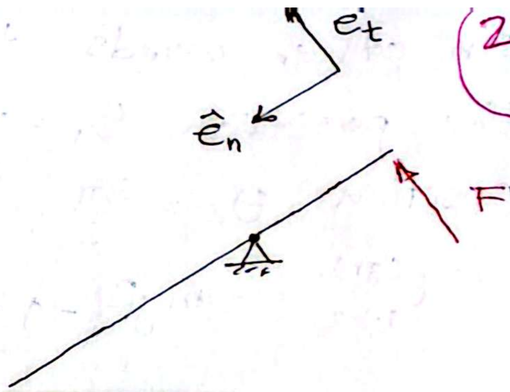
$$\underbrace{\mathcal{U}_{1\to2}}_{\text{Work done by ALL forces}} = \underbrace{T_2 - T_1}_{\text{difference of kinetic energy between positions ① and ②}} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \frac{1}{2}v_2^2$$

($\frac{1}{2}mv_1^2$ → 0, body at rest)

- Now, for the work term $\mathcal{U}_{1\to2}$: we have two forces: F & mg.
- We must compute the work for each force.
- Remember that, for a force $\vec{R}$, the work is $\mathcal{U}_{1\to2} = \int_{s_1}^{s_2} \vec{R} \cdot d\vec{s}$
- "$d\vec{s}$" is the projection of the small displacement ds onto $\hat{e}_t$

Let's start with the force F:

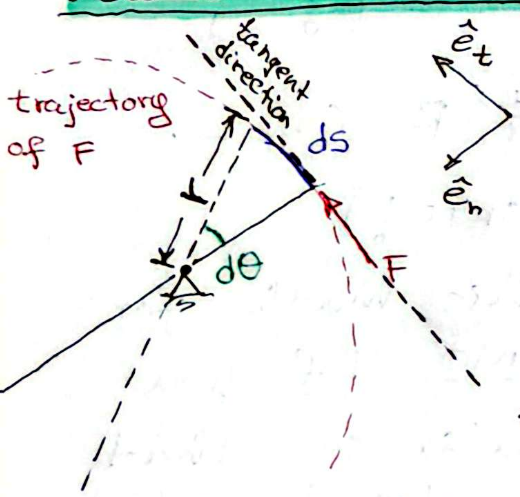
$$\mathcal{U}^{(F)}_{1\to 2} = \int_{s_1}^{s_2} \vec{F}\cdot d\vec{s}$$



How do we write F as a vector?

See the sketch on the right. For the coordinate system we defined, $\boxed{\vec{F} = F_t\hat{e}_t + F_n\hat{e}_n}$, where F_t and F_n are the tangential and normal components of $\vec{F}$. From the drawing, we can see that $F_t = F$ (because $\hat{e}_t$ is perfectly aligned with the direction of F) and $F_n = 0$ (because we were told that F is always normal to the rod).

How do we write the vector $d\vec{s}$?



First of all, what is $d\vec{s}$? Note that "F" is following a trajectory, we usually call it $s(t)$. The vector "$d\vec{s}$" is the projection of an infinitesimally small piece of that trajectory, "ds", onto the tangential direction of the trajectory. The infinitesimally small piece of trajectory, "ds", appears on the sketch above. To that small piece "ds" corresponds an infinitesimally small angle, "dθ". From the arc-length formula: $ds = r\,d\theta$. Since "ds" is infinitesimally small, it happens to approximate the tangent direction (see the slides from Lecture 2 to remember why this is true).

Therefore, we can express $d\vec{s}$ as $d\vec{s} = ds\, e_t$ and, finally, as $d\vec{s} = r\, d\theta\, \hat{e}_t$.

Going back to the work equation for $\vec{F}$:

We now have all the ingredients for $U_{1\to 2}^{(F)}$:

$$U_{1\to 2}^{(F)} = \int_{s_1}^{s_2} \vec{F} \cdot d\vec{s} = \int_{\theta_1}^{\theta_2} (F\hat{e}_t) \cdot (r\, d\theta\, \hat{e}_t)$$

→ Note the limits change!

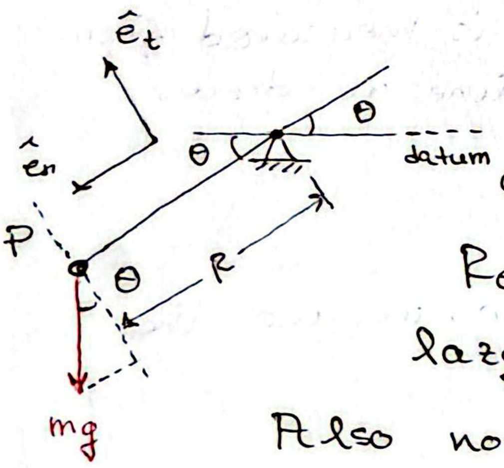
$$= \underbrace{Fr}_{\text{constants out}} \int_{\theta_1}^{\theta_2} (\hat{e}_t \cdot \hat{e}_t)\, d\theta = Fr \int_{\theta_1}^{\theta_2} d\theta = Fr(\theta_2 - \theta_1)$$

($\hat{e}_t \cdot \hat{e}_t = 1$)

$$= Fr\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2} Fr$$

Work done by the gravitational force

Now we want: $U_{1\to 2}^{(gr)} = \int_{s_1 \to \theta_1}^{s_2 \to \theta_2} (-mg\cos\theta\, \hat{e}_t) \cdot (R\, d\theta\, \hat{e}_t)$



Note that, in this case, the force inside the integral is the projection of "mg" onto the tangent axis.

Remember: the normal component is lazy, it does not contribute to the work.

Also notice the sign, which comes from "mg"'s projection acting opposite to $\hat{e}_t$.

Finally, here $d\vec{s} = R\, d\theta\, \hat{e}_t$ using a similar procedure as before. Integrating:

$$U_{1\to 2}^{(gr)} = \int_{\theta_1}^{\theta_2} (-mg\cos\theta\, R\, d\theta) = -mgR(\sin\theta_2 - \sin\theta_1)$$

Note that, using the "datum" as reference for the angles, now $\theta_2 = 270° = \frac{3\pi}{2}$ and $\theta_1 = \pi$ (for the force "mg"!)

In other words, the force "mg" started at the position $\theta_1 = \pi$ and traveled to the position $\theta_2 = \frac{3\pi}{2}$. Thus:

4/5

$$\mathcal{U}^{(gr)}_{1\to 2} = -mgR(-1-0) = mgR$$

Go back to the work equation

Don't forget why we were doing all this!

$$\mathcal{U}_{1\to 2} = \mathcal{U}^{(F)}_{1\to 2} + \mathcal{U}^{(gr)}_{1\to 2} = \frac{\pi}{2}Fr + mgR = \frac{1}{2}mv_2^2$$

3. Kinematics: not today!

4. Solve: we were looking for v_2:

$$v_2 = \sqrt{\pi Fr + 2mgR} = \sqrt{0.4L\pi F + 1.2Lmg},$$

where I substituted $r = 0.4L$ & $R = 0.6L$.

VERY IMPORTANT NOTES (DON'T SKIP!)

Manuel, do we need to do all this math for every problem? Short answer: NO. If you look at the solution video, you'll see that it doesn't take more than 5 minutes to solve (and way less writing!).

I wanted to show you:

1. the "general" way of finding $\vec{F}$ and $d\vec{s}$ without shortcuts, with all the steps
2. the formulation of the work done by gravitational energy, which is a formula you can now use for every problem

5/5

For example, note that since each force is describing a circle, we know that the path or trajectory traveled by our force "F" when going from $\theta_1 = 0$ to $\theta_2 = 90^\circ$ is just 1/4 of a circle's circumference:

same as our integral!

$$U^{(F)}_{1\to 2} = F \cdot \begin{pmatrix} \text{trajectory} \\ \text{traveled by} \\ \text{force } F \end{pmatrix} = F \cdot \begin{pmatrix} \text{1/4 of a} \\ \text{circumference} \\ \text{of radius "r"} \end{pmatrix} = F\left(\frac{2\pi r}{4}\right) = \frac{Fr\pi}{2}$$

For the gravity force, note that the work is only the force "mg" times the difference in height between position (1) of the force "mg" (i.e., of P) and position (2), taking the "datum" line as a reference:

$$U^{(gr)}_{1\to 2} = mg \cdot \begin{pmatrix} \text{height} \\ \text{difference} \\ \text{between (1) \& (2)} \end{pmatrix} = mg\begin{pmatrix} \text{height} & - & \text{height} \\ (2) & & (1) \end{pmatrix}$$

same as integral!

$$= mg\left(-R - 0\right) = -mgR = mgh, \quad h = -R - 0$$

$$h = h_2 - h_1$$

note the "–" sign: height is below datum!

In general, the work of gravitational forces depend only on the difference of height between positions (1) and (2). We call forces with such a property conservative forces. Next class we will see that spring forces are also conservative. We call their "formulas" "potential energy functions". For gravit. forces:

denoted by "V"

$$U^{(g)}_{1\to 2} = mgh_2 - mgh_1 = V^{(gr)}_2 - V^{(gr)}_1$$