# Important Announcements

1. Homework Schedule:
   - H.3.C, H.3.D: ~~Wednesday, 02/11~~ → Friday, 02/13
   - H.3.E, H.3.F: Monday, 02/16
   - H.3.G, H.3.H: Wednesday, 02/18
2. No class on Wednesday, 02/11
3. Class on Friday, 02/13 (online option, recording available)

# ME 274: Basic Mechanics II

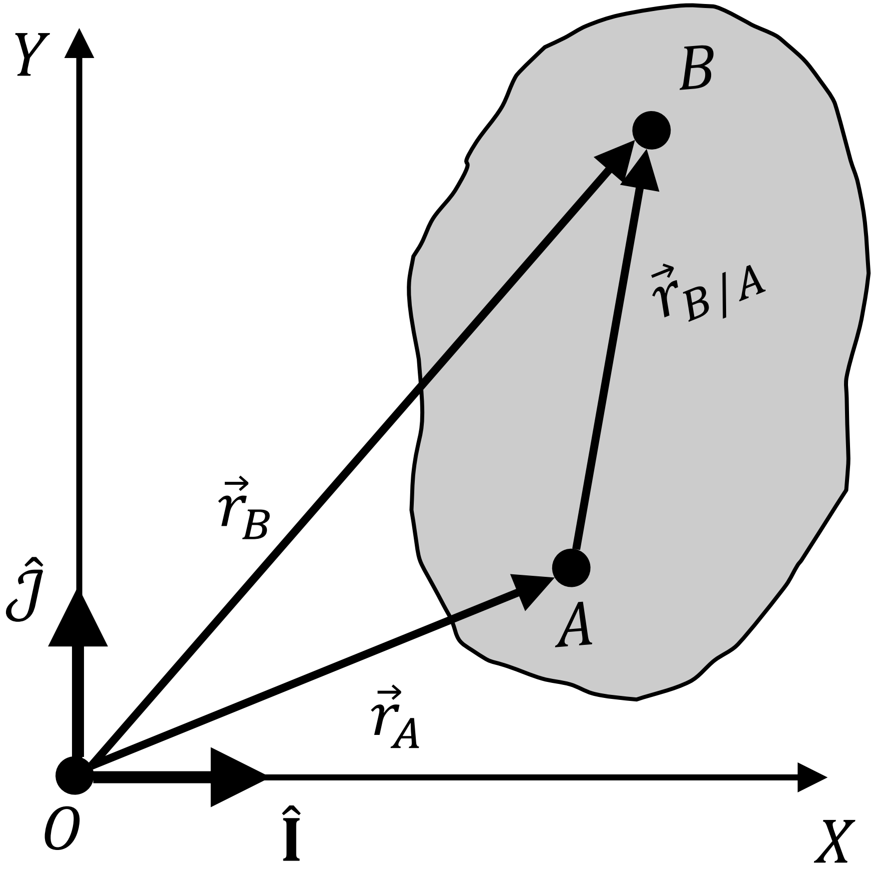
***Week 5*** *– Monday, February 9*

Particle kinematics: 2D Rotating Reference Frames

Instructor: Manuel Salmerón

# Today’s Agenda

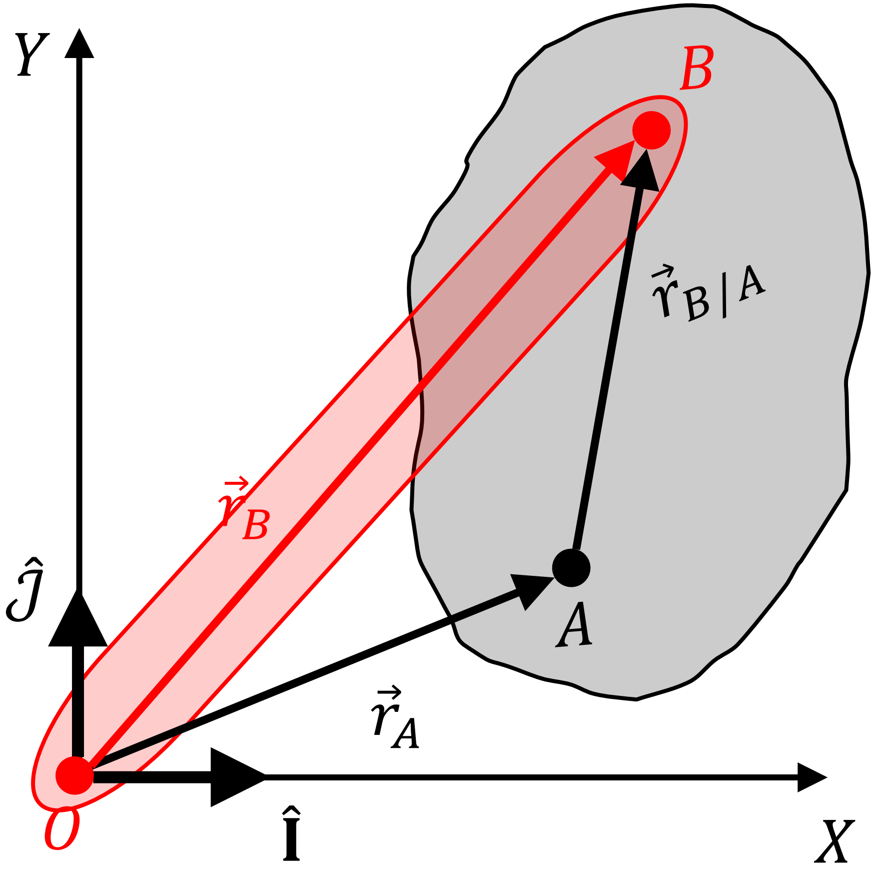
1. Recap: 2D Rotating Reference Frames
2. Example 3.A.1
3. Review for Exam 1




**How does $B$ move with respect to $A$?**

$A$ and $B$ are in the same rigid body:

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A = \underbrace{\vec{\omega} \times \vec{r}_{B/A}}_{\text{Describes how } B \text{ rotates around } A}$$

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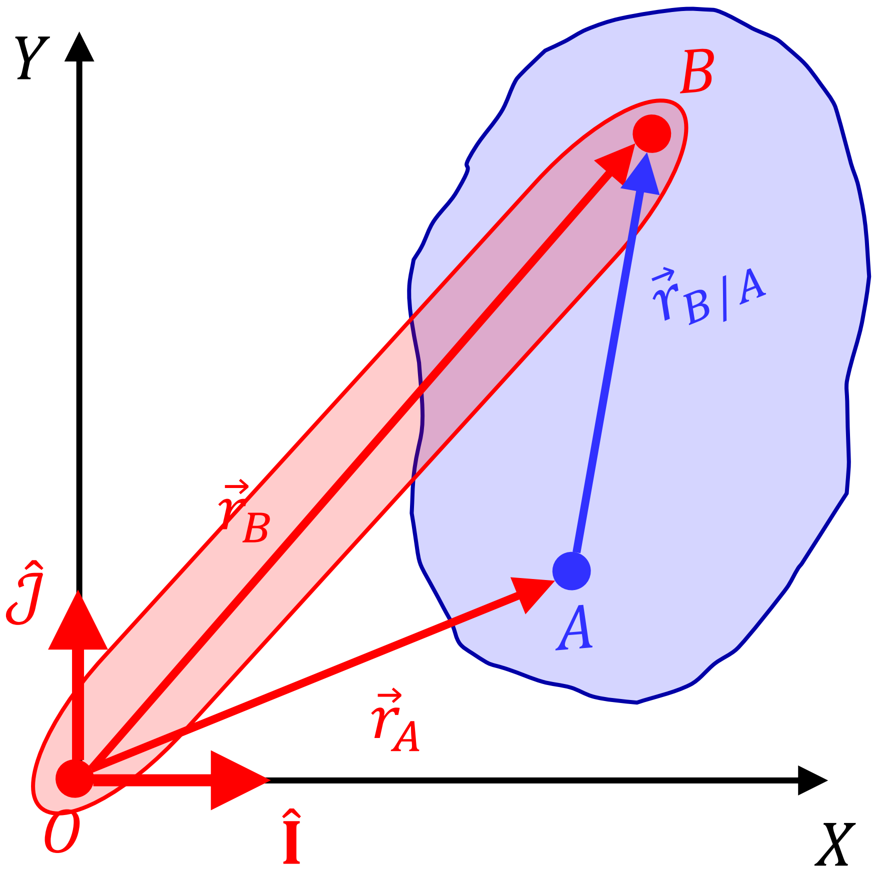
$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A = \underbrace{\vec{\omega} \times \vec{r}_{B/A}}_{\text{Describes how } B \text{ rotates around } A} + \left(\vec{v}_{B/A}\right)_{rel}$$

Describes how $B$ moves away and closer to $A$




## How does $B$ move with respect to $A$?

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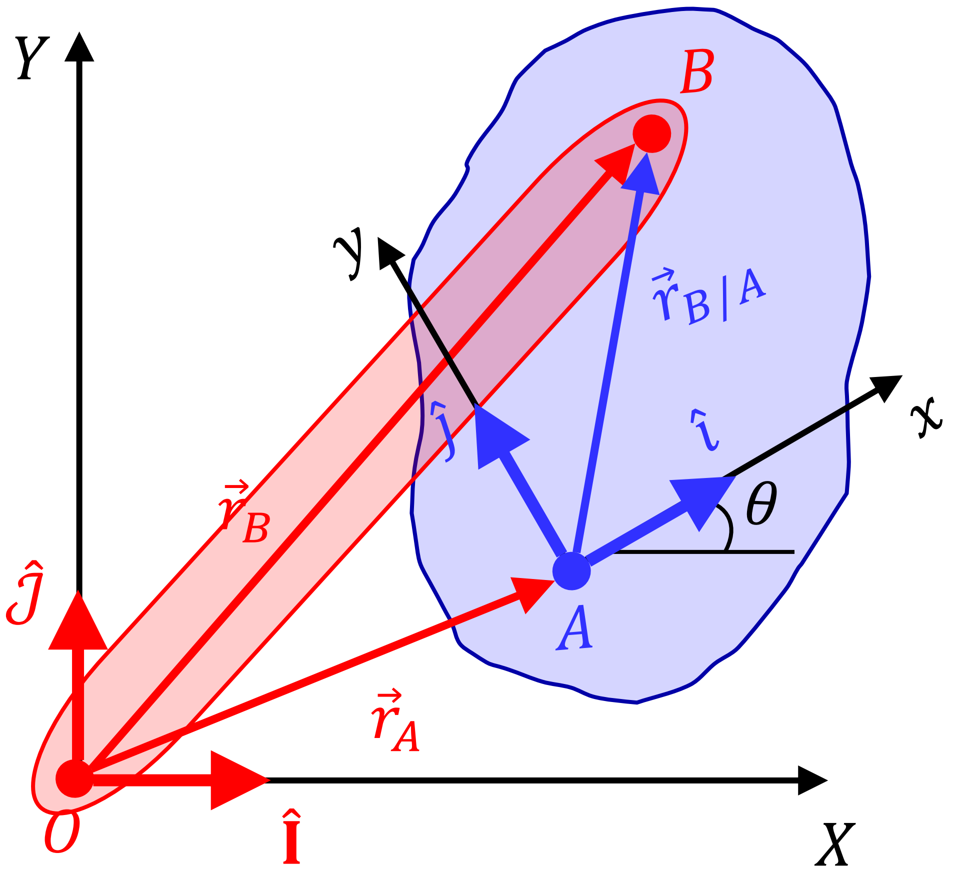
Describes how $B$ moves away and closer to $A$

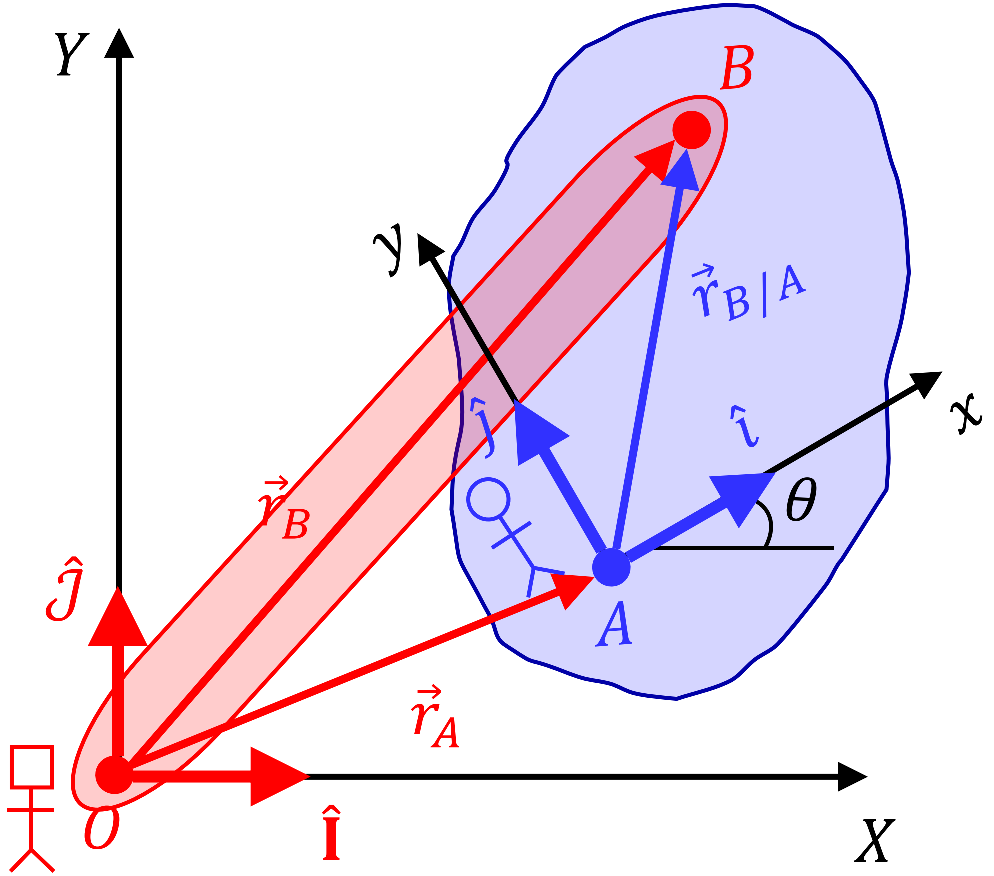



## How does $B$ move with respect to $A$?

$A$ and $B$ are in the same rigid body:

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A = \underbrace{\vec{\omega} \times \vec{r}_{B/A}}_{\text{Describes how } B \text{ rotates around } A} + \left(\vec{v}_{B/A}\right)_{rel}$$

$\left(\vec{v}_{B/A}\right)_{rel}$: Describes how $B$ moves away and closer to $A$




**How does $B$ move with respect to $A$?**

$A$ and $B$ are in the same rigid body:

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A = \underbrace{\vec{\omega} \times \vec{r}_{B/A}}_{\text{Describes how } B \text{ rotates around } A} + \left(\vec{v}_{B/A}\right)_{rel}$$

Describes how $B$ moves away and closer to $A$

**TIP:** $\vec{\omega}$ will likely have $A$ on its name

**REMARK:** the fixed referent frame does not need to be attached to a body

MOVING OBSERVER

FIXED OBSERVER

# (Typical) Solution Steps

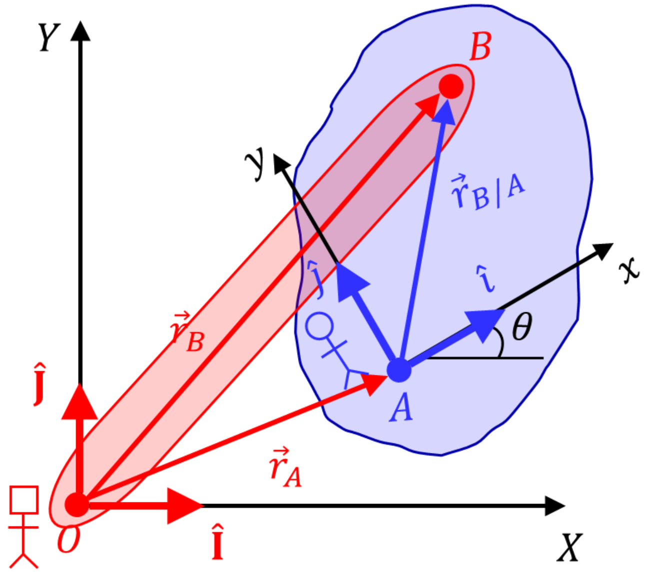
1. Find the moving reference frame
2. Locate the axes of the moving and fixed reference frames
3. Describe the angular motion of the moving reference frame: $\vec{\omega}$ and $\vec{\alpha}$
4. Describe the relative motion of the moving reference frame: $\left(\vec{v}_{B/A}\right)_{rel}$ and $\left(\vec{a}_{B/A}\right)_{rel}$
5. Solve the fundamental equations:

$$\vec{v}_B = \vec{v}_O + \vec{\omega} \times \vec{r}_{B/O}$$

$$\vec{a}_B = \vec{a}_O + \vec{\alpha} \times \vec{r}_{B/O} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/O}\right)$$

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right)$$



## Example 3.A.6

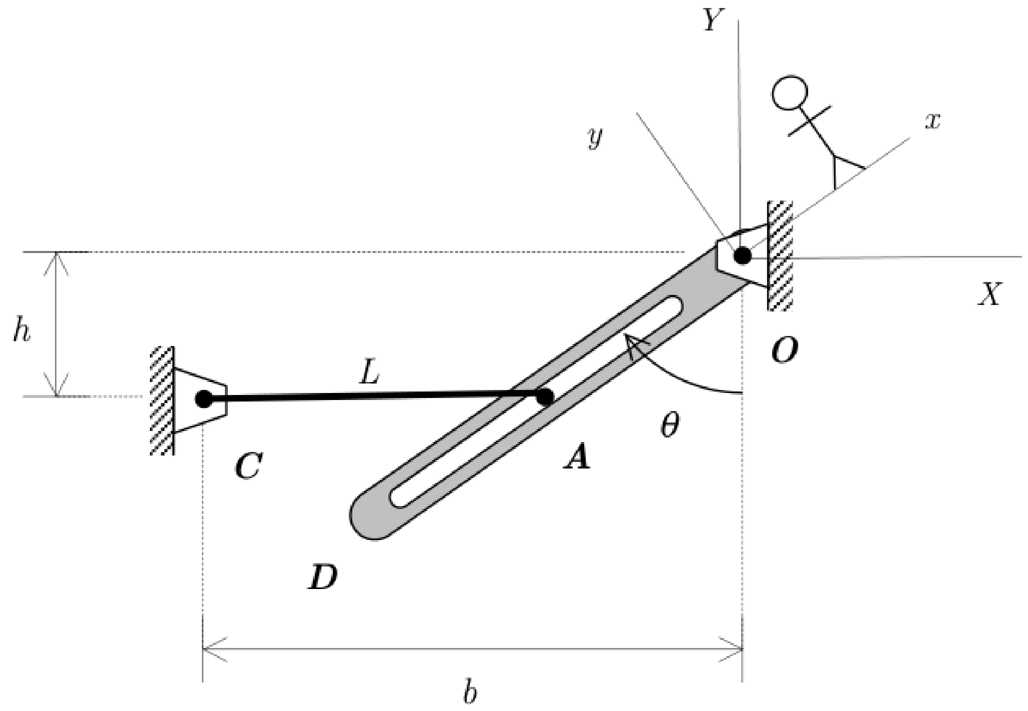
**Given:** Link OD is rotating clockwise at a constant rate of $\dot{\theta} = 2$ rad/s. When $\theta = 45°$, link CA is horizontal.

**Find:** Determine:

(a) The velocity of A when $\theta = 45°$; and

(b) The acceleration of A at the same position

Use the following parameters in your analysis: $L = 0.225$ m, $h = 0.225$ m and $b = 0.45$ m.



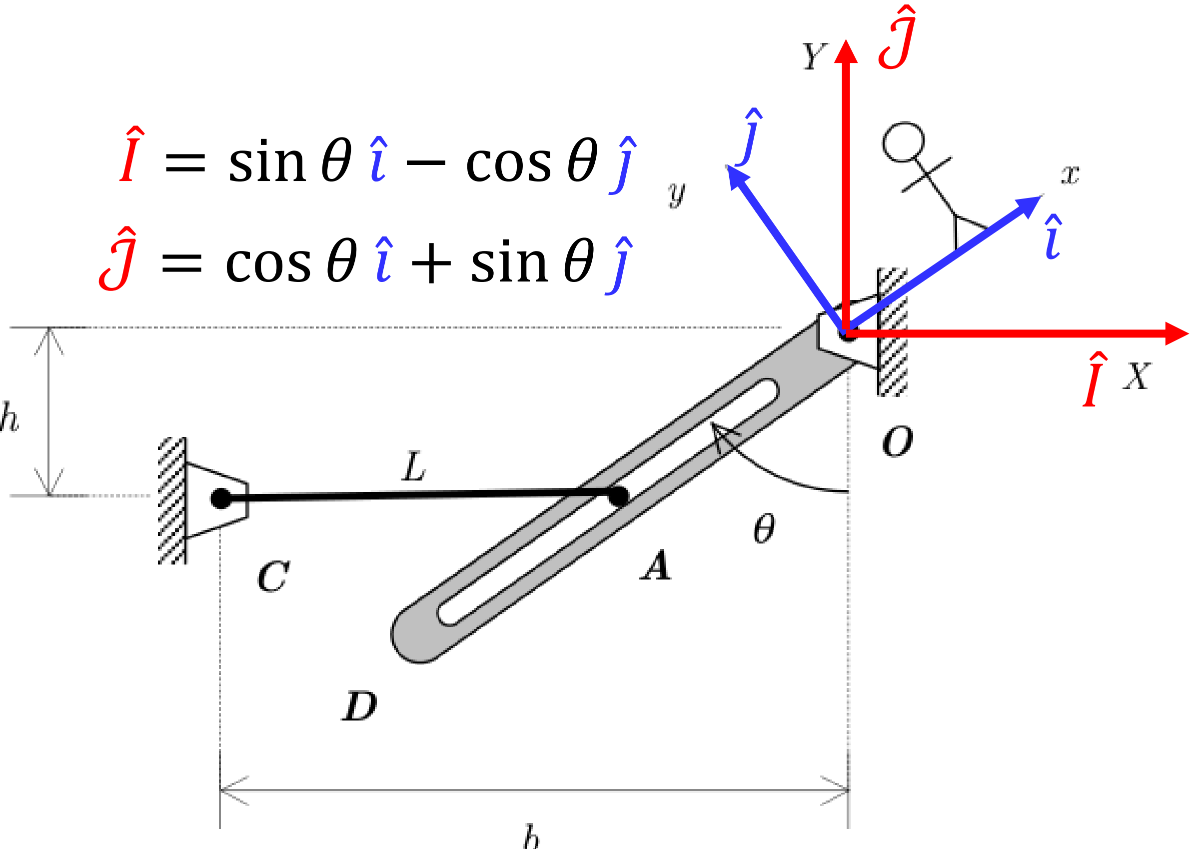
# Example 3.A.6

**Given:** $\dot{\theta} = 2$ rad/s, $\theta = 45°$

**Find:** (a) $\vec{v}_A$, (b) $\vec{a}_A$

$$\hat{I} = \sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}$$

$$\hat{J} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$




**Solution:**

1. Moving reference (given)
2. Locate axes (moving and fixed)

We go for velocity first:

$\vec{v}_A = \vec{v}_C + \vec{\omega}_{AC} \times \vec{r}_{A/C}$ (same RB)

$\vec{v}_A = \vec{v}_O + \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}_{OD} \times \vec{r}_{A/O}$ (different RB)

3. Angular motion of moving reference:

$$\omega_{OD} = \dot{\theta} \qquad \vec{\omega}_{OD} = -\dot{\theta}\hat{k}$$

4. Relative motion of moving reference:

$\left(\vec{v}_{A/O}\right)_{rel} = v_{rel}$ (unknown)

5. Solve fundamental equations ($\vec{v}_A = \vec{v}_A$)

$$\vec{\omega}_{AC} \times \vec{r}_{A/C} = \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}_{OD} \times \vec{r}_{A/O}$$

$$\omega_{AC} L \hat{J} = \omega_{AC} L\left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right) = v_{rel}\hat{\imath} + \dot{\theta} d\hat{\jmath}$$

# Example 3.A.6

Solution:

Now, acceleration:

$$\vec{a}_A = \vec{\alpha}_{AC} \times \vec{r}_{A/C} - \omega_{AC}^2 \vec{r}_{A/C} \qquad \text{(same RB)}$$

$$\vec{a}_A = \left(\vec{a}_{A/O}\right)_{rel} + 2\vec{\omega}_{OD} \times \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}_{OD} \times \left(\vec{\omega}_{OD} \times \vec{r}_{A/O}\right) \qquad \text{(different RB)}$$

3. Angular motion of moving reference:

$$\alpha_{OD} = 0 \qquad \vec{\alpha}_{OD} = \vec{0} \qquad (\dot{\theta} \text{ is constant})$$

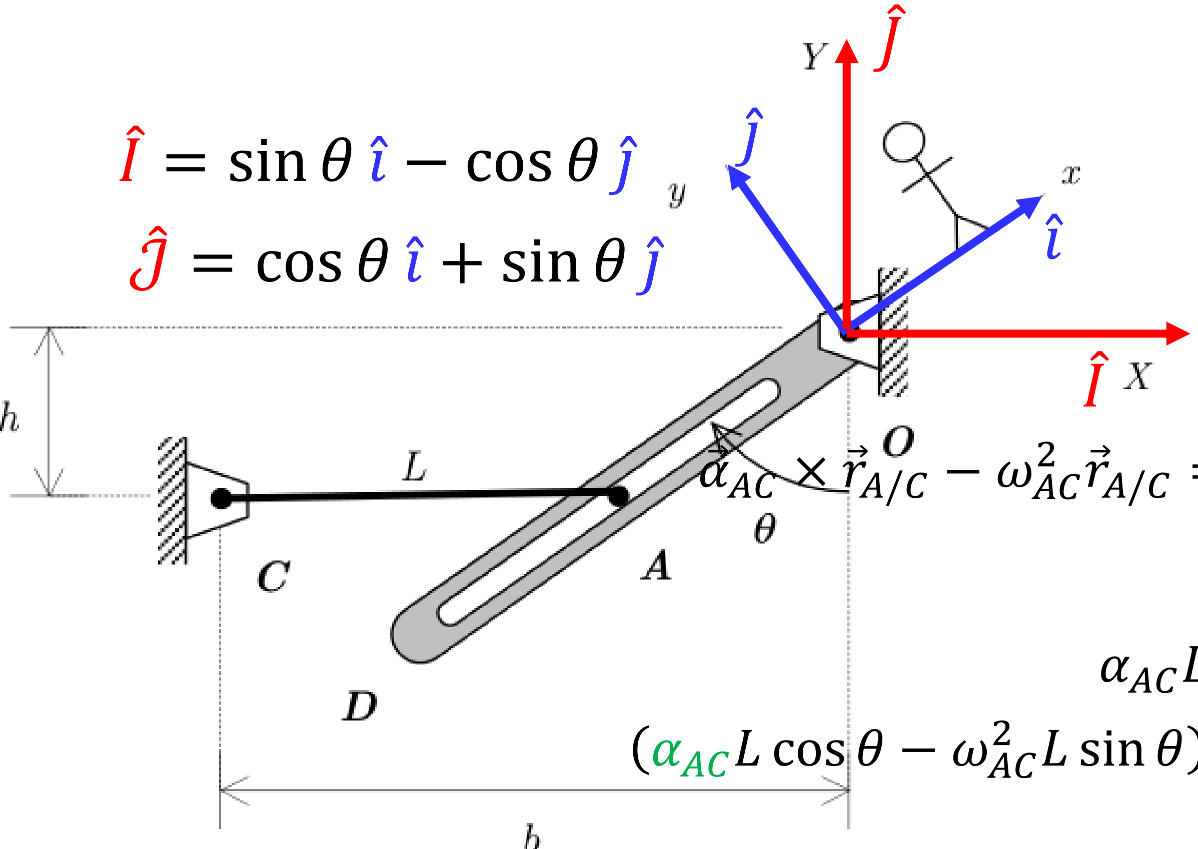
4. Relative motion of moving reference:

$$\left(\vec{a}_{A/O}\right)_{rel} = a_{rel} \qquad \text{(unknown)}$$

5. Solve fundamental equations ($\vec{a}_A = \vec{a}_A$)

$$\hat{I} = \sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}$$

$$\hat{J} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$



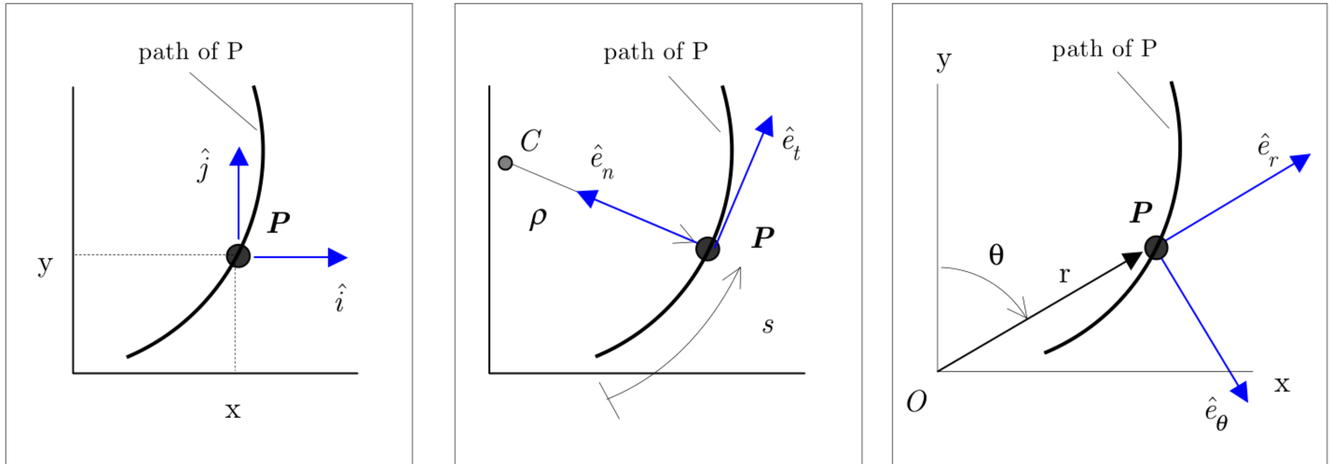

$$\vec{\alpha}_{AC} \times \vec{r}_{A/C} - \omega_{AC}^2 \vec{r}_{A/C} = \left(\vec{a}_{A/O}\right)_{rel} + 2\vec{\omega}_{OD} \times \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}_{OD} \times \left(\vec{\omega}_{OD} \times \vec{r}_{A/O}\right)$$

…long development…

$$\alpha_{AC} L \hat{J} - \omega_{AC}^2 L \hat{I} = a_{rel}\hat{\imath} - 2\dot{\theta} v_{rel}\hat{\jmath} + \dot{\theta}^2 d\hat{\imath}$$

$$(\alpha_{AC} L\cos\theta - \omega_{AC}^2 L\sin\theta)\hat{\imath} + (\alpha_{AC} L\sin\theta + \omega_{AC}^2 L\cos\theta)\hat{\jmath} = \left(a_{rel} + \dot{\theta}^2 d\right)\hat{\imath} - 2\dot{\theta} v_{rel}\hat{\jmath}$$

# Particle Kinematics




*Cartesian description* | *Path description* | *Polar description*

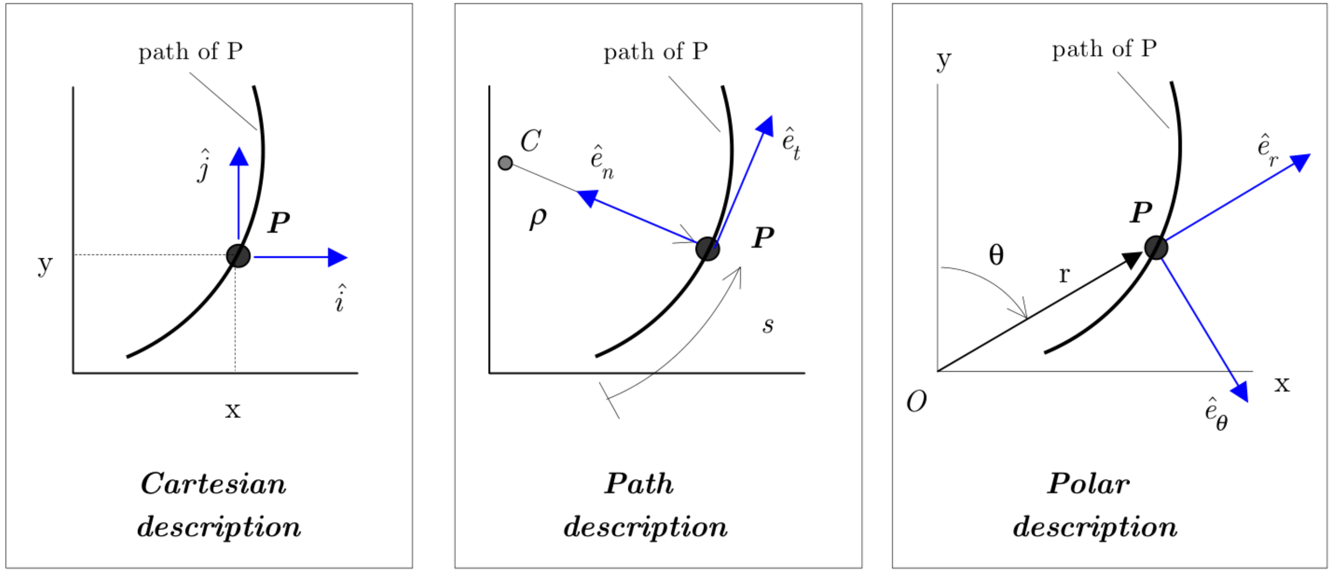
$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$
$$= v\hat{e}_t$$
$$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$
$$= \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$
$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

**Things to remember:**

- $\vec{v}$ and $\vec{a}$ are the same, we just describe them differently (balancing coefficients, vector equality)
- $\vec{v}$ is tangent to the path; $|\vec{v}|$ is the speed
- $\frac{v^2}{\rho}\hat{e}_n$ points INWARD to the path; $\dot{v}\hat{e}_t$ is tangent to the path
- $\hat{e}_\theta$ points in the same direction to which $\theta$ “grows”; $\hat{e}_r$ points in the direction where $r$ points

# Particle Kinematics



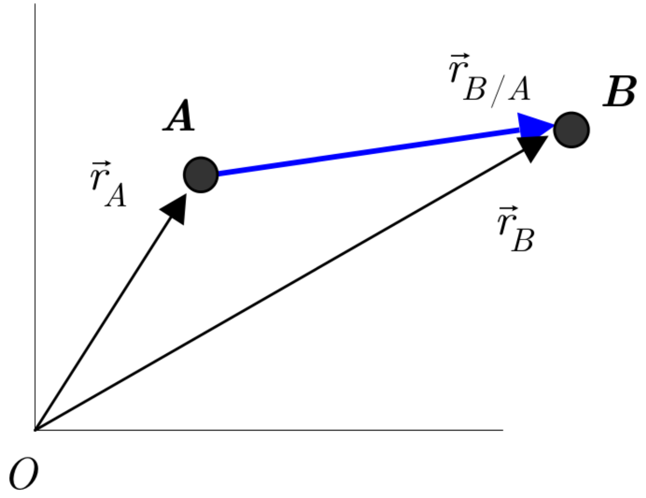

$$
\begin{aligned}
\vec{v} &= \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} \\
&= v\hat{e}_t \\
&= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta
\end{aligned}
$$

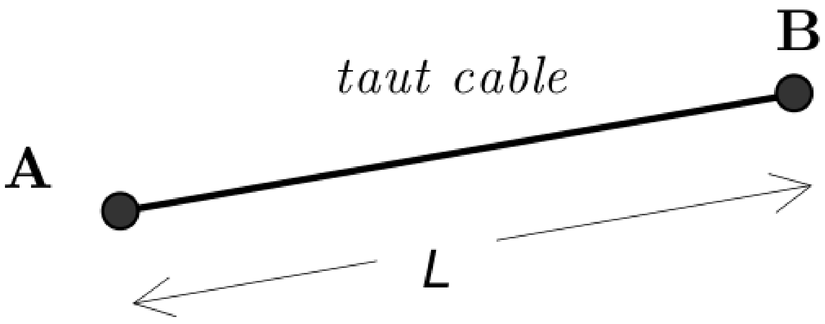
$$
\begin{aligned}
\vec{a} &= \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} \\
&= \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta
\end{aligned}
$$

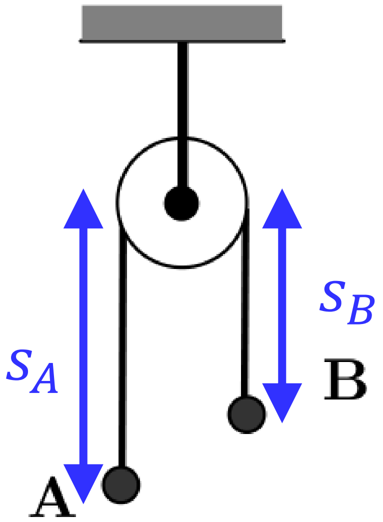
**What else would you add?**

- **--**
- **--**

# Relative and Constrained Motion



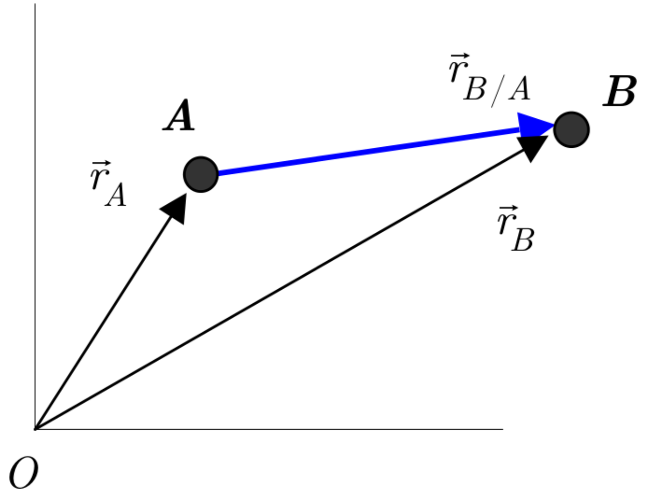



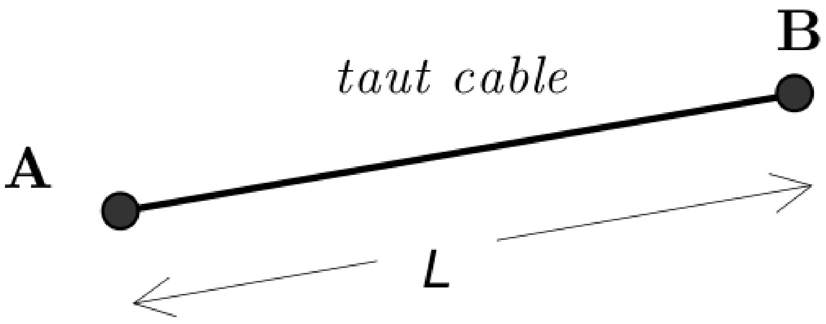



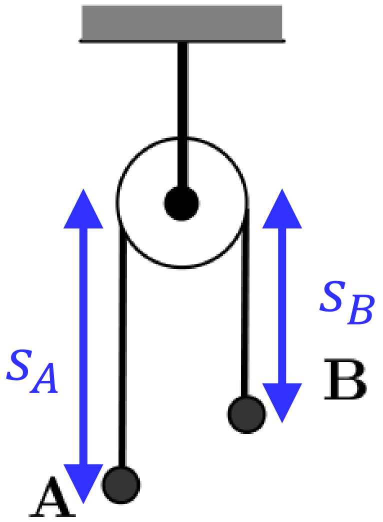

Things to remember:

- $\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$ : how does $B$ look like if I am standing at $A$?
- $\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$ : what is the velocity of $B$ relative to $A$?
- $L = s_A + s_B + \cdots$ : sum of all the little pieces of cable/rope.
- $L$ remains constant: $\frac{dL}{dt} = v_A + v_B + \cdots = 0$ and $\frac{d^2L}{dt^2} = a_A + a_B + \cdots = 0$

# Relative and Constrained Motion



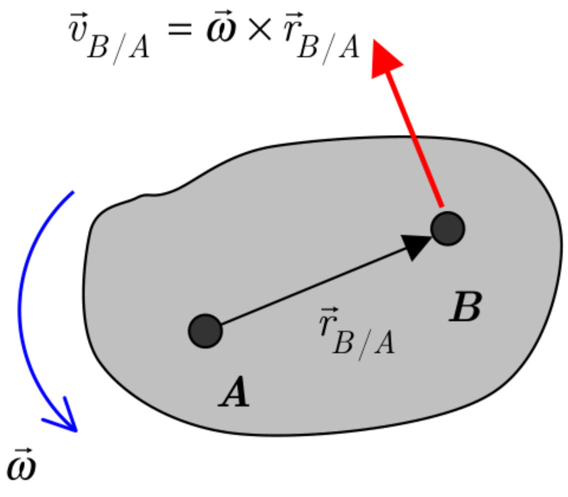



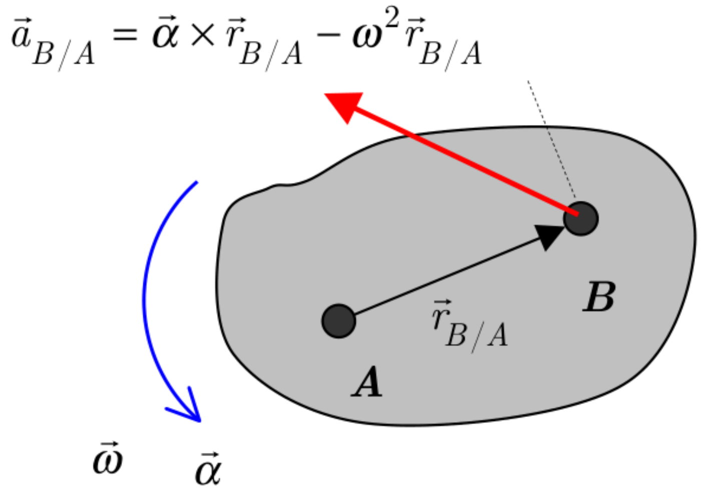




What else would you add?

- ..
- ..

# Planar Rigid Body Motion

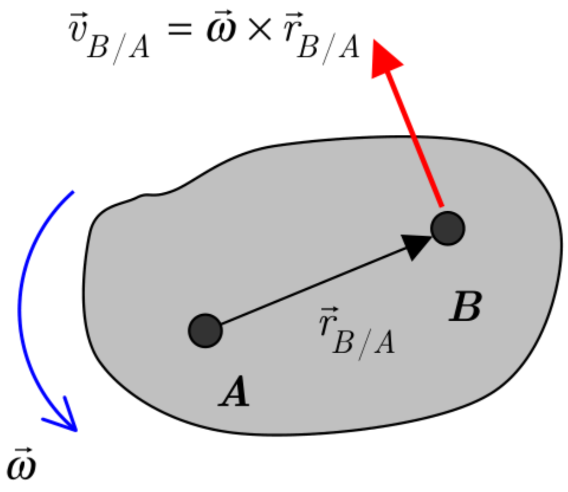


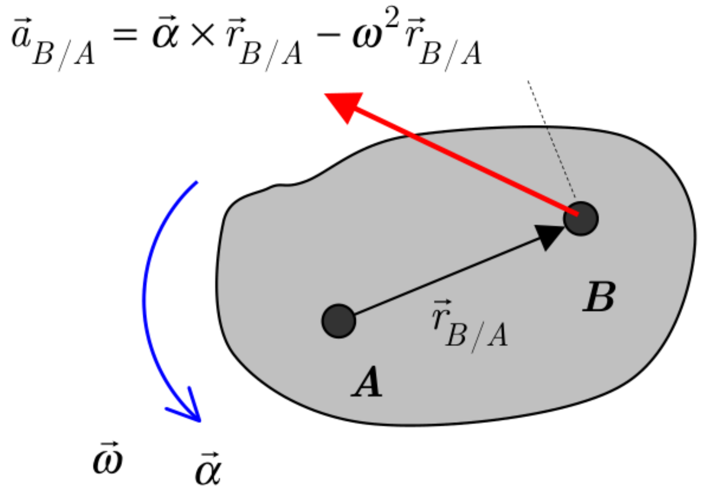




Things to remember:

- $\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$ , $\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$ , $\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A$
- $A$ and $B$ must be inside the same rigid body
- $\vec{\omega}$ and $\vec{\alpha}$ rotate around $\hat{k}$ (perpendicular to the plane)
- $\vec{\omega}$ and $\vec{\alpha}$ are the same for all the points inside the same body
- Counterclockwise (CCW) is positive, clockwise (CW) is negative
- Start with less complex (velocity), proceed with more complex (acceleration)

# Planar Rigid Body Motion

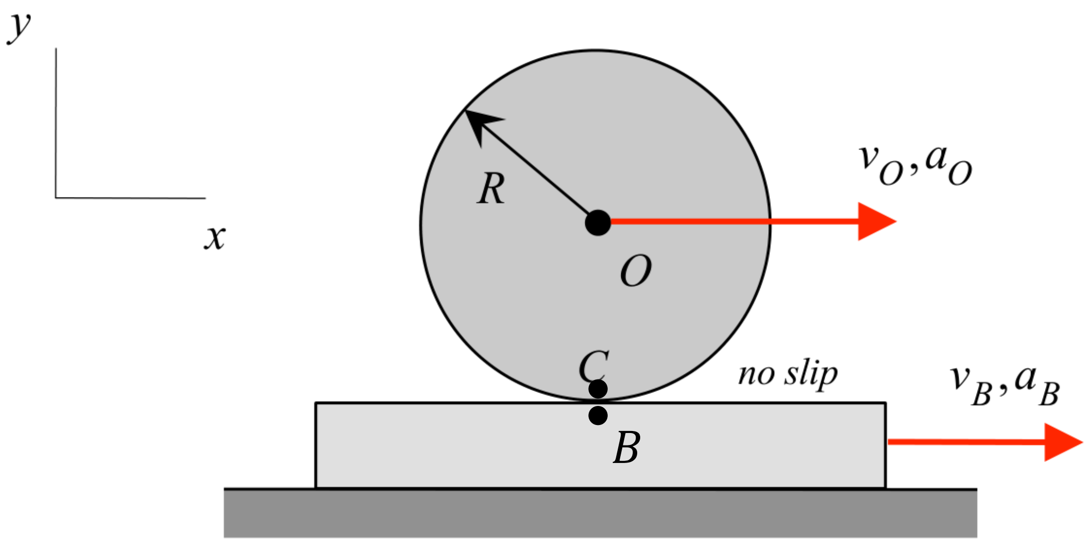



What else?

- ..

# No slipping

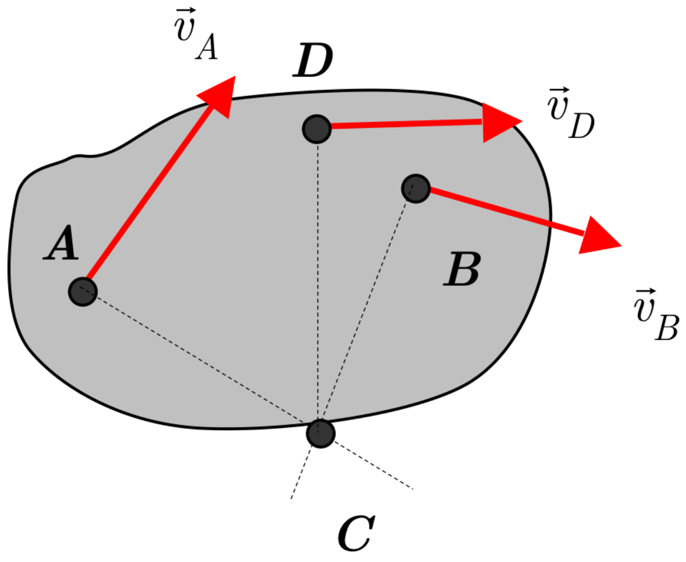



Things to remember:

- In $x$: $C$ “inherits” the motion of $B$: $v_{Cx} = v_B$, $a_{Cx} = a_B$ (if $B$ doesn’t move, $v_B = a_B = 0$)
- In $y$:
  - $C$ can not “go beyond” the ground (or beyond $B$): $v_{Cy} = 0$
  - $C$ can still have the “intention” to move up: $a_{Cy} \neq 0$

**Heuristic descriptions, see lecture book for rigor.**

# Instantaneous Centers of Rotation




Things to remember:

- The ICR is at the intersection of the lines perpendicular to all velocity vectors in a rigid body
- Knowing the ICR, we can find the direction of the velocity of any point in the RB
- $\vec{v} = \vec{0}$
- $\omega = \dfrac{|\vec{v}_A|}{|\vec{r}_{A/C}|} = \dfrac{|\vec{v}_B|}{|\vec{r}_{B/C}|} = \dfrac{|\vec{v}_D|}{|\vec{r}_{D/C}|}$
- If the lines intersect at ∞ (parallel velocities): no rotation, pure translation

# ME 274: Basic Mechanics II

***Week 5*** – *Friday, February 13*

Particle kinematics: 3D Rotating Reference Frames

Instructor: Manuel Salmerón

# Today’s Agenda

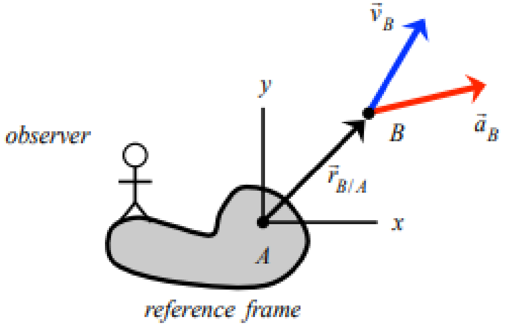
1. Recap: 2D Rotating Reference Frames
2. 3D Rotating Reference Frames
3. Example 3.B.1
4. Summary

# *Summary*: 2D Moving Reference Frame Kinematics 2

*PROBLEM*: A person attached to a moving body (reference frame) is observing the motion of point B.

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right)$$




*APPLICATION*: Using 2D MRF equations in solving problems in the kinematics of mechanisms.

AP (rigid body):

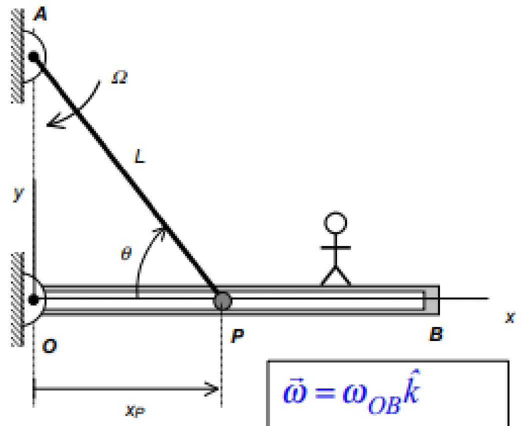
$$\vec{v}_P = \left(-\Omega\hat{k}\right) \times \vec{r}_{P/A}$$

$$\vec{a}_P = \left(-\dot{\Omega}\hat{k}\right) \times \vec{r}_{P/A} + \left(-\Omega\hat{k}\right) \times \left[\left(-\Omega\hat{k}\right) \times \vec{r}_{P/A}\right]$$

OP (*not* a rigid body):

$$\vec{v}_P = \dot{x}_P\hat{i} + \left(\omega_{OB}\hat{k}\right) \times \vec{r}_{P/A}$$

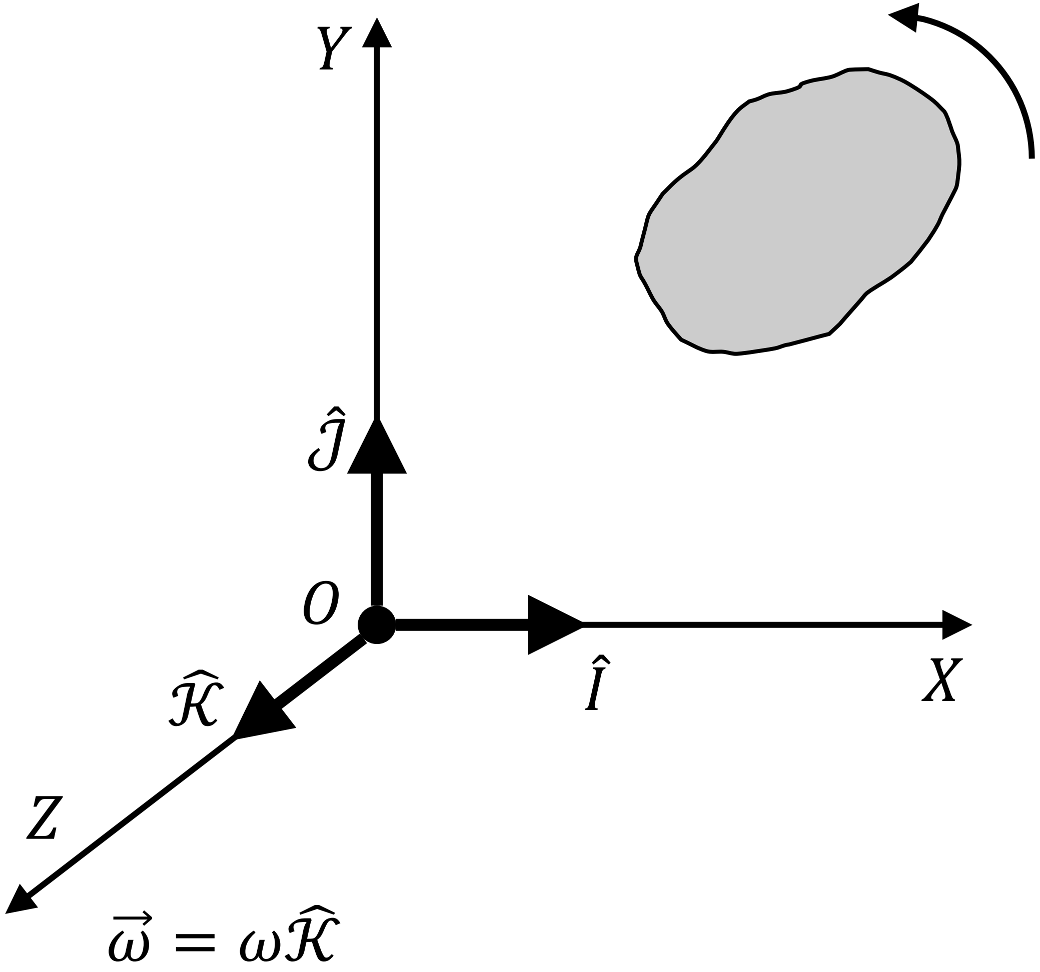
$$\vec{a}_P = \ddot{x}_P\hat{i} + \left(\alpha_{OB}\hat{k}\right) \times \vec{r}_{P/A} + 2\left(\omega_{OB}\hat{k}\right) \times \left(\dot{x}_P\hat{i}\right) + \left(\omega_{OB}\hat{k}\right) \times \left[\left(\omega_{OB}\hat{k}\right) \times \vec{r}_{P/A}\right]$$




With observer on OB:

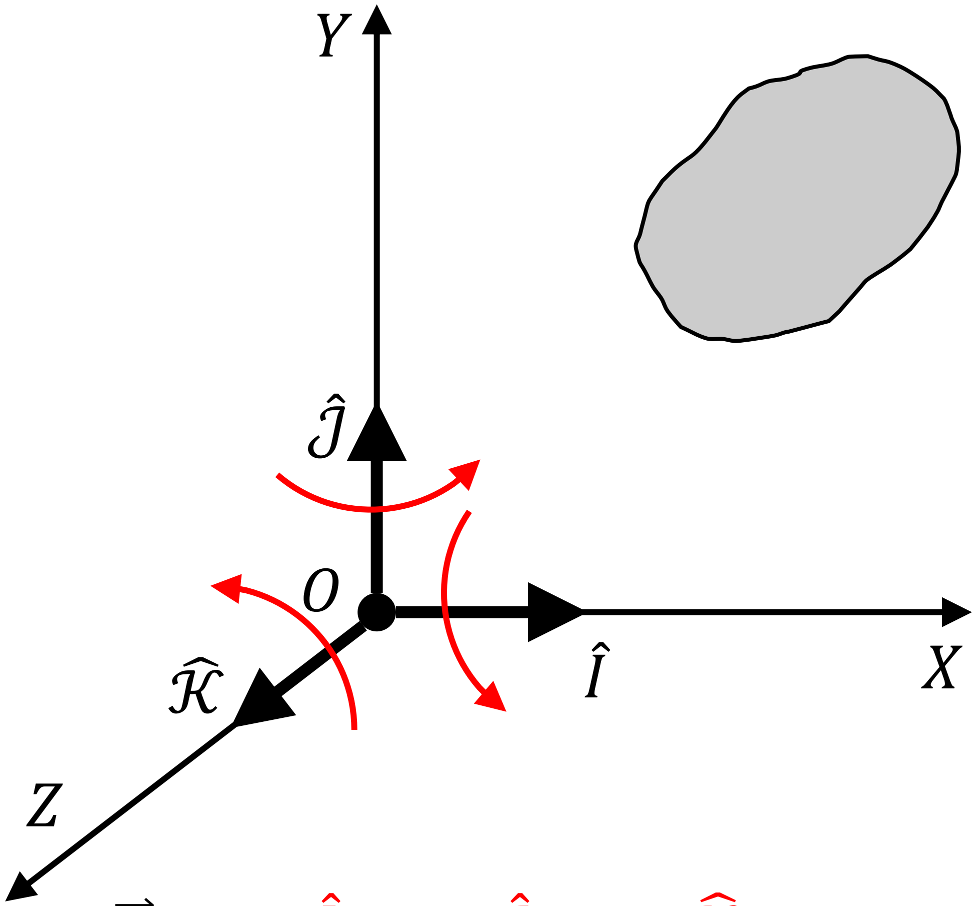
$$\vec{\omega} = \omega_{OB}\hat{k}$$
$$\vec{\alpha} = \alpha_{OB}\hat{k}$$
$$\left(\vec{v}_{P/A}\right)_{rel} = \dot{x}_P\hat{i}$$
$$\left(\vec{a}_{P/A}\right)_{rel} = \ddot{x}_P\hat{i}$$

# 3D Rotating Reference Frames




**In 2D**, any rigid body ALWAYS rotated around the unit vector $\hat{\mathcal{K}}$ (no matter what)

# 3D Rotating Reference Frames

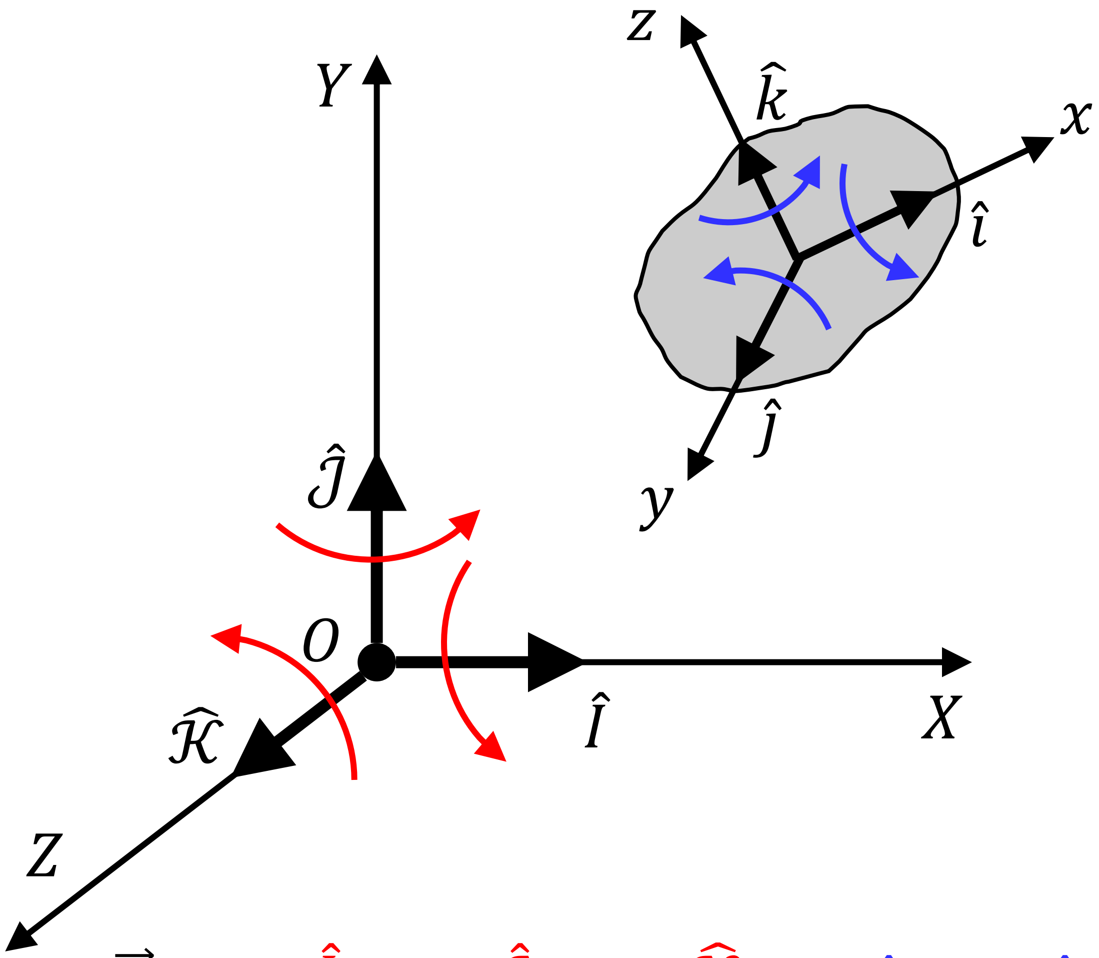



$$\vec{\omega} = \omega_1 \hat{I} + \omega_2 \hat{J} + \omega_3 \hat{\mathcal{K}}$$

**In 2D**, any rigid body ALWAYS rotated around the unit vector $\hat{\mathcal{K}}$ (no matter what)

**In 3D**, a rigid body can rotate around any axis, i.e., around $\hat{I}$, $\hat{J}$, or $\hat{\mathcal{K}}$…

# 3D Rotating Reference Frames



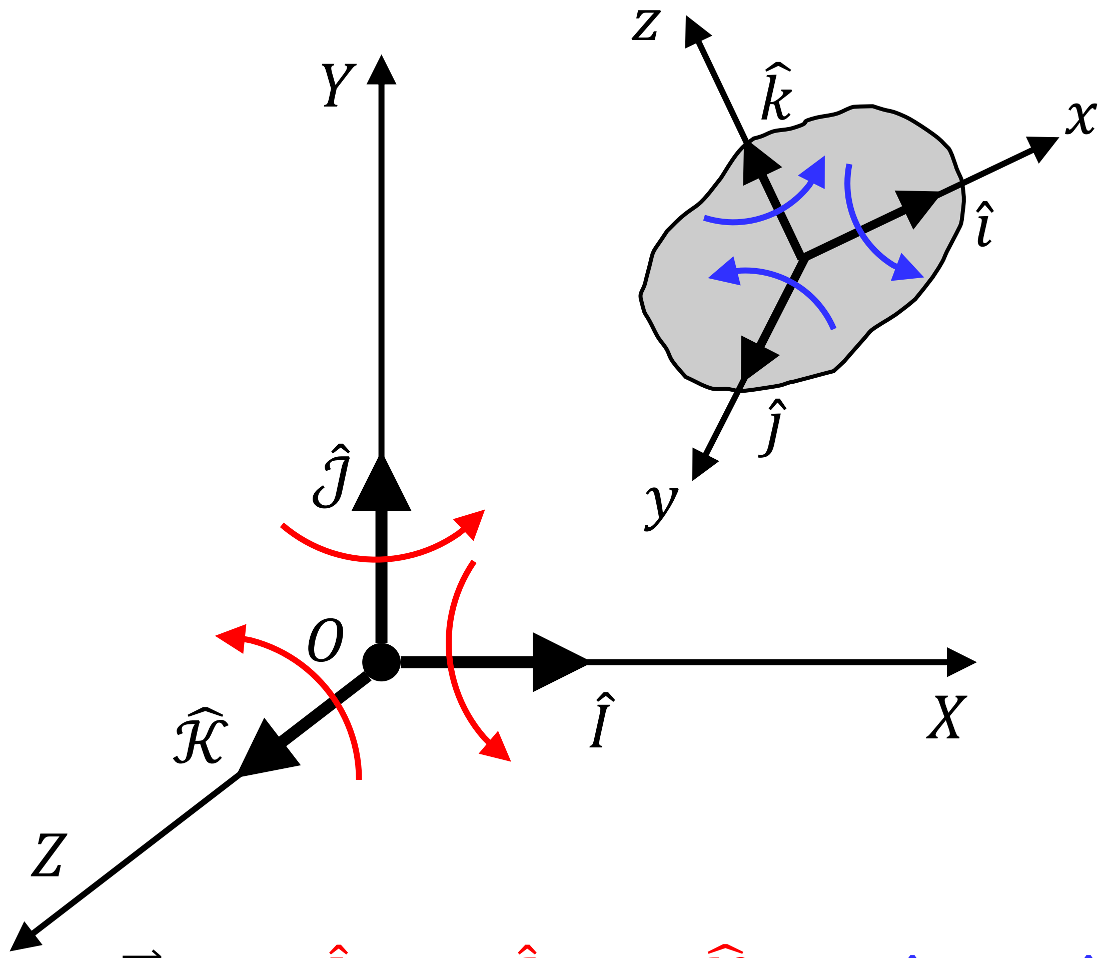

**In 2D**, any rigid body ALWAYS rotated around the unit vector $\hat{\mathcal{K}}$ (no matter what)

**In 3D**, a rigid body can rotate around any axis, i.e., around $\hat{I}$, $\hat{J}$, or $\hat{\mathcal{K}}$…

… and around the three axes of a **moving** reference frame

$$\vec{\omega} = \omega_1\hat{I} + \omega_2\hat{J} + \omega_3\hat{\mathcal{K}} + \omega_4\hat{\imath} + \omega_5\hat{\jmath} + \omega_6\hat{k}$$

# 3D Rotating Reference Frames




$$\vec{\omega} = \omega_1 \hat{I} + \omega_2 \hat{\mathcal{J}} + \omega_3 \hat{\mathcal{K}} + \omega_4 \hat{\imath} + \omega_5 \hat{\jmath} + \omega_6 \hat{k}$$

We will use:

- cross-product properties,
- unit vector projections,
- the right-hand rule,
- the following derivatives:

$$\frac{d\hat{\imath}}{dt} = \vec{\omega} \times \hat{\imath}$$

$$\frac{d\hat{\jmath}}{dt} = \vec{\omega} \times \hat{\jmath}$$

$$\frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k} \qquad \textbf{(new!)}$$

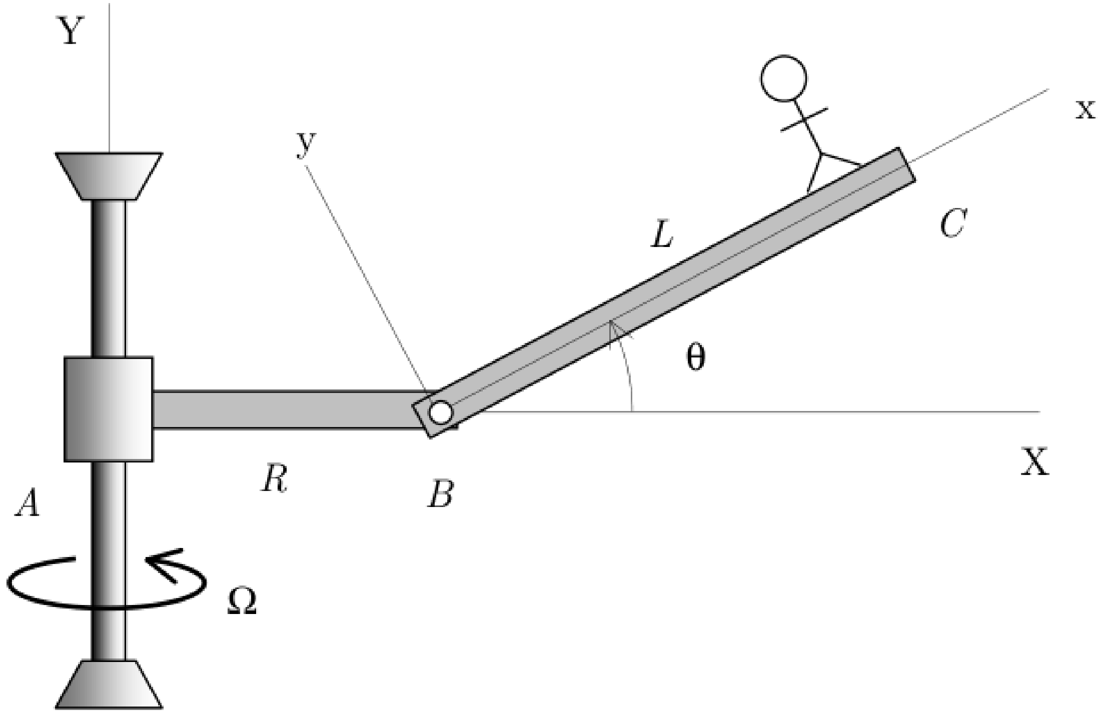
### Example 3.B.1

**Given:** Bar BC is pinned at end B to bar AB, which in turn rotates about a fixed vertical axis at a constant rate of $\Omega = 5$ rad/s. The angle $\theta$ is increasing at a constant rate of $\dot{\theta} = 4$ rad/s. The observer and the $xyz$ axes are attached to arm BC, and the $XYZ$ axes are fixed.

**Find:** Determine:

(a) The angular velocity of the observer when $\theta = 90°$; and

(b) The angular acceleration of the observer when $\theta = 90°$.

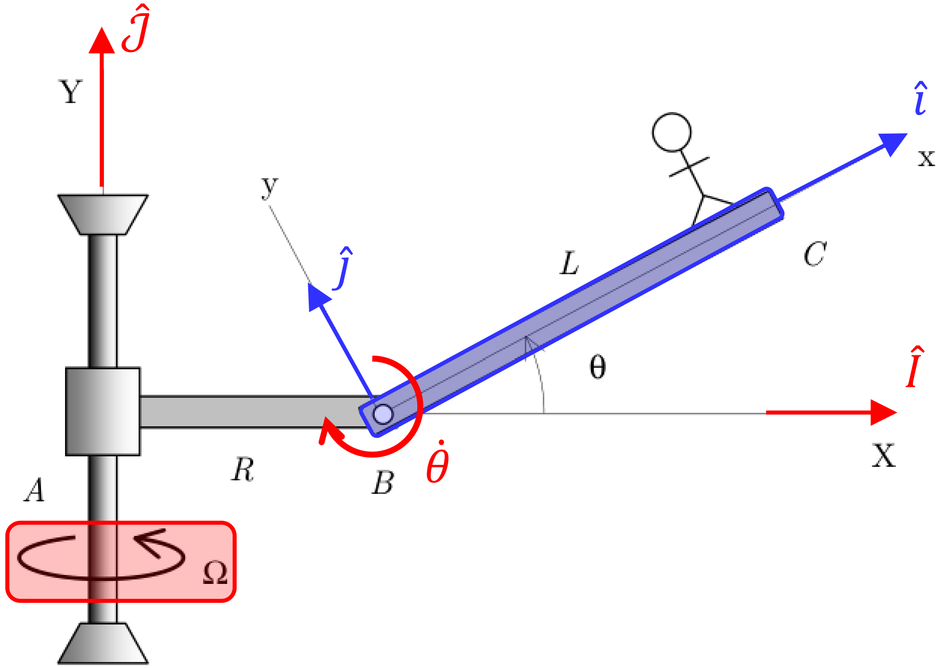


## Example 3.B.1

**Given:** $\Omega \equiv$ constant, $\dot{\theta} \equiv$ constant

**Find:** (a) $\vec{\omega}$ when $\theta = 90°$

(b) $\vec{\alpha}$ when $\theta = 90°$




**Solution:**

Find all the axes around which the observer rotates

$$\vec{\omega} = \Omega\hat{J} + \dot{\theta}\hat{k}$$

Don't substitute yet! Take derivatives first:

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \cancelto{0}{\dot{\Omega}}\hat{J} + \Omega\cancelto{0}{\frac{d\hat{J}}{dt}} + \dot{\theta}\frac{d\hat{k}}{dt} + \cancelto{0}{\ddot{\theta}}\hat{k}$$

Had we substituted $\hat{k}$ with $\hat{\mathcal{K}}$, the derivative would have been 0

Note that $\frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}$. Thus:

$$\vec{\alpha} = \dot{\theta}\left(\vec{\omega} \times \hat{k}\right) = \dot{\theta}\left[\left(\Omega\hat{J} + \dot{\theta}\hat{k}\right) \times \hat{k}\right]$$

$\vec{\alpha} = \dot{\theta}\left[\Omega\left(\hat{J} \times \hat{k}\right) + \dot{\theta}\left(\hat{k} \times \hat{k}\right)\right]$, but $\hat{k} \times \hat{k} = \vec{0}$…

$$\vec{\alpha} = \dot{\theta}\left[\Omega\left(\hat{J} \times \hat{k}\right)\right]$$

NOW we substitute. At this instant, $\hat{\mathcal{K}} = \hat{k}$

$\vec{\alpha} = \dot{\theta}\Omega\left(\hat{J} \times \hat{\mathcal{K}}\right)$, but $\hat{J} \times \hat{\mathcal{K}} = \hat{I}$ (right-hand rule: $+\hat{I}$)

(a) $\vec{\omega} = \Omega\hat{J} + \dot{\theta}\ \hat{\mathcal{K}}$ (b) $\vec{\alpha} = \dot{\theta}\Omega\hat{I}$

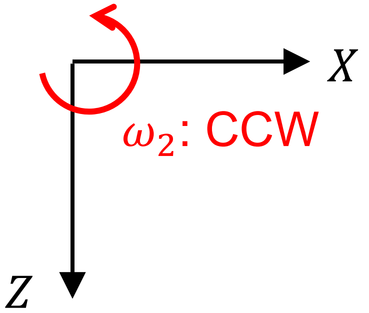
**Example 3.B.3**

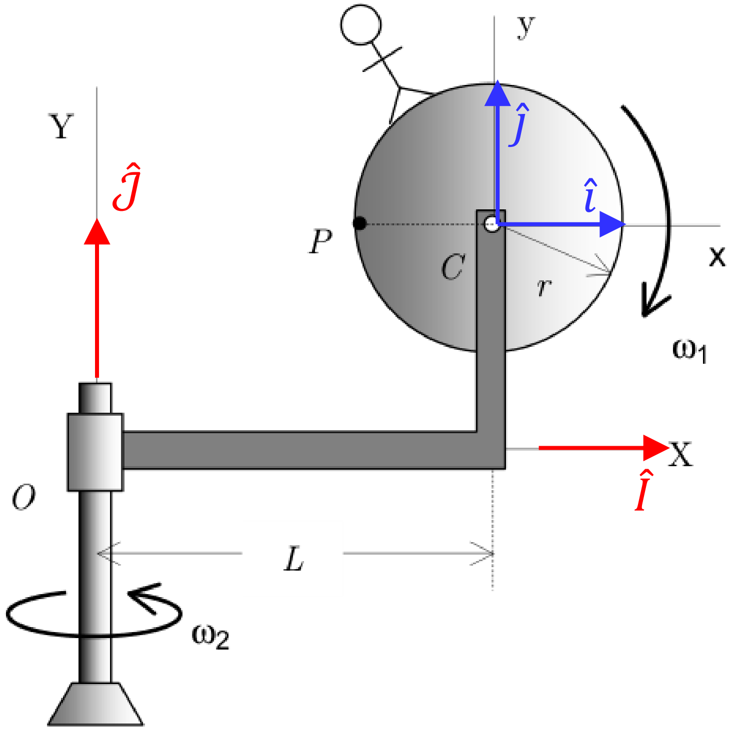
**Given:** A disk rotates with a constant rate of $\omega_1 = 20$ rad/s with respect to the arm OC as the arm OC rotates about a fixed vertical axis with a constant rate of $\omega_2 = 5$ rad/s. The observer and the $xyz$ axes are attached to the disk, while the $XYZ$ axes are fixed. At this instant, the $XYZ$ and $xyz$ axes are aligned.

**Find:** Determine:

(a) The angular velocity of the observer at the instant shown; and

(b) The angular acceleration of the observer at the instant shown.




Solution:

Find all the axes around which the observer rotates:

$$\vec{\omega} = -\omega_1 \hat{k} + \omega_2 \hat{J}$$

Take derivatives:

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = -\omega_1 \frac{d\hat{k}}{dt} - 0 + 0 + 0$$ (derivatives of constant variables and axes: 0)

$$\vec{\alpha} = -\omega_1 (\vec{\omega} \times \hat{k}) = -\omega_1 [(-\omega_1 \hat{k} + \omega_2 \hat{J}) \times \hat{k}]$$

$$\vec{\alpha} = -\omega_1 \omega_2 (\hat{J} \times \hat{\mathcal{K}}) = -\omega_1 \omega_2 \hat{I}$$ ($+\hat{I}$ because is CCW)

$$\vec{\omega} = -\omega_1 \hat{\mathcal{K}} + \omega_2 \hat{J}$$ (don't forget to substitute in $\vec{\omega}$!)