ME 274: Basic Mechanics II

Week 3 *– Monday, January 26*

Particle kinematics: Planar Rigid Body Motion

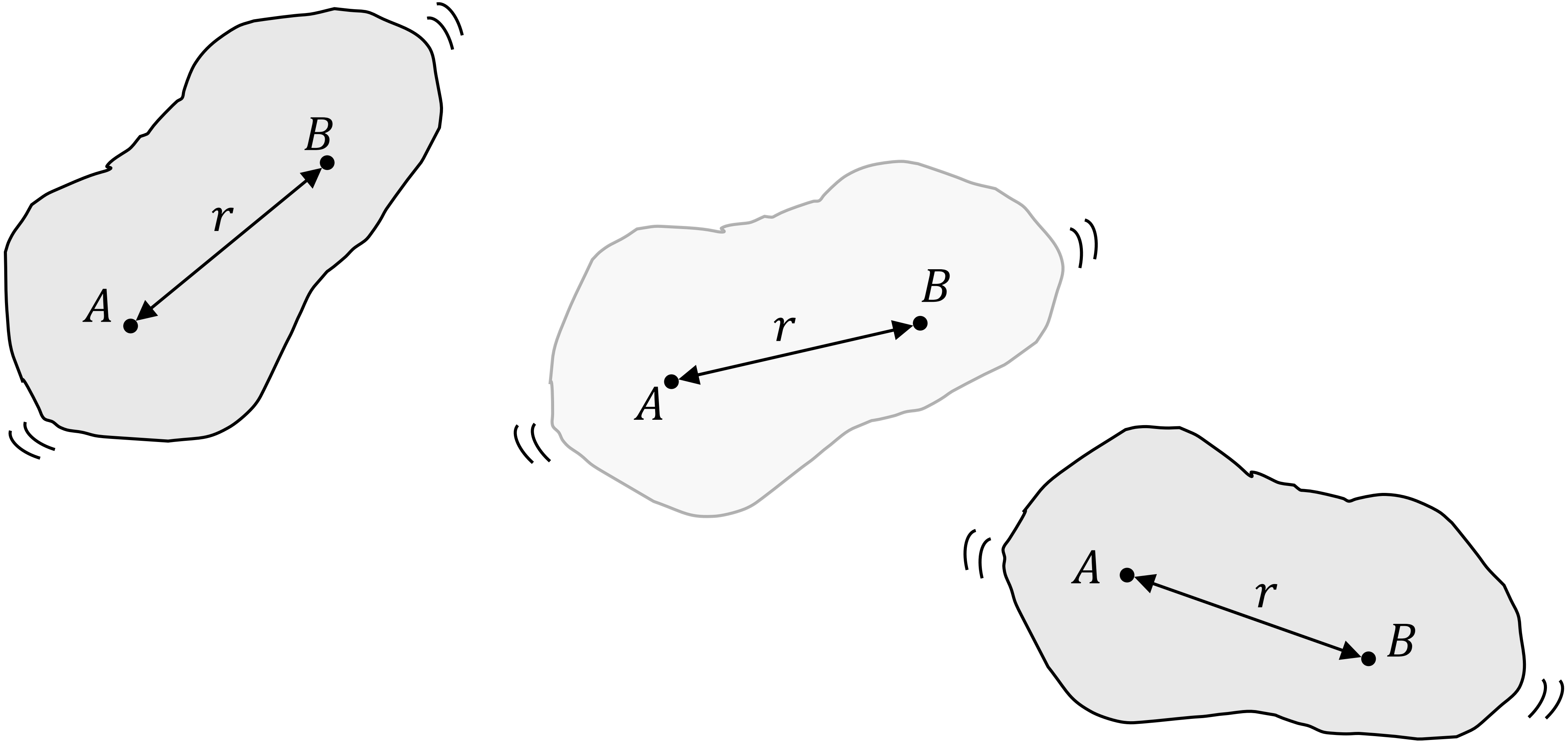
Instructor: Manuel Salmerón

Today’s Agenda

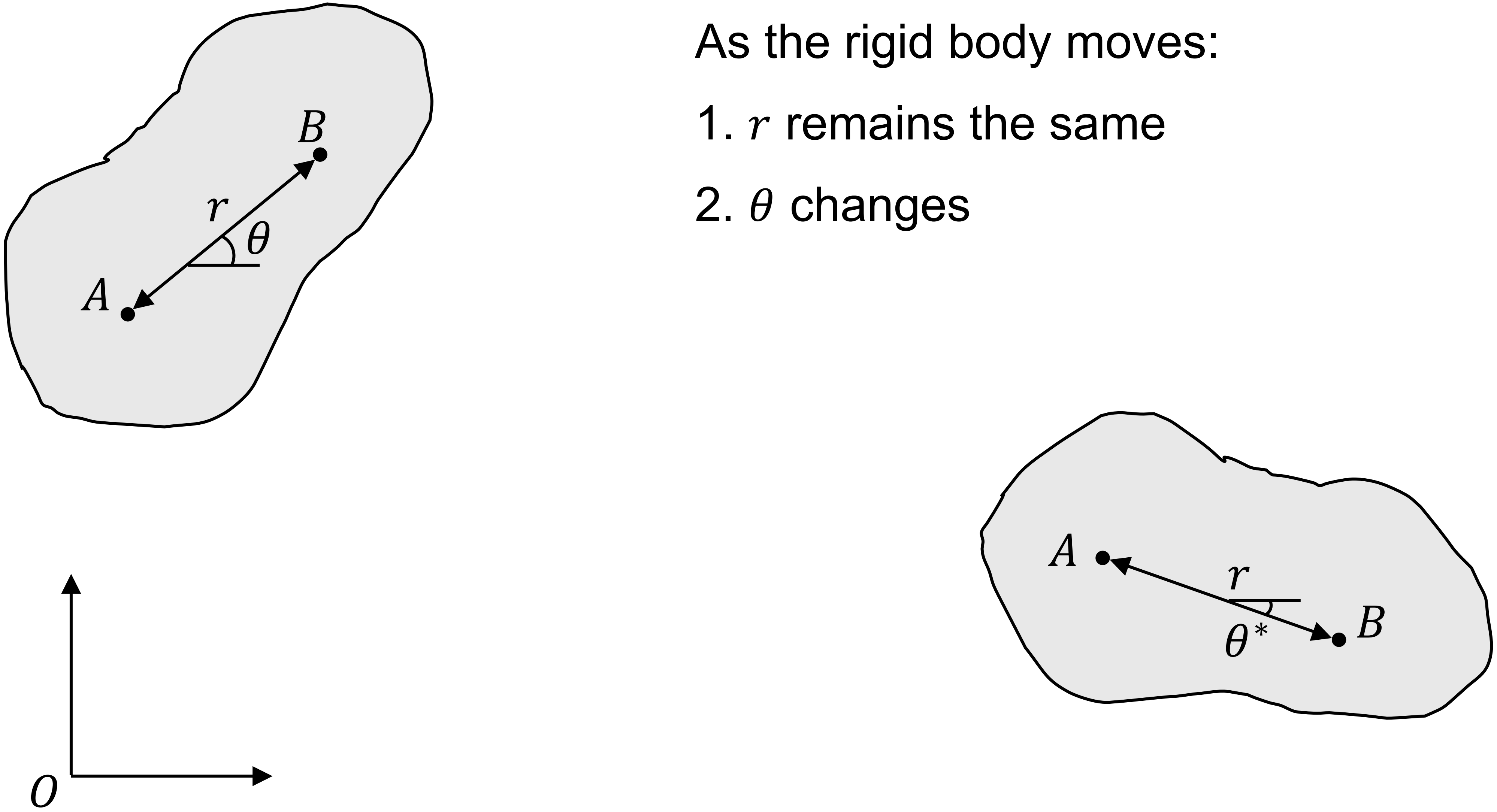
1. Demonstration: rigid vs flexible
2. Kinematic Equations For Rigid Bodies
3. Example 2.A.1
4. Example 2.A.3
5. Summary

1. Demonstration

2. Kinematic Equations for Rigid Bodies



2. Kinematic Equations for Rigid Bodies

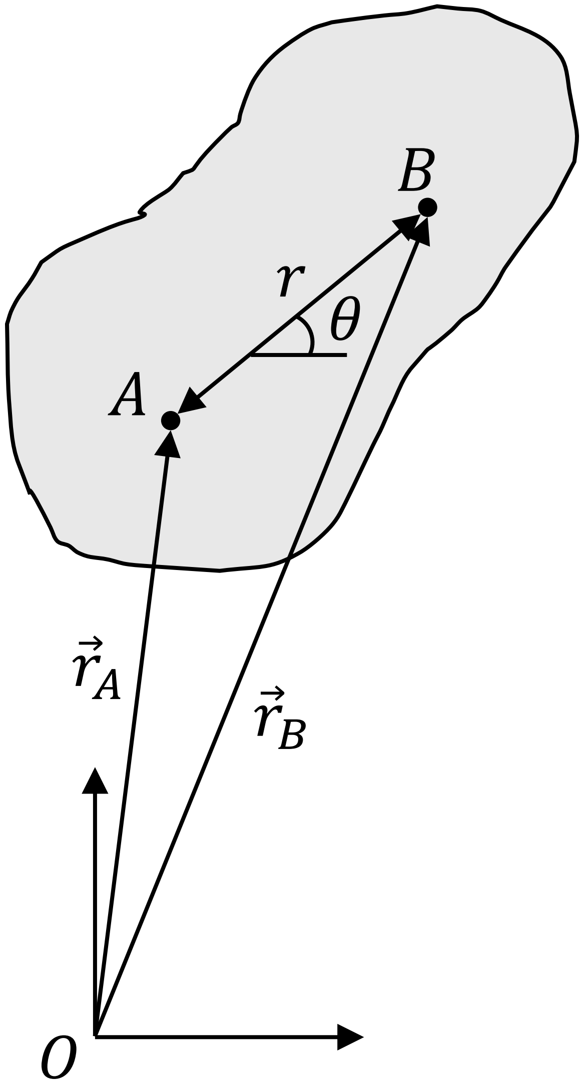


As the rigid body moves:

1. r remains the same

2. θ changes

2. Kinematic Equations for Rigid Bodies



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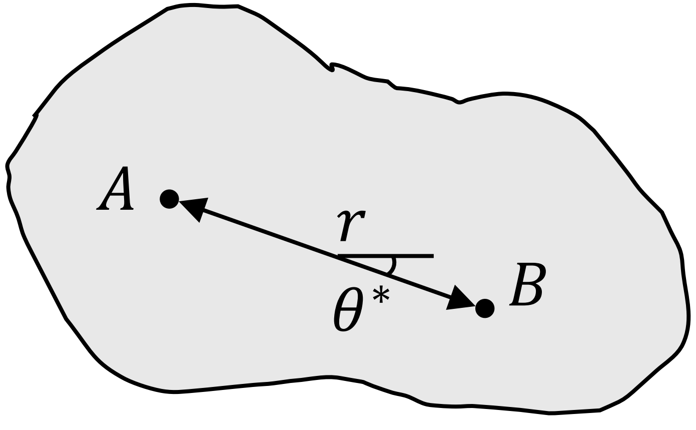

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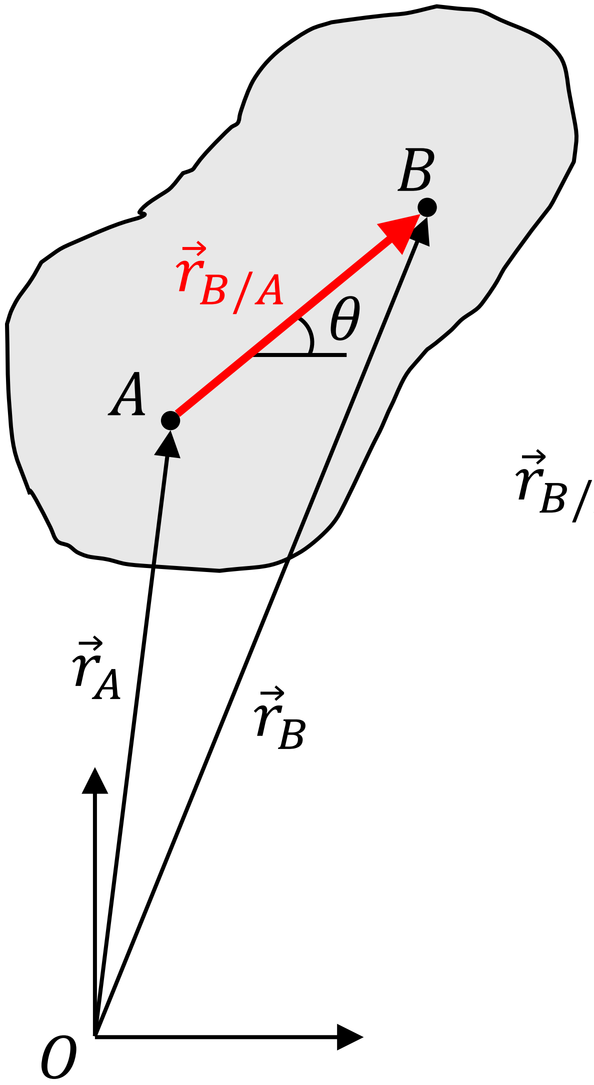

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

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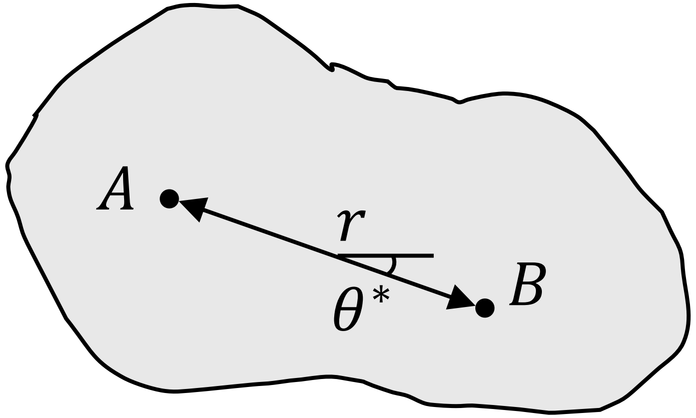

2. Kinematic Equations for Rigid Bodies

As the rigid body moves:

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3. $\vec{r}_{B/A}$ changes

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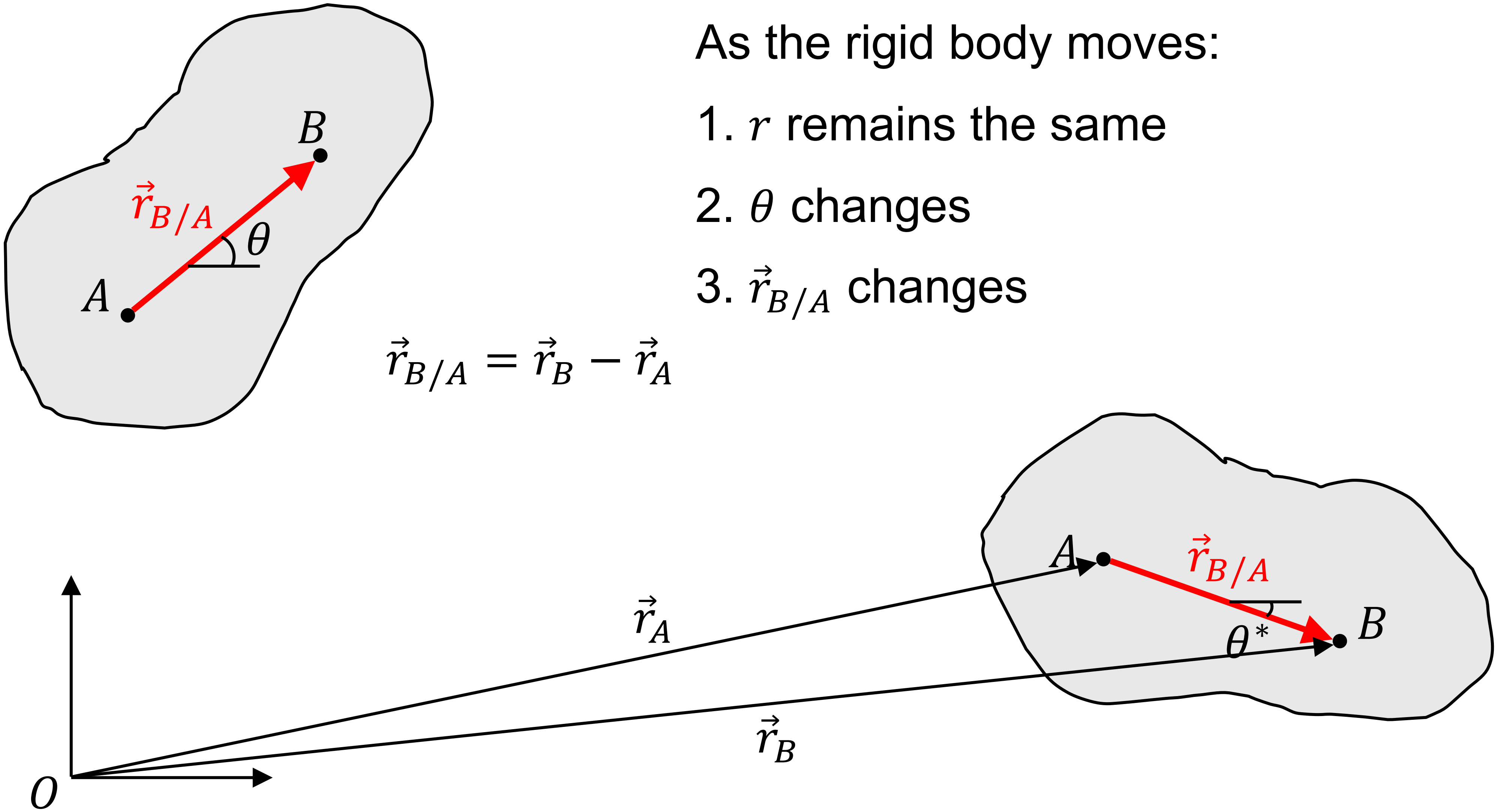

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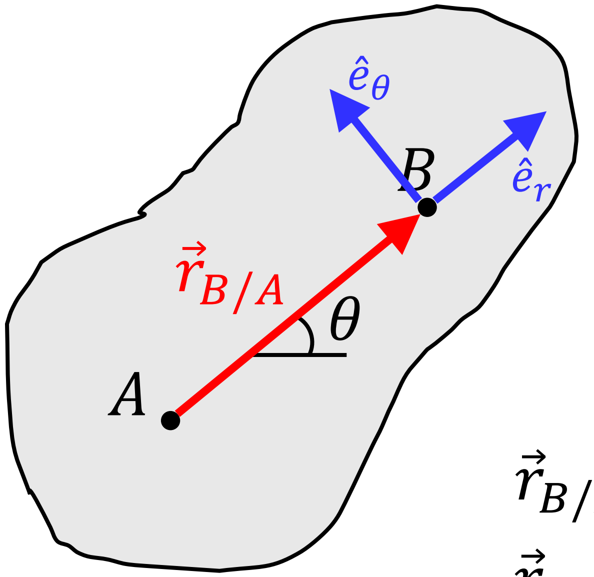

As the rigid body moves:

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3. $\vec{r}_{B/A}$ changes

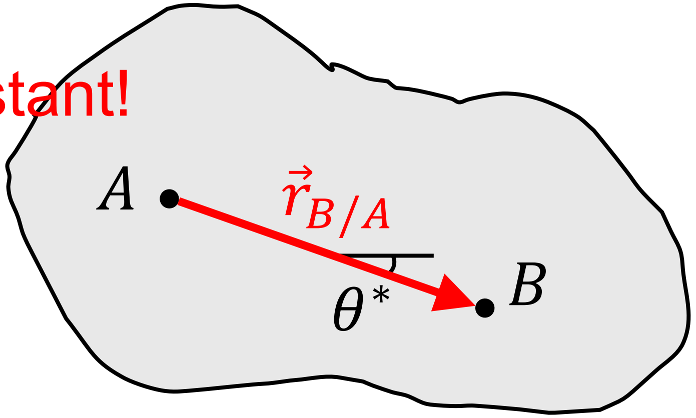
$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

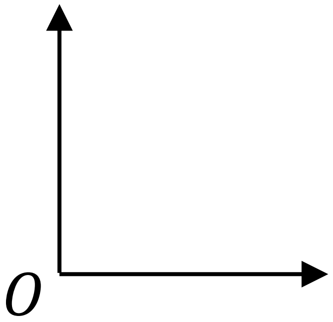
$$\vec{r}_{B/A} = r\hat{e}_r$$

0, r is constant!

$$\vec{v}_{B/A} = \frac{d\vec{r}_{B/A}}{dt} = \cancel{\dot{r}}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$= r\dot{\theta}\hat{e}_\theta$$

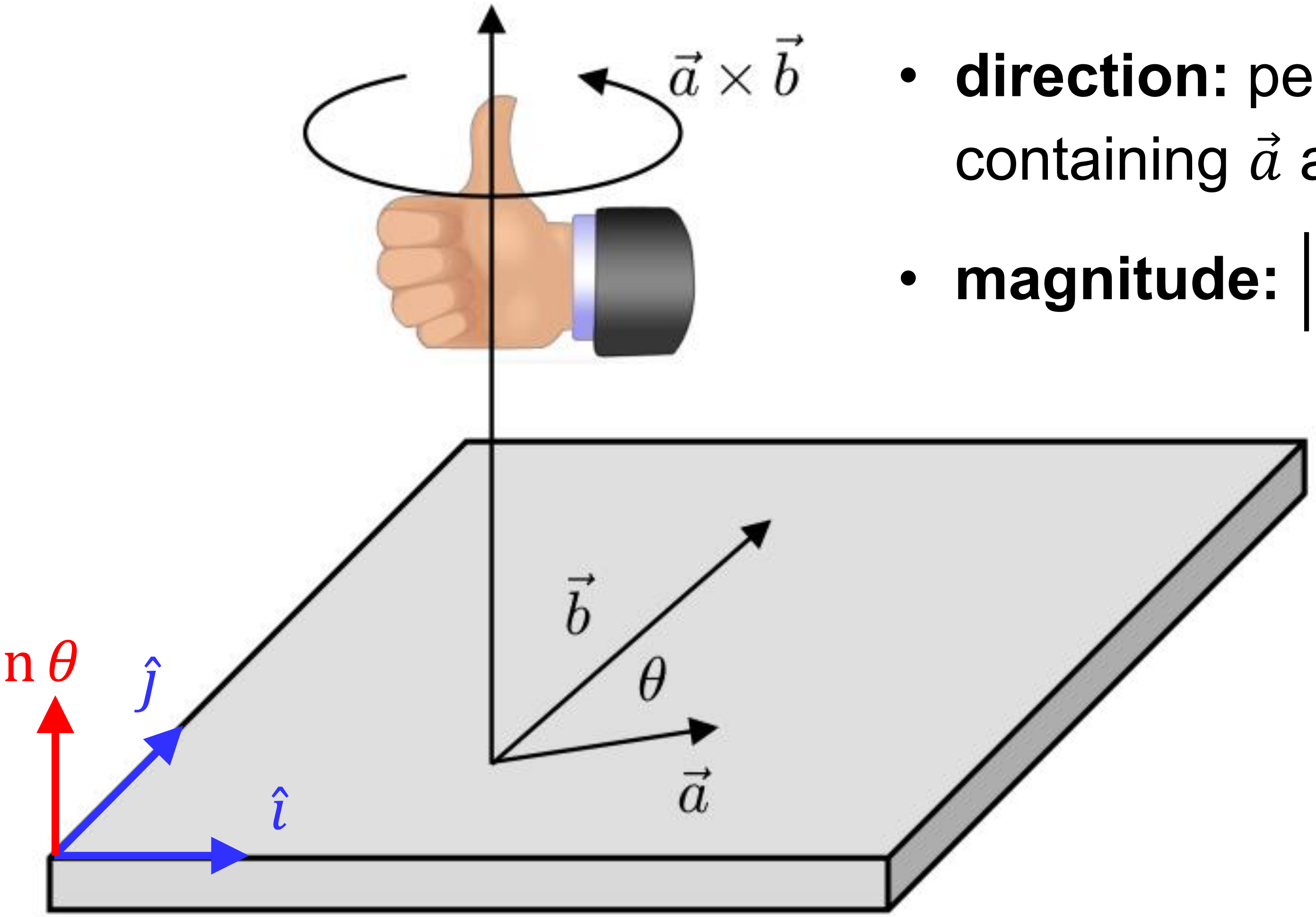




Brief Reminder: Cross Product

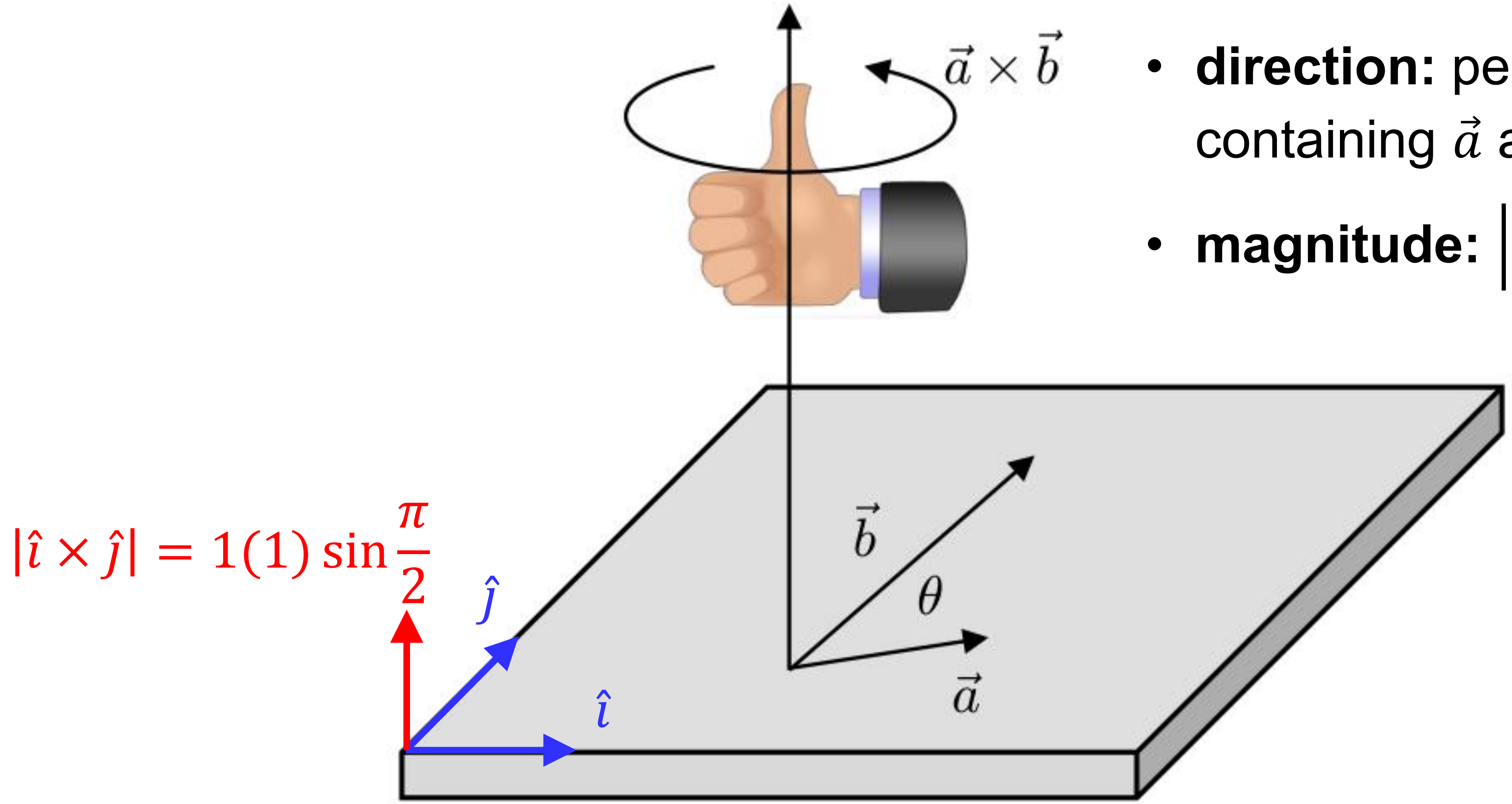
$\vec{a} \times \vec{b}$ is a vector with:

- **direction:** perpendicular to the plane containing $\vec{a}$ and $\vec{b}$
- **magnitude:** $\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \left|\vec{b}\right| \sin\theta$



$|\hat{\imath} \times \hat{\jmath}| = |\hat{\imath}||\hat{\jmath}| \sin\theta$

Brief Reminder: Cross Product



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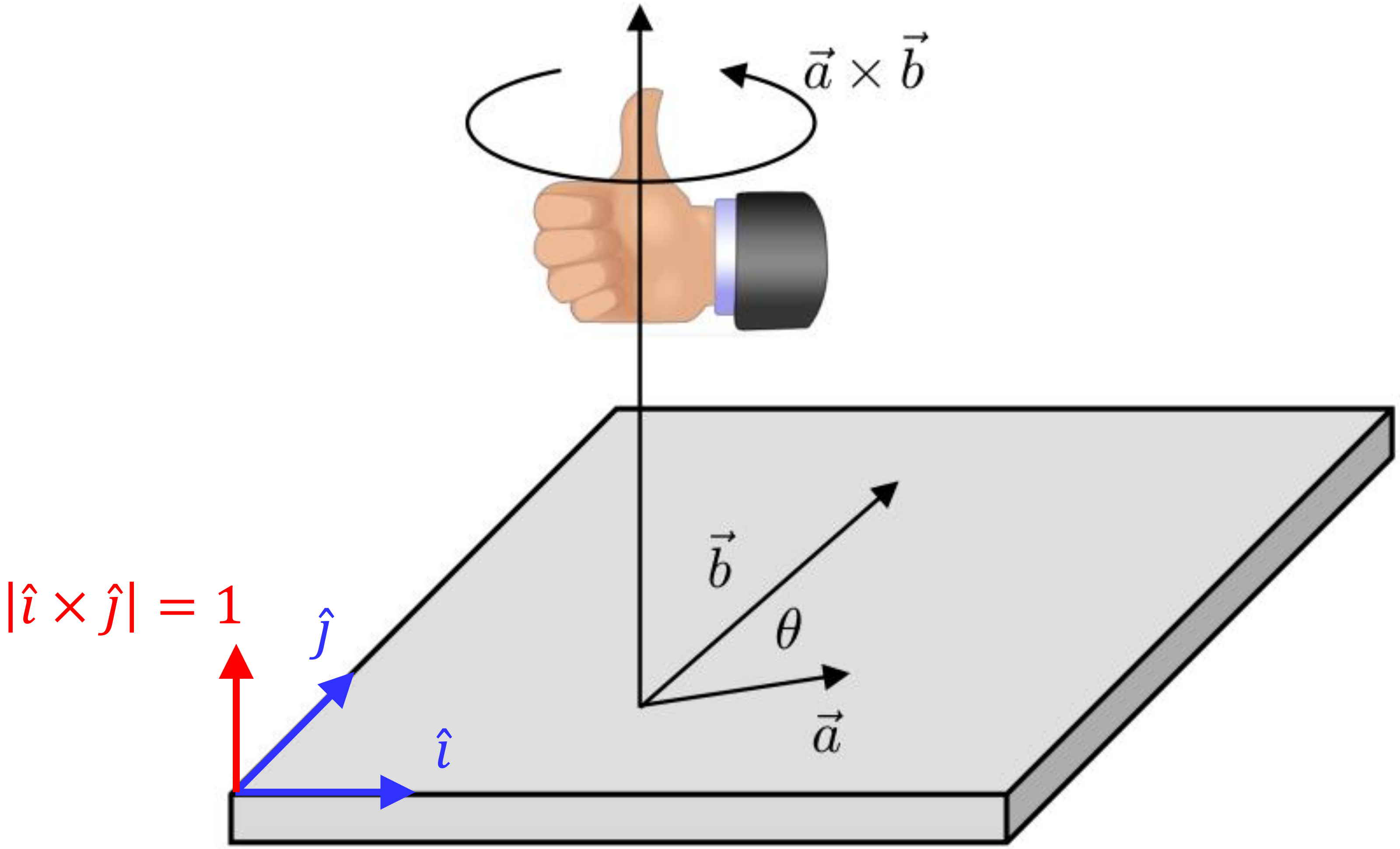
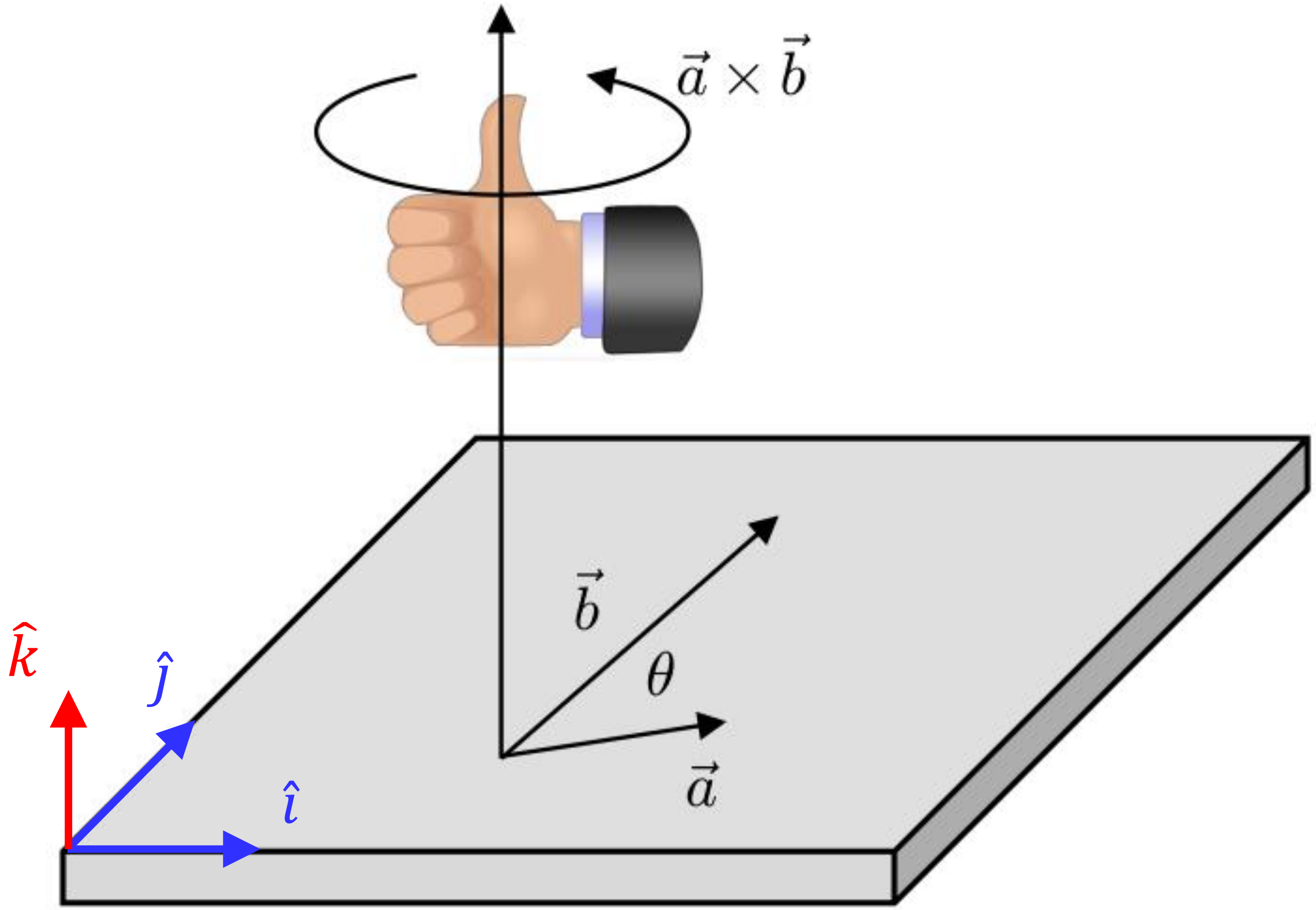
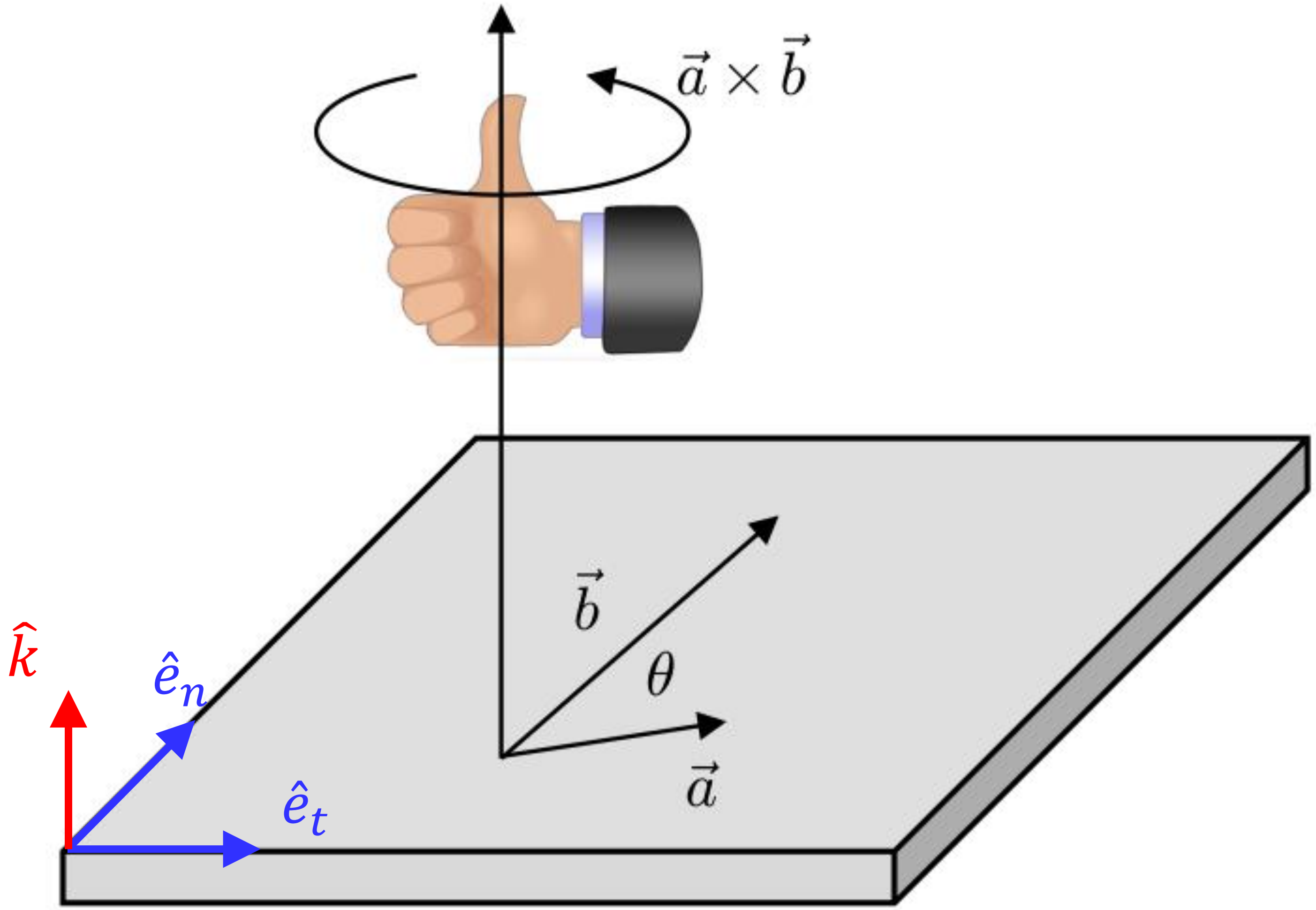
Brief Reminder: Cross Product



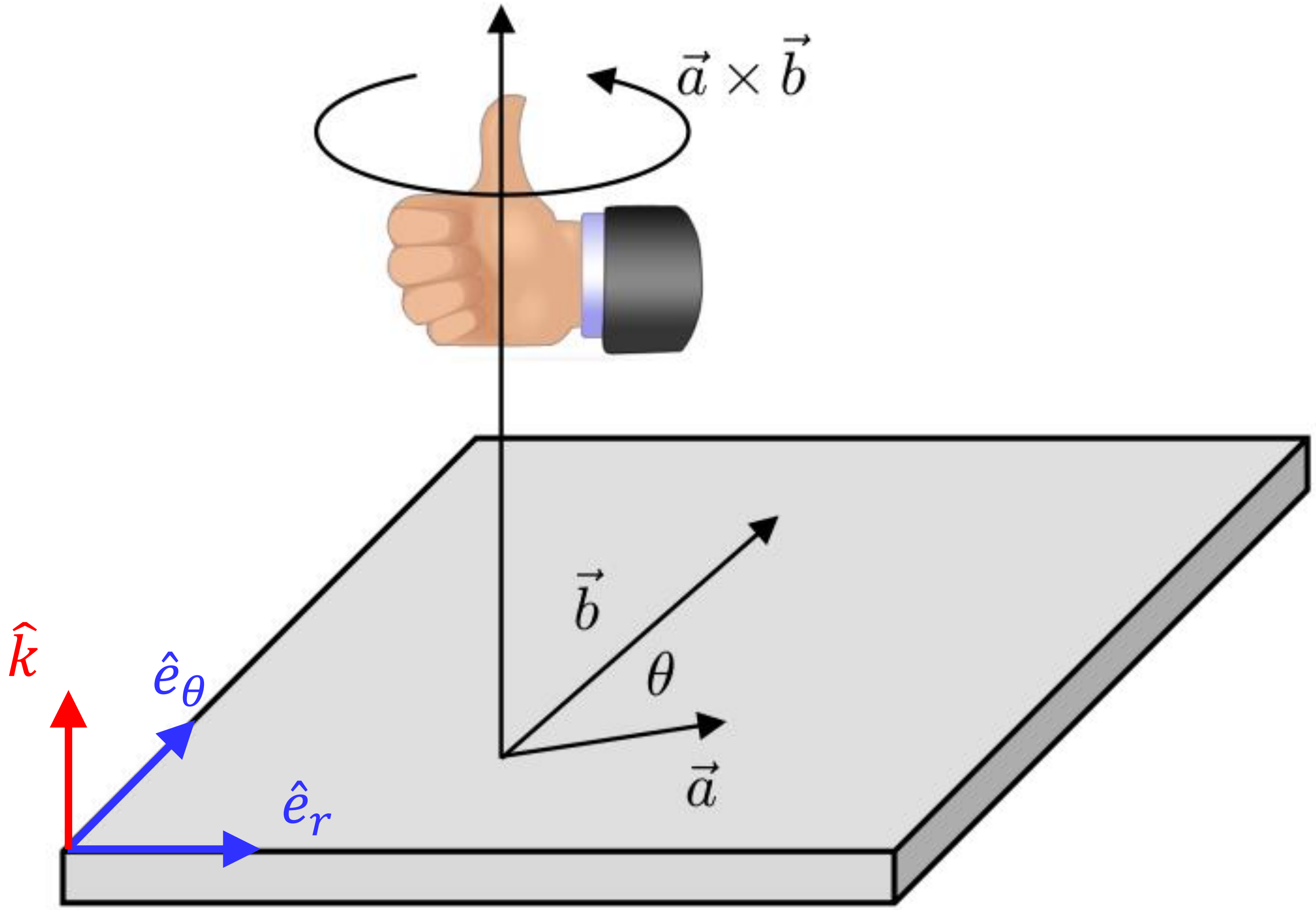
Brief Reminder: Cross Product



Brief Reminder: Cross Product

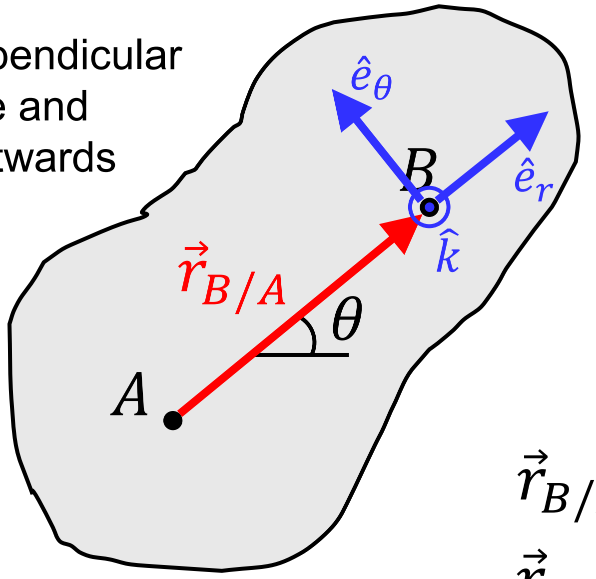


Brief Reminder: Cross Product



2. Kinematic Equations for Rigid Bodies

⊙:vector perpendicular to the plane and pointing outwards



As the rigid body moves:

1. r remains the same
2. θ changes
3. $\vec{r}_{B/A}$ changes

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

$$\vec{r}_{B/A} = r\hat{e}_r$$

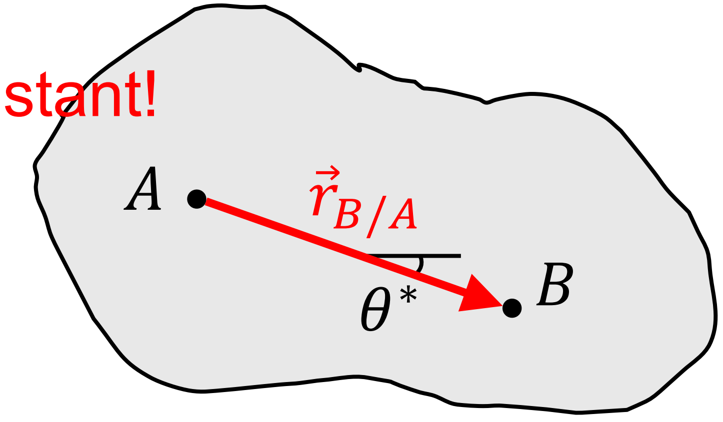
0, r is constant!

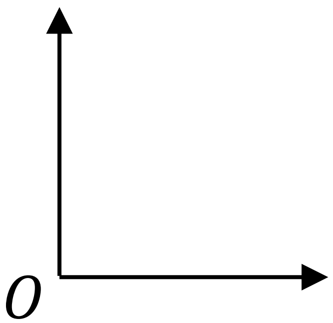
$$\vec{v}_{B/A} = \frac{d\vec{r}_{B/A}}{dt} = \cancel{\dot{r}}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$= r\dot{\theta}\hat{e}_\theta = r\dot{\theta}\left(\hat{k} \times \hat{e}_r\right)$$

$$= \left(\dot{\theta}\hat{k}\right) \times \cancelto{\vec{r}_{B/A}}{(r\hat{e}_r)}$$

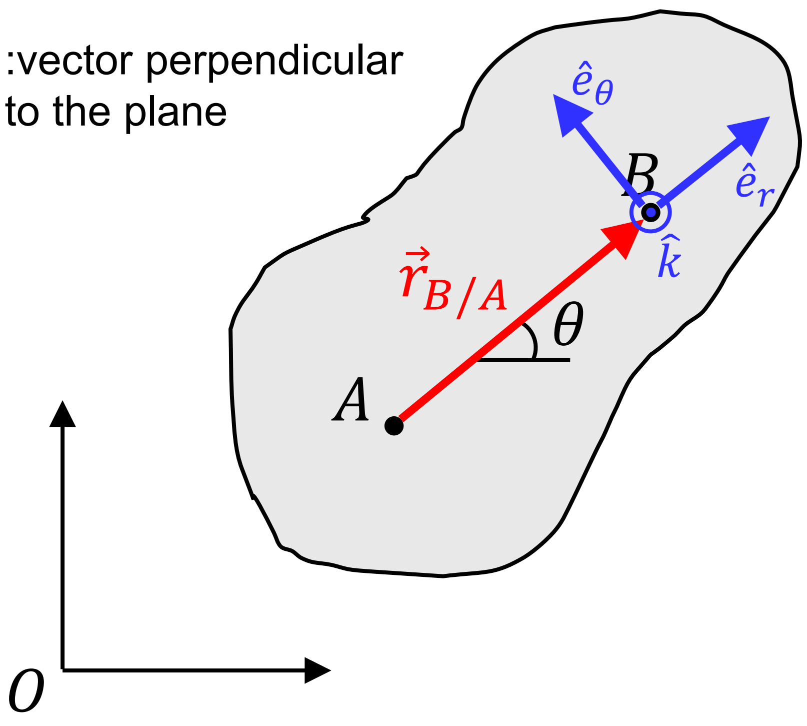
$\vec{\omega}$: angular velocity of the rigid body





2. Kinematic Equations for Rigid Bodies

⊙ :vector perpendicular to the plane



For a point B referenced to point A on the same rigid body:

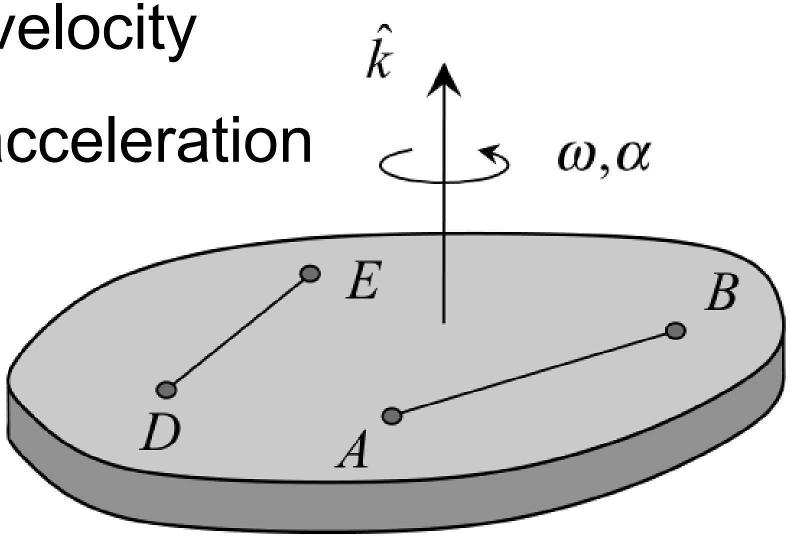
$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A},$$

$$\vec{a}_B = \vec{a}_A + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right) + \vec{\alpha} \times \vec{r}_{B/A},$$

where:

$\vec{\omega} = \omega\hat{k} = \dot{\theta}\hat{k}$: angular velocity

$\vec{\alpha} = \alpha\hat{k} = \ddot{\theta}\hat{k}$: angular acceleration



Note that $\vec{\omega}$ and $\vec{\alpha}$ are the same for any pair of points on the rigid body

Example 2.A.1

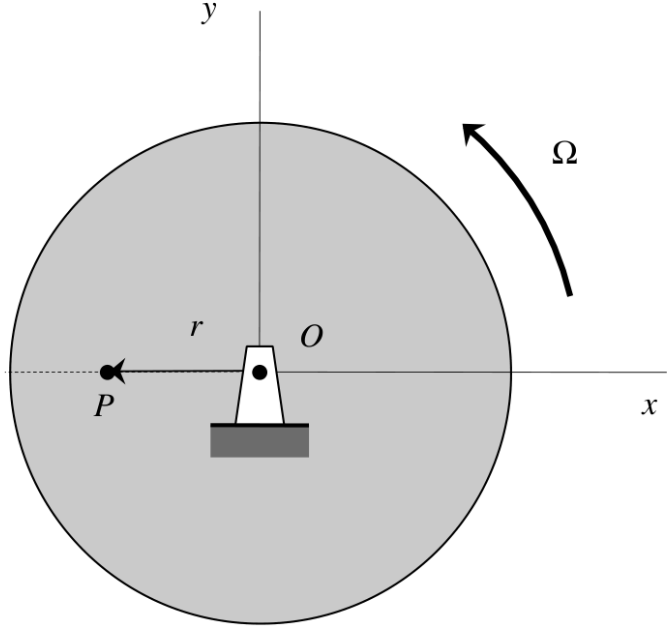
Given: The disk shown is rotating at a non-constant rate of Ω about a fixed axis passing through its center O. At a particular instant, the acceleration vector of point P on the disk is $\vec{a}_P$.

Find: Determine:

(a) The angular velocity of the disk at this instant; and

(b) The angular acceleration of the disk at this instant.

Use the following parameters in your analysis: $\vec{a}_P = 3\hat{\imath} + 4\hat{\jmath}$ m/s^2 and $r = 0.4$ m. Also, be sure to write your answers as vectors.



Example 2.A.1

Given: $\omega = \Omega$, $\vec{a}_P = 3\hat{\imath} + 4\hat{\jmath}$ m/s², $r = 0.4$ m

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

Solution:

Write down the fundamental equations:

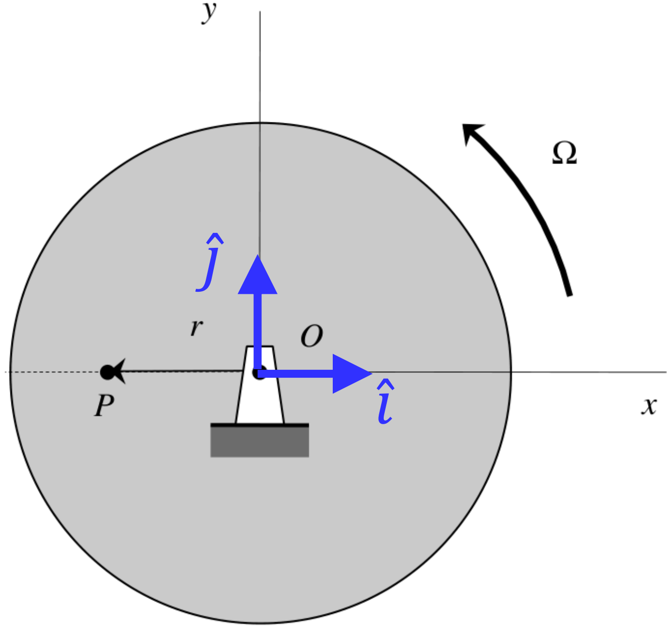
$$\vec{v}_P = \vec{v}_O + \vec{\omega} \times \vec{r}_{P/O}$$

$$\vec{a}_P = \vec{a}_O + \vec{\alpha} \times \vec{r}_{P/O} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right) \quad \textbf{(known!)}$$

Note that O is fixed: $\vec{v}_O = \vec{0}$, $\vec{a} = \vec{0}$, $\vec{0} = 0\hat{\imath} + 0\hat{\jmath}$

Define:

$$\vec{r}_{P/O} = \vec{r}_P - \vec{r}_O = r_x\hat{\imath} + r_y\hat{\jmath}$$



Example 2.A.1

Given: $\omega = \Omega$, $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ m/s², $r = 0.4$ m

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

Solution:

We take what we know:

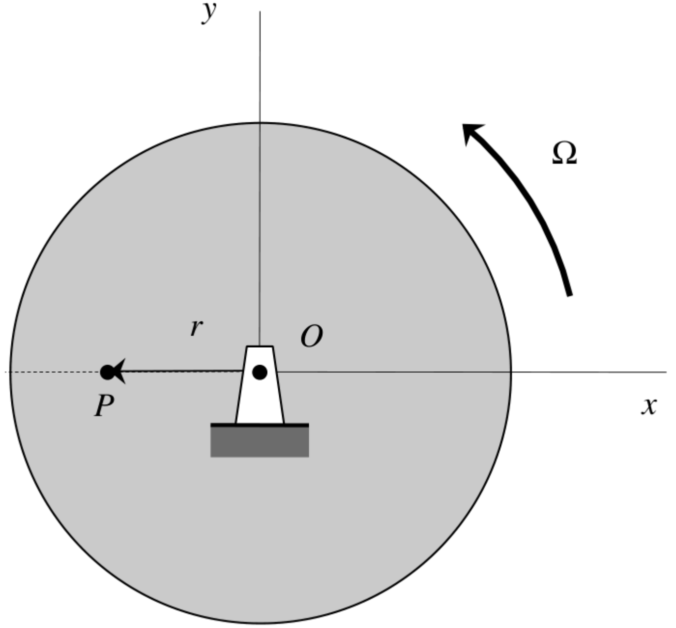
$$\vec{a}_P = \vec{a}_O + \vec{\alpha} \times \vec{r}_{P/O} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right) = \textbf{known}$$

We need the following cross products:

$$\vec{\alpha} \times \vec{r}_{P/O} \qquad \vec{\omega} \times \vec{r}_{P/O} \qquad \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right)$$

Remember that:

$$\vec{\omega} = \omega\hat{k} \quad \text{and} \quad \vec{\alpha} = \alpha\hat{k}$$



Example 2.A.1

$$\vec{r}_{P/O} = \vec{r}_P - \vec{r}_O = r_x\hat{\imath} + r_y\hat{\jmath}$$

$$\vec{\omega} = \omega\hat{k}$$

$$\vec{\alpha} = \alpha\hat{k}$$

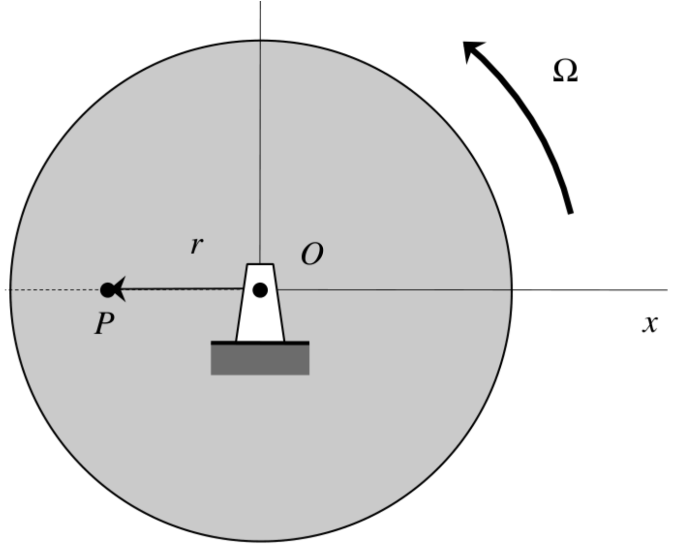
Given: $\omega = \Omega$, $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ m/s^2, $r = 0.4$ m

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

Solution:

$$\vec{\alpha} \times \vec{r}_{P/O} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \alpha \\ r_x & r_y & 0 \end{vmatrix} = \hat{\imath}\left(0 \cdot 0 - \alpha r_y\right) - \hat{\jmath}(0 \cdot 0 - \alpha r_x) + \hat{k}\left(0 \cdot r_y - 0 \cdot r_x\right)$$

$$\vec{\alpha} \times \vec{r}_{P/O} = -\omega r_y\hat{\imath} + \omega r_x\hat{\jmath}$$



Example 2.A.1

$$\vec{r}_{P/O} = \vec{r}_P - \vec{r}_O = r_x\hat{\imath} + r_y\hat{\jmath}$$

$$\vec{\omega} = \omega\hat{k}$$

$$\vec{\alpha} = \alpha\hat{k}$$

Given: $\omega = \Omega$, $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ m/s^2, $r = 0.4$ m

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

Solution:

$$\vec{\omega} \times \vec{r}_{P/O} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \omega \\ r_x & r_y & 0 \end{vmatrix} = \hat{\imath}\left(0 \cdot 0 - \omega r_y\right) - \hat{\jmath}(0 \cdot 0 - \omega r_x) + \hat{k}\left(0 \cdot r_y - 0 \cdot r_x\right)$$

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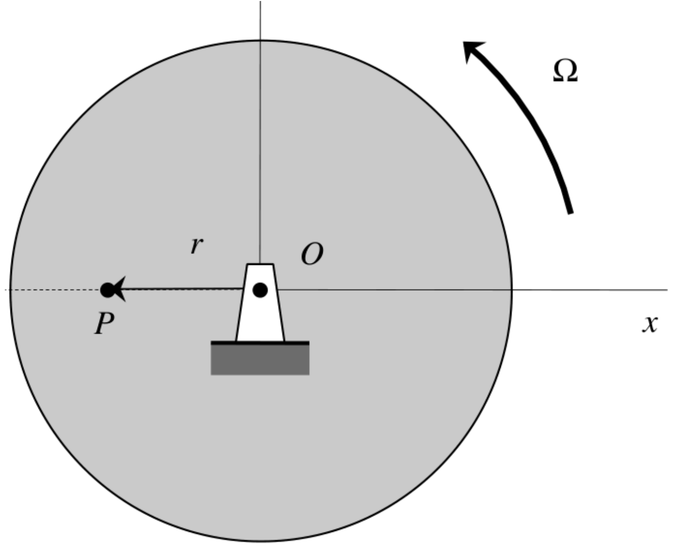


Example 2.A.1

Given: $\omega = \Omega$, $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ m/s^2, $r = 0.4$ m

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

Solution:

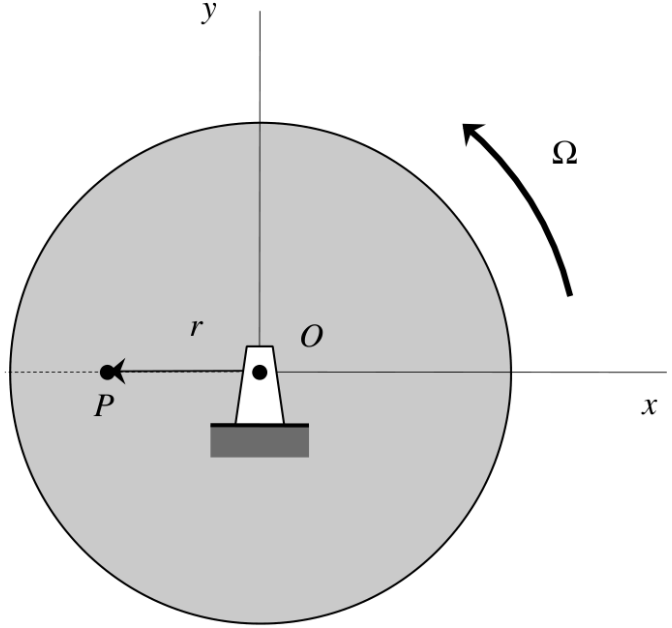
$$\vec{\alpha} \times \vec{r}_{P/O} = -r_y \alpha \hat{\imath} + r_x \alpha \hat{\jmath}$$

$$\vec{\omega} \times \vec{r}_{P/O} = -r_y \omega \hat{\imath} + r_x \omega \hat{\jmath}$$

$$\vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right) = ?$$

Attendance:

https://forms.gle/p2QFNkr54EhLmHr5A



Example 2.A.1

$$\vec{r}_{P/O} = \vec{r}_P - \vec{r}_O = r_x\hat{\imath} + r_y\hat{\jmath}$$

$$\vec{\omega} = \omega\hat{k}$$

$$\vec{\alpha} = \alpha\hat{k}$$

Given: $\omega = \Omega$, $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ m/s^2, $r = 0.4$ m

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

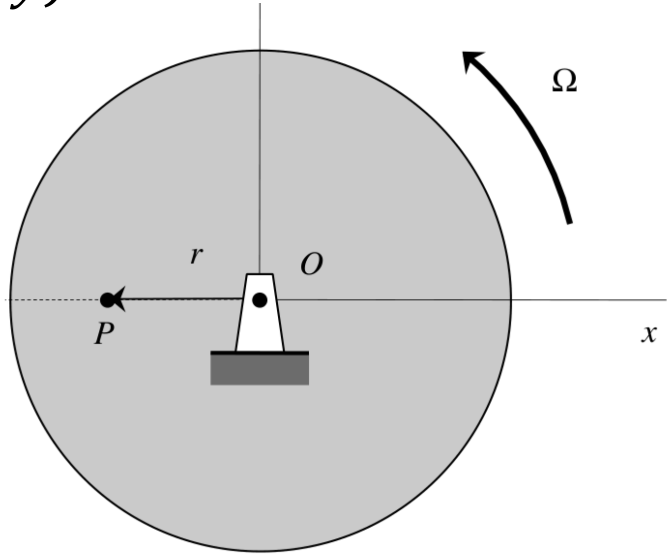
Solution:

$$\vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \omega \\ -\omega r_y & \omega r_x & 0 \end{vmatrix} = \hat{\imath}(0 - \omega^2 r_x) - \hat{\jmath}\left(0 - \omega^2 r_y\right) + \hat{k}(0 - 0)$$

$$\vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right) = -\omega^2 r_x\hat{\imath} - \omega^2 r_y\hat{\jmath}$$

$$\vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right) = -\omega^2\left(r_x\hat{\imath} + r_y\hat{\jmath}\right)$$

$$\vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right) = -\omega^2\vec{r}_{P/O}$$



Example 2.A.1

Given: $\omega = \Omega$, $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ m/s^2, $r = 0.4$ m

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

Solution:

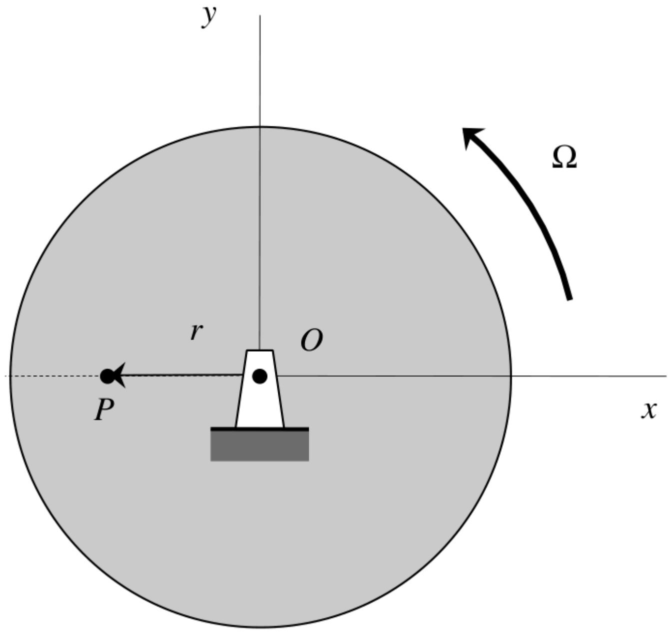
$$\vec{\alpha} \times \vec{r}_{P/O} = -r_y \alpha \hat{\imath} + r_x \alpha \hat{\jmath}$$

$$\vec{\omega} \times \vec{r}_{P/O} = -r_y \omega \hat{\imath} + r_x \omega \hat{\jmath}$$

$$\vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right) = -r_x \omega^2 \hat{\imath} - r_y \omega^2 \hat{\jmath} = -\omega^2 \overbrace{\left(r_x \hat{\imath} + r_y \hat{\jmath}\right)}^{r_{P/O}}$$

General Formula: If points A and B line in a plane perpendicular to $\hat{k}$, then:

$$\vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{A/B}\right) = -\omega^2 \vec{r}_{B/A}$$



Example 2.A.1

Given: $\omega = \Omega$, $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ m/s², $r = 0.4$ m

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

Solution:

$\vec{\alpha} \times \vec{r}_{P/O} = -r_y\alpha\hat{\imath} + r_x\alpha\hat{\jmath}$

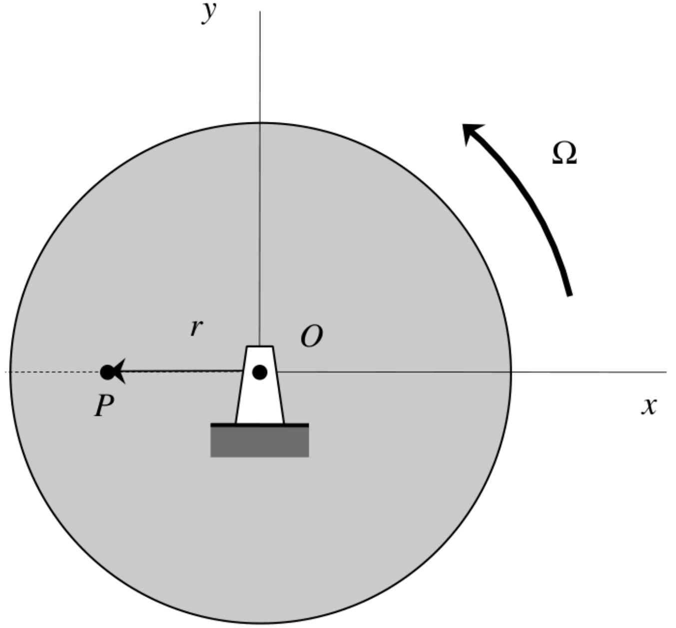
$\vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right) = -\omega^2\left(r_x\hat{\imath} + r_y\hat{\jmath}\right)$

Substitute cross-products into $\vec{a}_P$:

$\vec{a}_P = \vec{\alpha} \times \vec{r}_{P/O} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right)$

$\vec{a}_P = \alpha\left(-r_y\hat{\imath} + r_x\hat{\jmath}\right) - \omega^2\left(r_x\hat{\imath} + r_y\hat{\jmath}\right)$

$\vec{a}_P = \left(-\alpha r_y - \omega^2 r_x\right)\hat{\imath} + \left(\alpha r_x - \omega^2 r_y\right)\hat{\jmath} = 3\hat{\imath} + 4\hat{\jmath}$



Example 2.A.1

Given: $\omega = \Omega$, $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ m/s², $r = 0.4$ m

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

Solution:

$$\vec{a}_P = \left(-\alpha r_y - \omega^2 r_x\right)\hat{\imath} + \left(\alpha r_x - \omega^2 r_y\right)\hat{\jmath} = 3\hat{\imath} + 4\hat{\jmath}$$

Component $\hat{\imath}$: $-\alpha r_y - \omega^2 r_x = 3$

Component $\hat{\jmath}$: $\alpha r_x - \omega^2 r_y = 4$

From the figure: $r_x = -r$ and $r_y = 0$

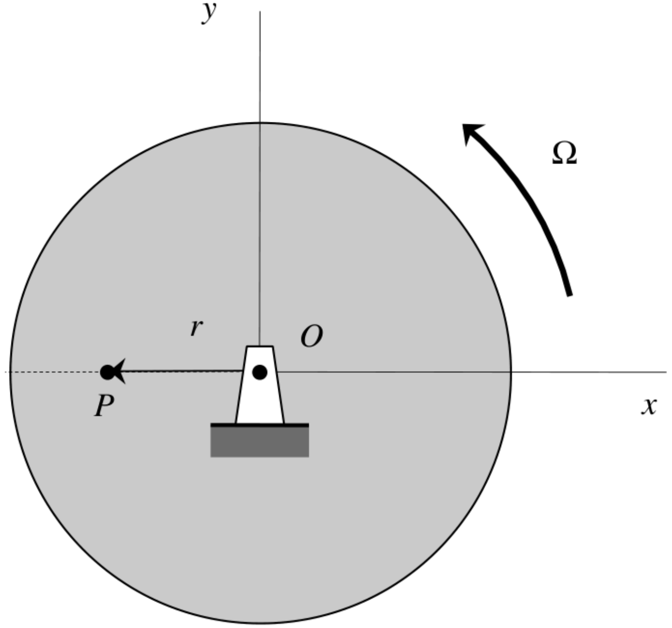
Thus: $\alpha = -4/r$ (from $\hat{\jmath}$)

$\omega = \pm\sqrt{3/r}$ (from $\hat{\imath}$)

We are asked for **vectors**:

$$\vec{\omega} = \pm\left(\sqrt{3}/r\right)\hat{k}$$

$$\vec{\alpha} = -(4/r)\hat{k}$$



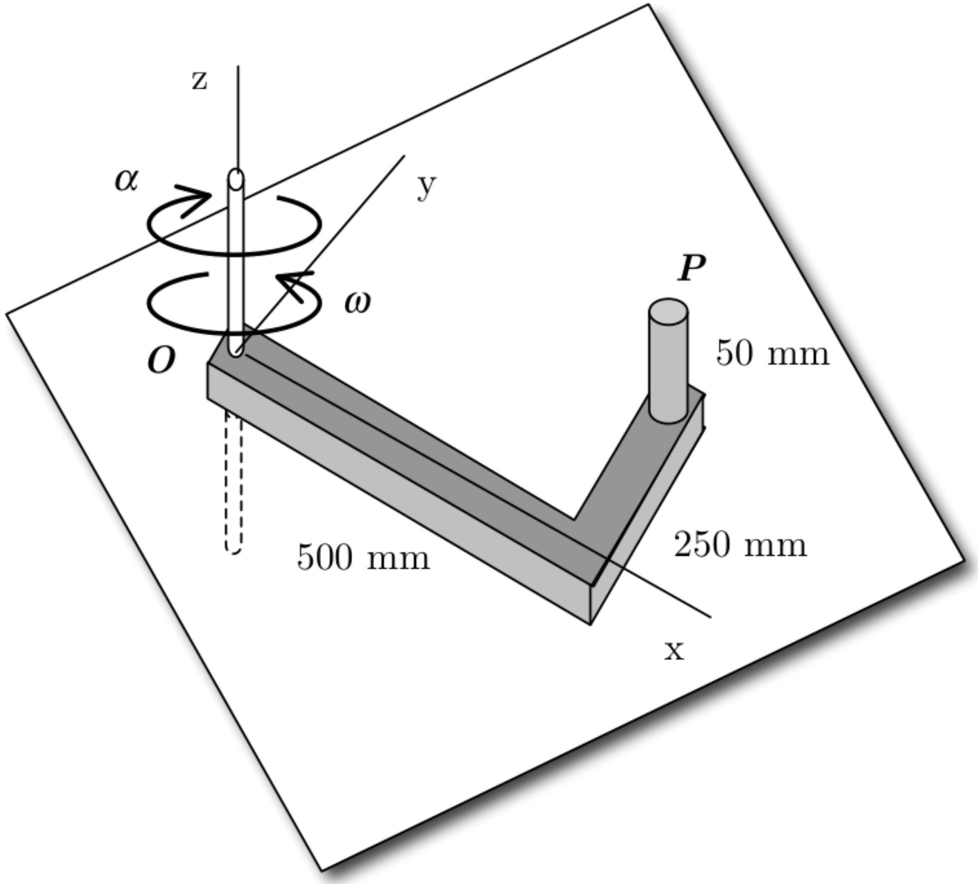
Example 2.A.3

Given: The system shown below rotates about a vertical shaft at point O, such that $\omega = 2$ rad/s and $\alpha = 3$ rad/s^2.

Find: Determine:

(a) The velocity of point P; and

(b) The acceleration of point P.

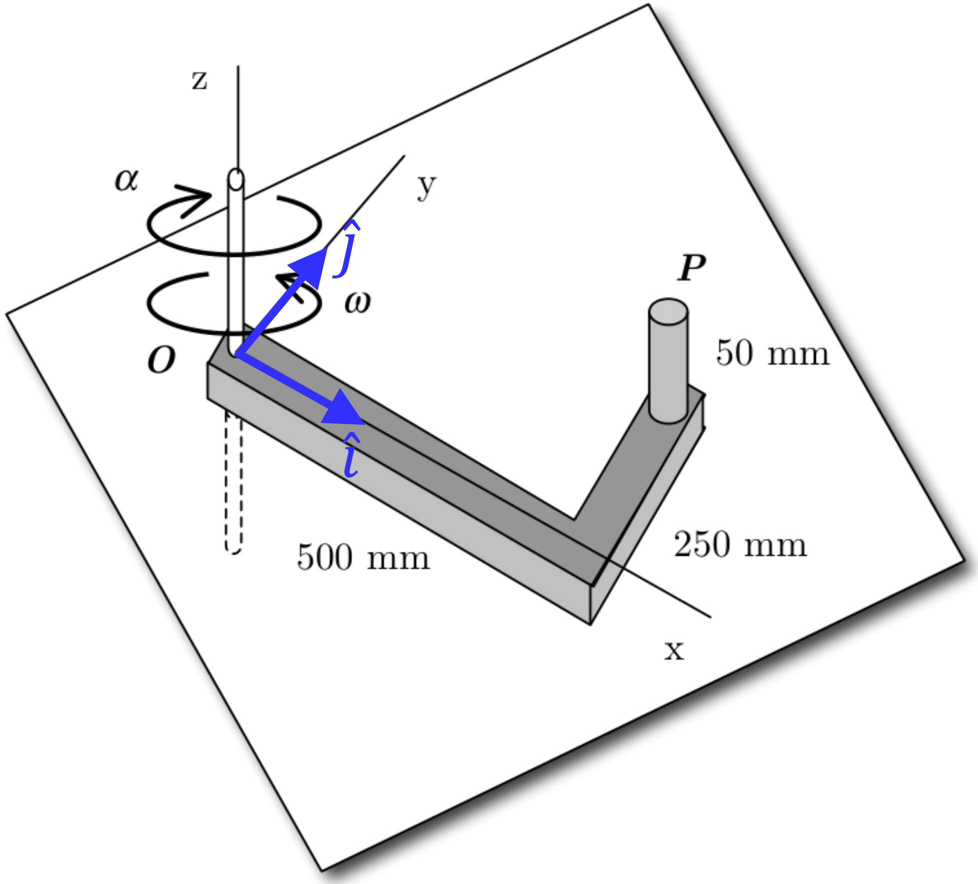


Example 2.A.3

Given: rotation about O, $\omega = 2$rad/s, $\alpha = 3$rad/s^2

Find: (a) $\vec{v}_P$, (b) $\vec{a}_P$

Solution:



Write down the fundamental equations:

$$\vec{v}_P = \vec{v}_O + \vec{\omega} \times \vec{r}_{P/O}$$

$$\vec{a}_P = \vec{a}_O + \vec{\alpha} \times \vec{r}_{P/O} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right)$$

Fixed point O: $\vec{v}_O = \vec{a}_O = \vec{0} = 0\hat{\imath} + 0\hat{\jmath}$

The position of P is:

$$\vec{r}_{P/O} = \vec{r}_P - \vec{r}_O = 0.5\hat{\imath} + 0.25\hat{\jmath} + 0.05\hat{k}$$

Example 2.A.3

Given: rotation about O, $\omega = 2$rad/s, $\alpha = 3$rad/s^2

Find: (a) $\vec{v}_P$, (b) $\vec{a}_P$

Solution:

Remember that:

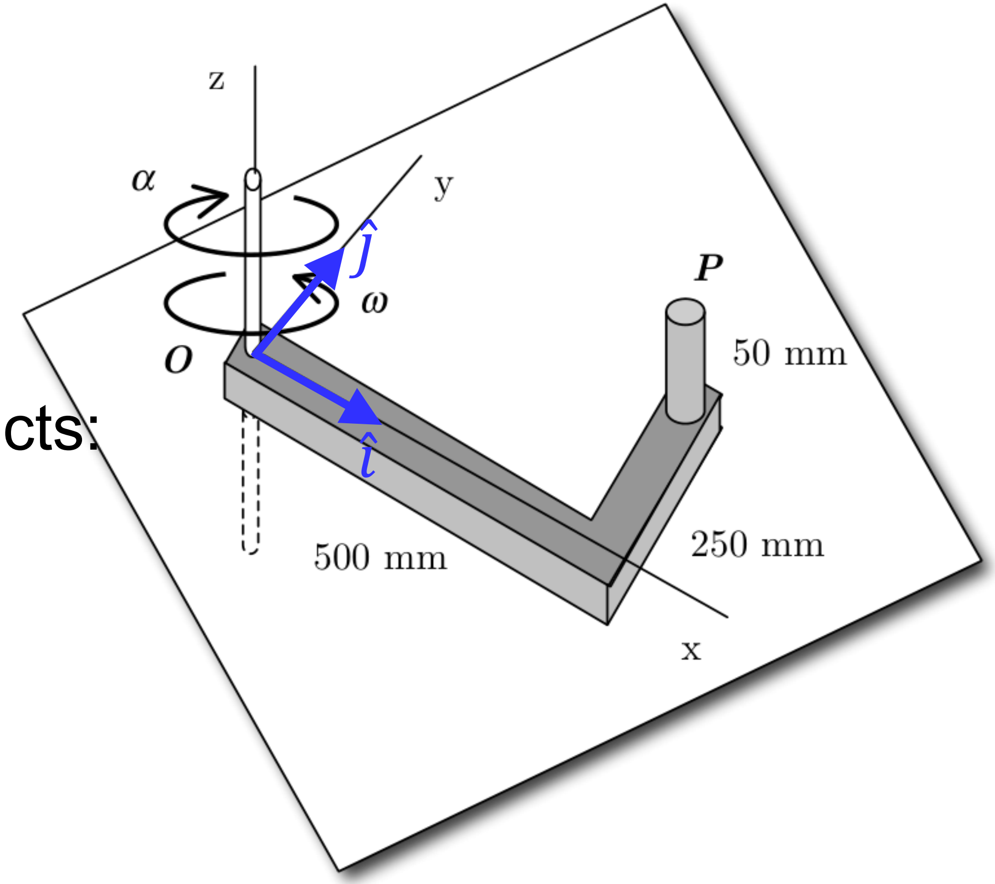
$\vec{\omega} = \omega\hat{k}$ and $\vec{\alpha} = \alpha\hat{k}$ **(known)**

Thus, we only need to compute the cross products:

$\vec{\omega} \times \vec{r}_{P/O}$,

$\vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right)$, and

$\vec{\alpha} \times \vec{r}_{P/O}$

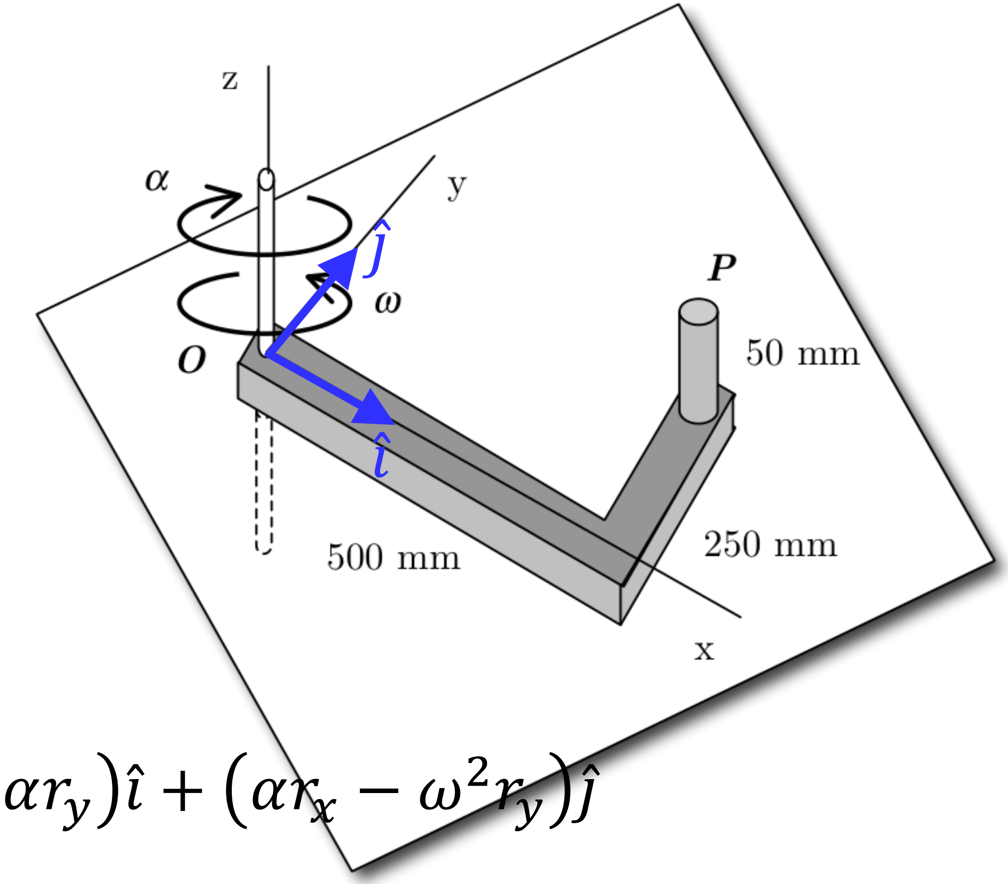


Example 2.A.3

Given: rotation about O, $\omega = 2$rad/s, $\alpha = 3$rad/s^2

Find: (a) $\vec{v}_P$, (b) $\vec{a}_P$

Solution:



$$\vec{\omega} \times \vec{r}_{P/O} = \omega r_y \hat{\imath} + \omega r_x \hat{\jmath}$$

$$\vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right) = -\omega^2\left(r_x \hat{\imath} + r_y \hat{\jmath}\right)$$

$$\vec{\alpha} \times \vec{r}_{P/O} = -\alpha r_y \hat{\imath} + \alpha r_x \hat{\jmath}$$

Thus:

$$\vec{v}_P = \vec{\omega} \times \vec{r}_{P/O} = \omega r_y \hat{\imath} + \omega r_x \hat{\jmath}$$

$$\vec{a}_P = \vec{\alpha} \times \vec{r}_{P/O} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{P/O}\right) = \left(-\omega^2 r_x - \alpha r_y\right)\hat{\imath} + \left(\alpha r_x - \omega^2 r_y\right)\hat{\jmath}$$

ME 274: Basic Mechanics II

Week 3 *– Wednesday, January 28*

Particle kinematics: Planar Rigid Body Motion

Instructor: Manuel Salmerón

Today’s Agenda

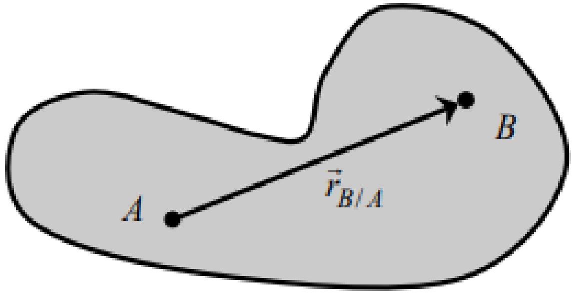
1. Recap: Kinematic Equations for Rigid Bodies
2. Example 2.A.7
3. Rolling without slipping
4. Example 2.A.5
5. Summary

Summary: Rigid Body Kinematics 1

PROBLEM: Two points A and B on the same rigid body undergoing planar motion.

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A} \right)$$



COMMENTS:

- $\vec{\omega}$ and $\vec{\alpha}$ are the *__angular velocity__* and *__angular acceleration__* vectors of the body. These are the same for ANY two points A and B.
- $\vec{r}_{B/A}$ points *__FROM__* point A *__TO__* point B.
- If A and B lie in the same plane, then: $\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}$
- From where did these equations come? From the general motion of two points (Chapter 1) with the constraint that $\left| \vec{r}_{B/A} \right|$.

Example 2.A.7

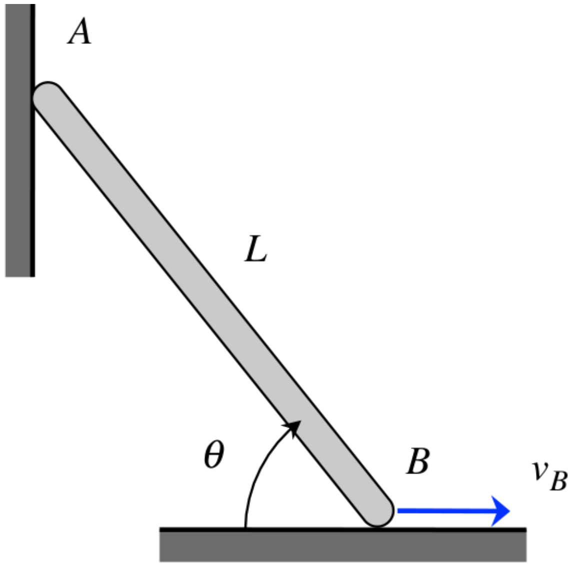
Given: End B of the link moves to the right with a constant speed v_B.

Find: Determine:

(a) The angular velocity of link AB; and

(b) The angular acceleration of link AB.

Use the following parameters in your analysis: $v_B = 3$ m/s, $L = 0.5$ m and $\theta = 36.87°$. Also, be sure to express your answers as vectors.



Example 2.A.7

Given: $v_B = 3$ m/s (constant), $L = 0.5$ m, $\theta = 36.87°$

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

Solution:

Fundamental equations:

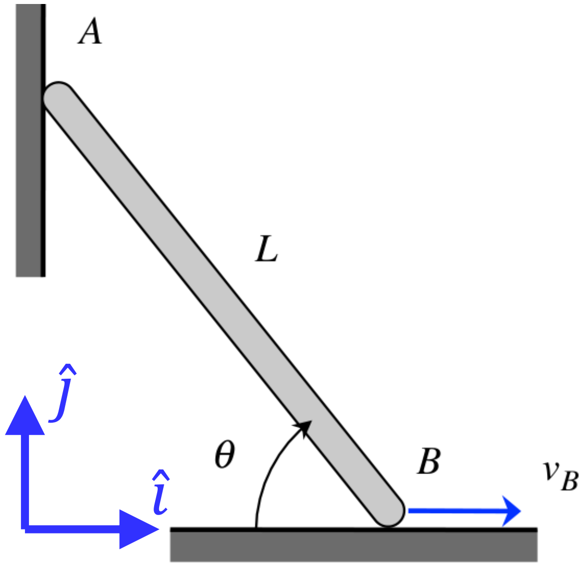
$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}$$

$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}_{A/B} - \omega^2 \vec{r}_{A/B}$$

Position vector:

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = -L\cos\theta\,\hat{\imath} + L\sin\theta\,\hat{\jmath}\;L\sin\theta\,\hat{\imath}$$

We know v_B, let's start with the velocity!



Example 2.A.7

Given: $v_B = 3$ m/s (constant), $L = 0.5$ m, $\theta = 36.87°$

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

Solution:

$\vec{\omega} \times \vec{r}_{A/B} =$… see the board!

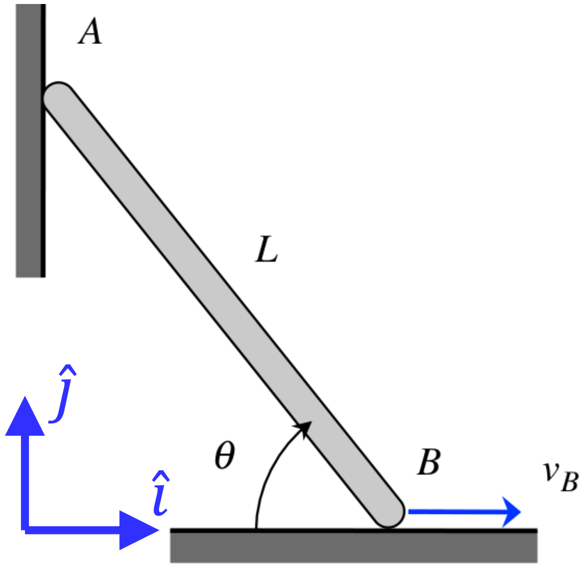
$$\vec{\omega} \times \vec{r}_{A/B} = -\omega L \sin\theta\,\hat{\imath} - \omega L \cos\theta\,\hat{\jmath}$$

Go back to the velocity:

$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B} = v_B\hat{\imath} - \omega L \sin\theta\,\hat{\imath} - \omega L \cos\theta\,\hat{\jmath}$$

We don't know anything about A, but note the constraint:

$$\vec{v}_A = \cancelto{0}{v_{Ax}}\hat{\imath} + v_{Ay}\hat{\jmath} = v_B\hat{\imath} - \omega L \sin\theta\,\hat{\imath} - \omega L \cos\theta\,\hat{\jmath}$$



Example 2.A.7

Given: $v_B = 3$ m/s (constant), $L = 0.5$ m, $\theta = 36.87°$

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

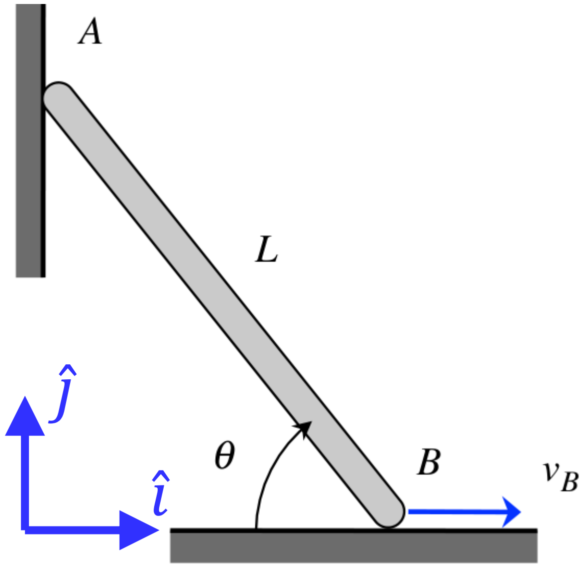
Solution:

$$\vec{v}_A = 0\hat{\imath} + v_{Ay}\hat{\jmath} = v_B\hat{\imath} - \omega L \sin\theta\,\hat{\imath} - \omega L \cos\theta\,\hat{\jmath}$$

Equality of vectors:

In $\hat{\imath}$: $v_B - \omega L \sin\theta = 0 \Rightarrow \omega = \dfrac{v_B}{L\sin\theta}$

We want a vector: $\vec{\omega} = \dfrac{v_B}{L\sin\theta}\hat{k}$

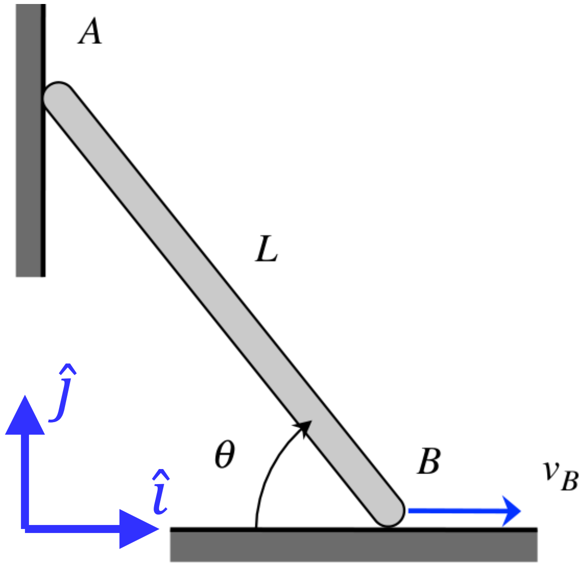


Example 2.A.7

Given: $v_B = 3$ m/s (constant), $L = 0.5$ m, $\theta = 36.87°$

Find: (a) $\vec{\omega}$, (b) $\vec{\alpha}$

Solution:



Now, for (b): $\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}_{A/B} - \omega^2 \vec{r}_{A/B}$

We need $\vec{\alpha} \times \vec{r}_{A/B}$ (see development in board)

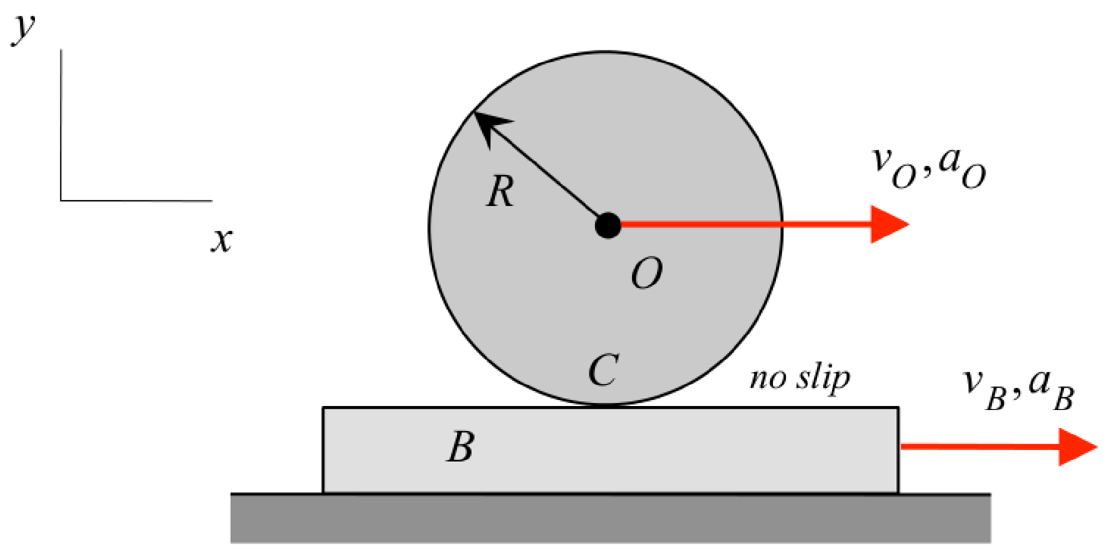
$$\vec{\alpha} \times \vec{r}_{A/B} = L\alpha(-\sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}) - \omega^2 L(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

Thus:

$$\vec{a}_A = \cancelto{0}{a_x}\hat{\imath} + a_y\hat{\jmath} = (-L\alpha\sin\theta + \omega^2 L\cos\theta)\hat{\imath} + (-L\alpha\cos\theta - \omega^2 L\sin\theta)\hat{\jmath}$$

$a_x = 0 = -L\alpha\sin\theta + \omega^2 L\cos\theta$, thus: $\vec{\alpha} = \omega^2 \dfrac{\cos\theta}{\sin\theta}\hat{k}$

Rolling without slipping



In equations:

1. $v_{Cx} = v_B$
2. $a_{Cx} = a_B$
3. $v_{Cy} = 0$
4. $a_{Cy} \neq 0$

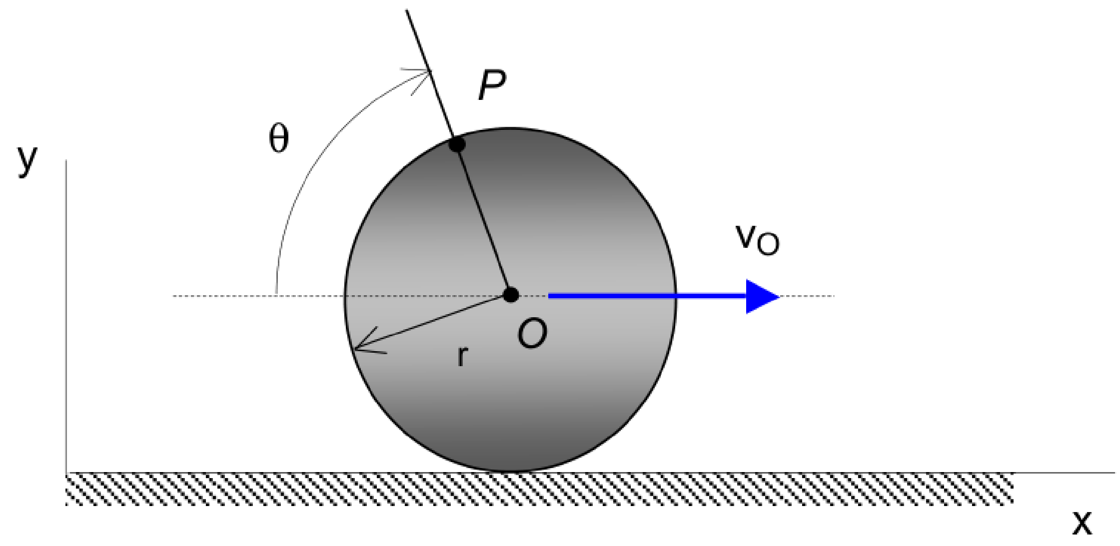
Assume the rigid rolling body and the rigid surface B move in X. Thus:

1. the speed of point C is equal to the speed of B in X
2. the rate of change of speed of point C is equal to the rate of change of speed of B in X
3. the speed of point C is 0 in Y
4. the rate of change of speed of point C is not 0 in Y

Example 2.A.5

Given: A wheel rolls without slipping on a rough horizontal surface. At one instant, when $\theta = 90°$, the center of the wheel O is moving to the right with a speed of $v_O = 5$ ft/s with this speed decreasing at a rate of 3 ft/s^2.

Find: Determine the acceleration of point P on the circumference of the wheel at this instant, if $r = 2$ ft. Make a sketch of this acceleration vector at P.



Example 2.A.5

Given: no slipping, $\theta = 90°$, $v_O = 5$ ft/s, $a_O = -3$ ft/s^2

Find: $\vec{a}_P$, if $r = 2$ ft

Solution:

$$\vec{v}_P = \vec{v}_O + \vec{\omega} \times \vec{r}_{P/O}$$

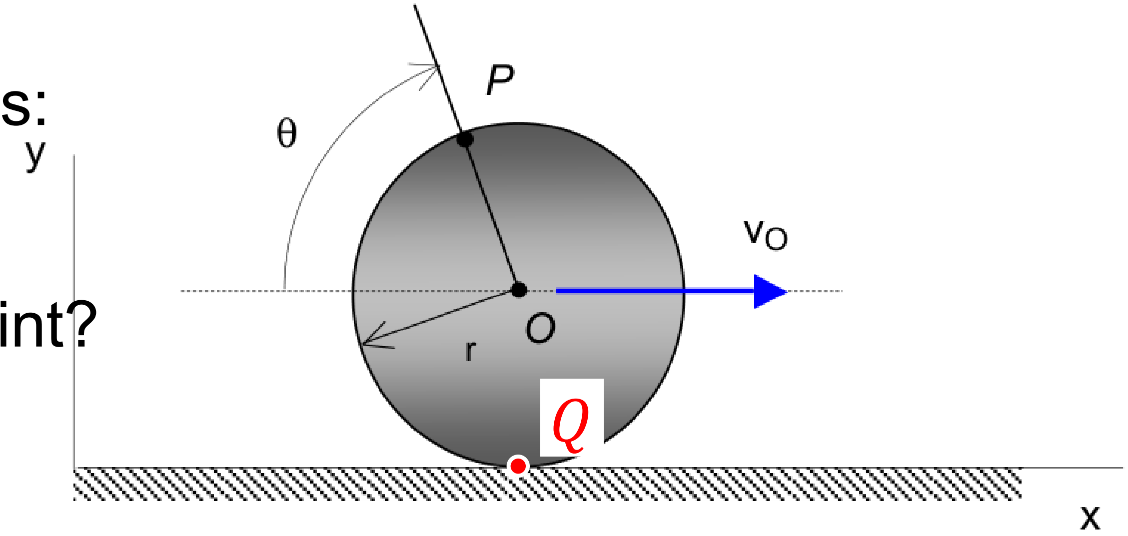
$$\vec{a}_P = \vec{a}_O + \vec{\alpha} \times \vec{r}_{P/O} - \omega^2 \vec{r}_{P/O}$$

The origin only moves forward at $\vec{v}_O = 5\hat{\imath}$ ft/s:

$$\vec{v}_P = 5\hat{\imath} + \omega\hat{k} \times (r\hat{\jmath}) = (5 - \omega r)\hat{\imath} = ???$$

Do we have information about any other point?

Yes: the point of contact, Q



Example 2.A.5

Given: no slipping, $\theta = 90°$, $v_O = 5$ ft/s, $a_O = -3$ ft/s^2

Find: $\vec{a}_P$, if $r = 2$ ft

Solution:

$\vec{v}_Q = \vec{v}_O + \vec{\omega} \times \vec{r}_{Q/O}$

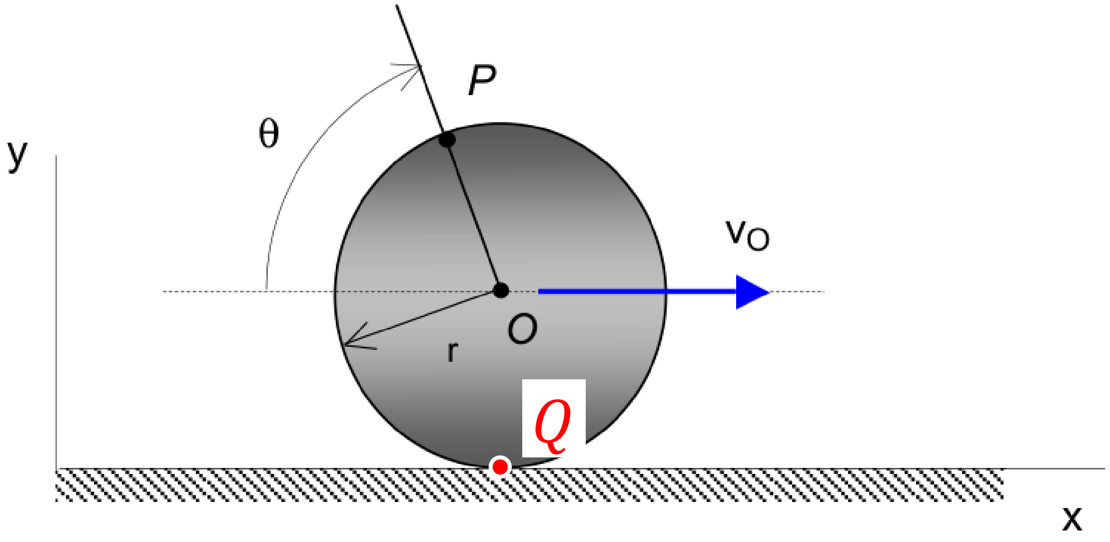
$\vec{v}_Q = v_{Qx}\hat{\imath} + v_{Qy}\hat{\jmath} = v_O\hat{\imath} + r\omega\hat{\imath} = (v_O + r\omega)\hat{\imath}$

No slipping: $v_{Qx} = 0$ and $v_{Qy} = 0$

Vector equality: $v_O + r\omega = 0$

Thus: $\omega = -\frac{v_O}{r}$

Problem asks for $\vec{a}_P$, so we will also need α!



Example 2.A.5

Given: no slipping, $\theta = 90°$, $v_O = 5$ ft/s, $a_O = -3$ ft/s^2

Find: $\vec{a}_P$, if $r = 2$ ft

Solution:

$$\vec{a}_Q = a_{Qx}\hat{\imath} + a_{Qy}\hat{\jmath} = \vec{a}_O + \vec{\alpha} \times \vec{r}_{Q/O} - \omega^2 \vec{r}_{Q/O}$$

Take advantage on two points with known information

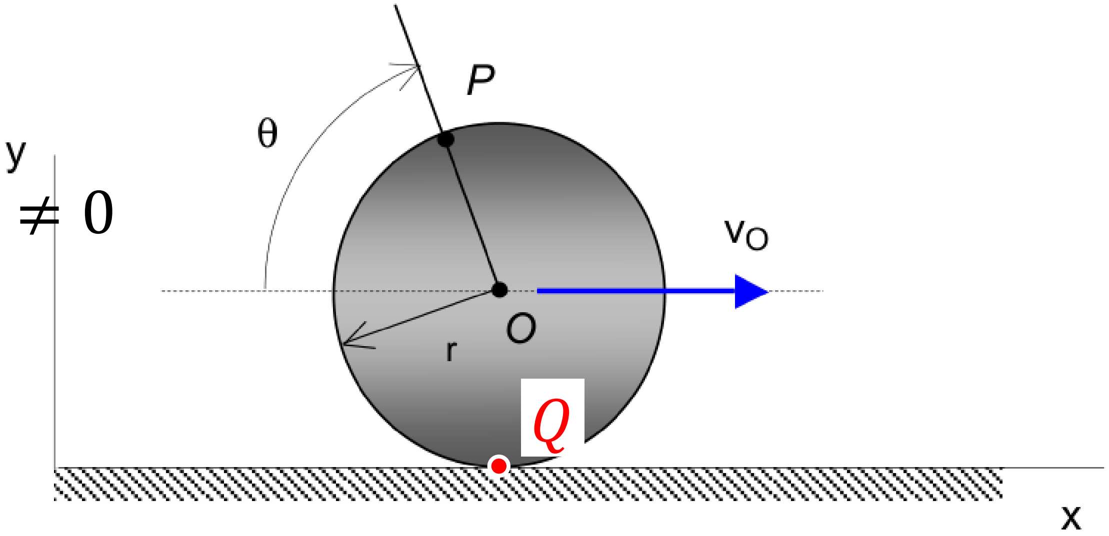
(…derivation in board…)

$$\vec{a}_Q = a_{Qx}\hat{\imath} + a_{Qy}\hat{\jmath} = a_O\hat{\imath} + \alpha r\hat{\imath} + \omega^2 r\hat{\jmath}$$

No slipping condition for Q: $a_{Qx} = 0$ and $a_{Qy} \neq 0$

In the X-component: $0 = a_O + \alpha r$

Thus: $\alpha = -a_O/r$



Example 2.A.5

Given: no slipping, $\theta = 90°$, $v_O = 5$ ft/s, $a_O = -3$ ft/s^2

Find: $\vec{a}_P$, if $r = 2$ ft

Solution:

We can finally go back to $\vec{a}_P$:

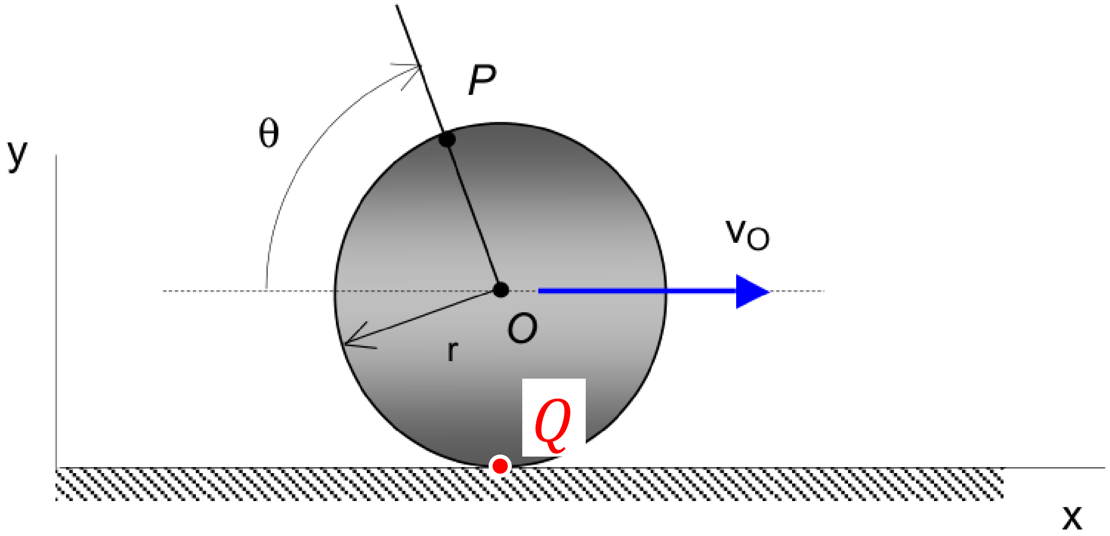
$$\vec{a}_P = \vec{a}_O + \vec{\alpha} \times \vec{r}_{P/O} - \omega^2 \vec{r}_{P/O}$$

(…cross product in board…)

$$\vec{a}_P = a_O \hat{\imath} - \alpha r \hat{\imath} - \omega^2 r \hat{\jmath}$$

$$\vec{a}_P = (a_O - \alpha r)\hat{\imath} - \omega^2 r \hat{\jmath}$$

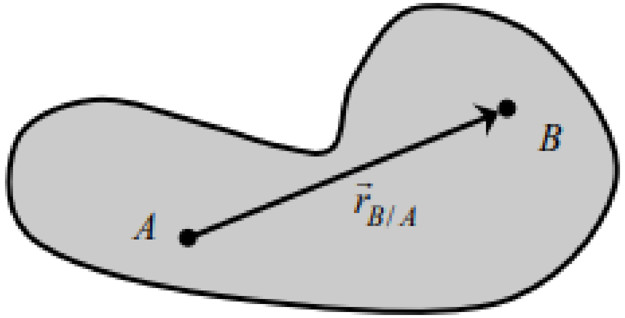
$$\vec{a}_P = 2a_O \hat{\imath} - \frac{v_O^2}{r} \hat{\jmath}$$



PROBLEM: Two points A and B on the same rigid body undergoing planar motion.

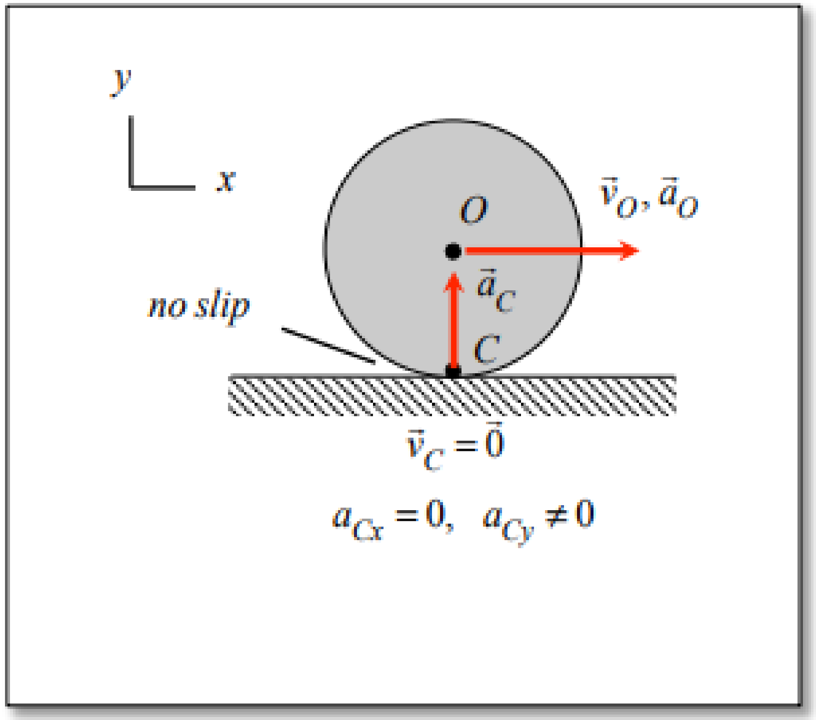
$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A} \right)$$

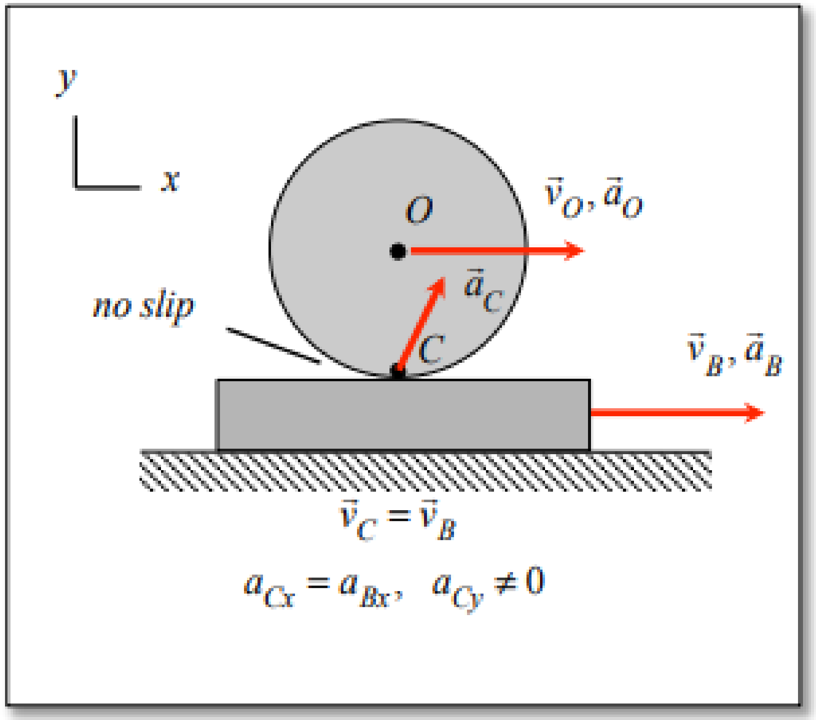


SPECIAL TOPIC: Rolling without slipping

rolling on fixed surface



rolling on moving surface



ME 274: Basic Mechanics II

Week 3 *– Friday, January 30*

Particle kinematics: Planar Rigid Body Motion

Instructor: Manuel Salmerón

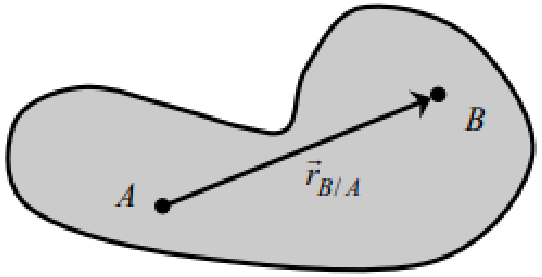
Today’s Agenda

1. Summary
2. Example 2.A.8
3. Example 2.A.10

Summary: Rigid Body Kinematics 2

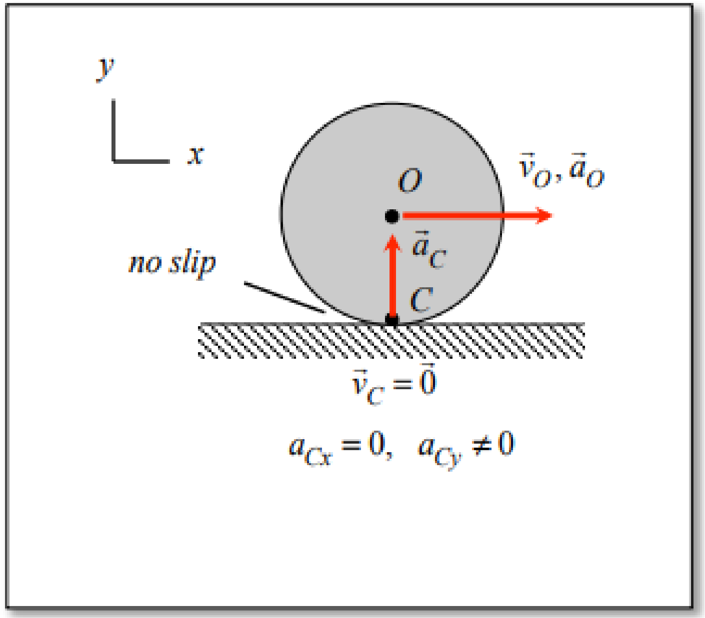
PROBLEM: Two points A and B on the same rigid body undergoing planar motion.

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$
$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right)$$

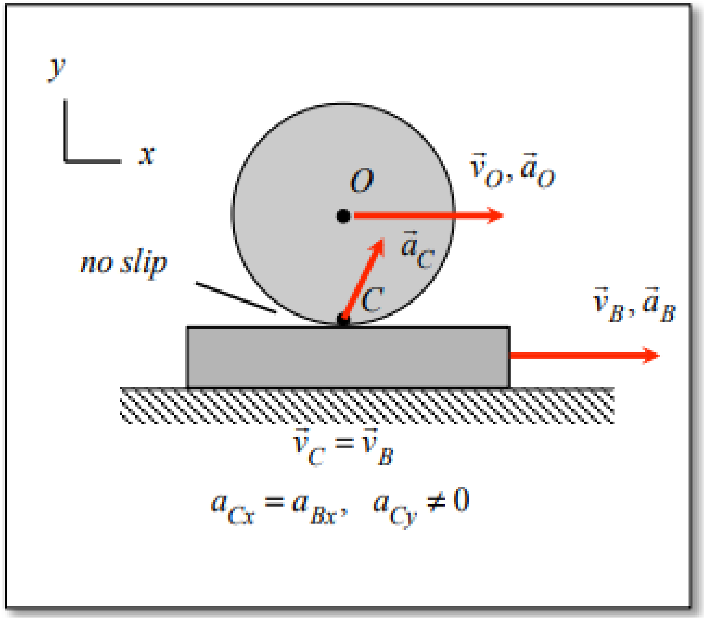


SPECIAL TOPIC: Rolling without slipping

rolling on fixed surface



rolling on moving surface



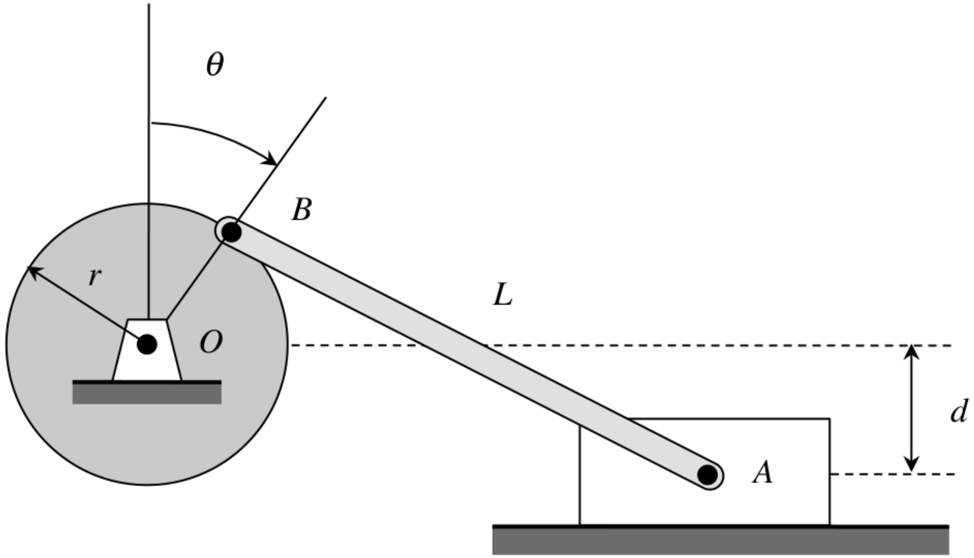
Example 2.A.8

Given: A flywheel rotates in the clockwise sense with a constant angular speed $\dot{\theta}$ about a shaft passing through its center O. The flywheel is connected to a piston A through connecting rod BA. The piston is constrained to slide along a horizontal surface.

Find: Determine:

(a) The acceleration of piston A; and
(b) The angular acceleration of connecting rod arm BA.

Use the following parameters in your analysis: $\dot{\theta} = 10$ rad/s, $\theta = 90^\circ$, $r = 0.1$ m, $d = 0.2$ m and $L = 0.45$ m.



Example 2.A.8

Given: $\dot{\theta} = 10$ rad/s (constant), $\theta = 90°$, $r = 0.1$ m, $d = 0.2$ m, $L = 0.45$ m

Find: (a) $\vec{a}_A$, (b) $\vec{\alpha}_{BA}$

Solution:

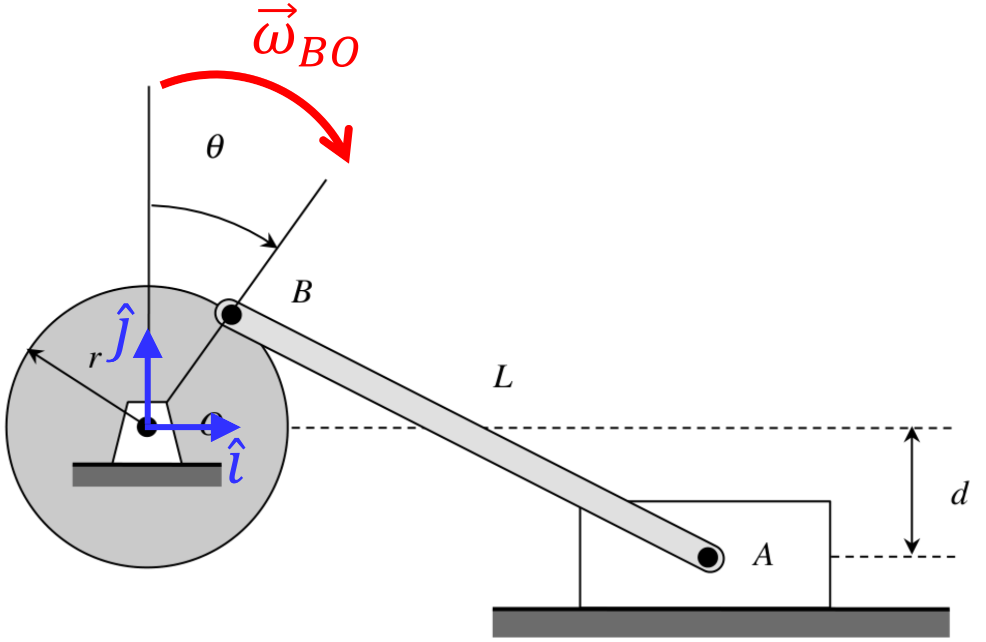
$$\vec{a}_A = \vec{a}_B + \vec{\alpha}_{BA} \times \vec{r}_{A/B} + \vec{\omega}_{BA} \times \left(\vec{\omega}_{BA} \times \vec{r}_{A/B}\right)$$

$\vec{\omega}_{BA}$ seems to be an important value to know!

O is fixed: good place to start

$$\vec{v}_B = \vec{v}_O + \vec{\omega}_{BO} \times \vec{r}_{B/O}$$

$$\vec{v}_B = \vec{0} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & -\dot{\theta} \\ r & 0 & 0 \end{vmatrix} = -\hat{\jmath}\left(0 + \dot{\theta} r\right) = -\dot{\theta} r \hat{\jmath}$$



Example 2.A.8

Given: $\dot{\theta} = 10$ rad/s (constant), $\theta = 90°$, $r = 0.1$ m, $d = 0.2$ m, $L = 0.45$ m

Find: (a) $\vec{a}_A$, (b) $\vec{\alpha}_{BA}$

Solution:

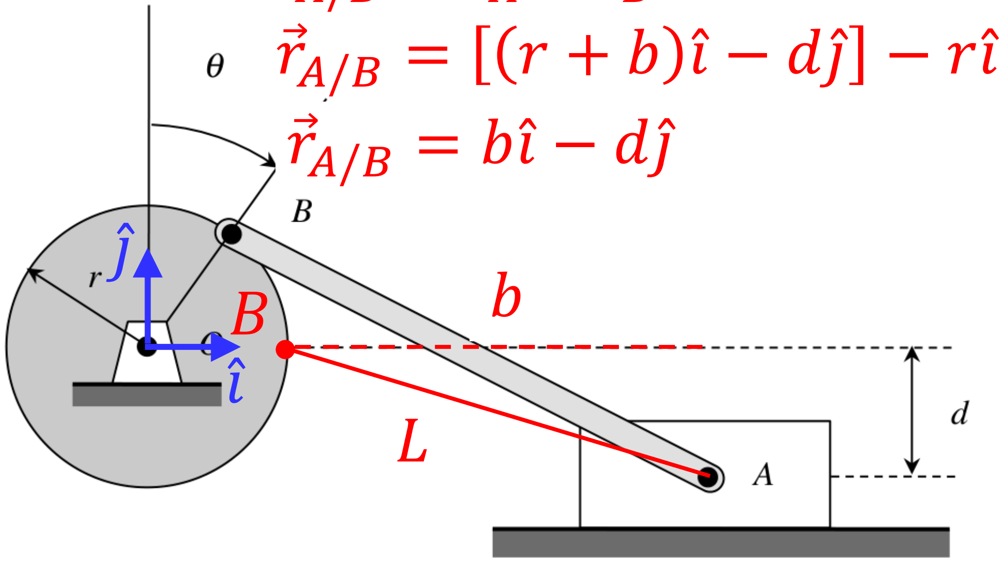
Knowing $\vec{v}_B$, we can move to the next body:

$$\vec{v}_A = \vec{v}_B + \vec{\omega}_{AB} \times \vec{r}_{A/B}$$

$$\vec{v}_A = -\dot{\theta} r \hat{\jmath} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \omega_{AB} \\ b & -d & 0 \end{vmatrix}$$

$$\vec{v}_A = -\dot{\theta} r \hat{\jmath} + (\omega_{AB} d)\hat{\imath} - (-\omega_{AB} b)\hat{\jmath}$$

$$\vec{v}_A = \omega_{AB} b \hat{\imath} + \left(\omega_{AB} d - \dot{\theta} r\right)\hat{\jmath}$$



$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

$$\vec{r}_{A/B} = [(r + b)\hat{\imath} - d\hat{\jmath}] - r\hat{\imath}$$

$$\vec{r}_{A/B} = b\hat{\imath} - d\hat{\jmath}$$

Example 2.A.8

Given: $\dot{\theta} = 10$ rad/s (constant), $\theta = 90°$, $r = 0.1$ m, $d = 0.2$ m, $L = 0.45$ m

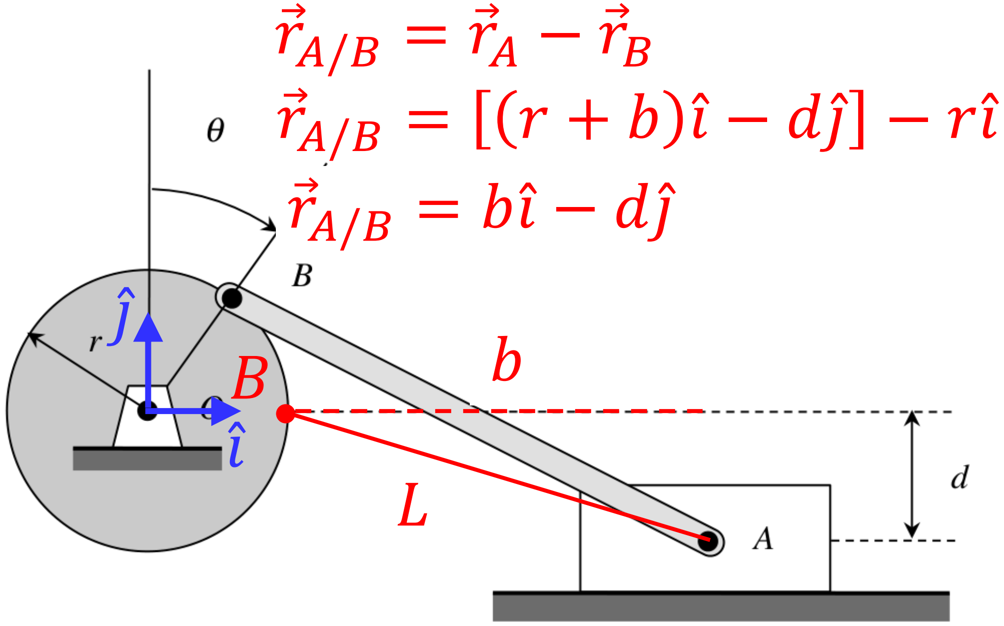
Find: (a) $\vec{a}_A$, (b) $\vec{\alpha}_{BA}$

Solution:

$$\vec{v}_A = \omega_{AB} b\hat{\imath} + \left(\omega_{AB} d - \dot{\theta} r\right)\hat{\jmath}$$

Note that A does not move in Y:

$$\omega_{AB} d - \dot{\theta} r = 0 \Rightarrow \vec{\omega}_{AB} = \left(\frac{\dot{\theta} r}{d}\right)\hat{k}$$



$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$
$$\vec{r}_{A/B} = [(r + b)\hat{\imath} - d\hat{\jmath}] - r\hat{\imath}$$
$$\vec{r}_{A/B} = b\hat{\imath} - d\hat{\jmath}$$

Example 2.A.8

Given: $\dot{\theta} = 10$ rad/s (constant), $\theta = 90°$, $r = 0.1$ m, $d = 0.2$ m, $L = 0.45$ m

Find: (a) $\vec{a}_A$, (b) $\vec{\alpha}_{BA}$

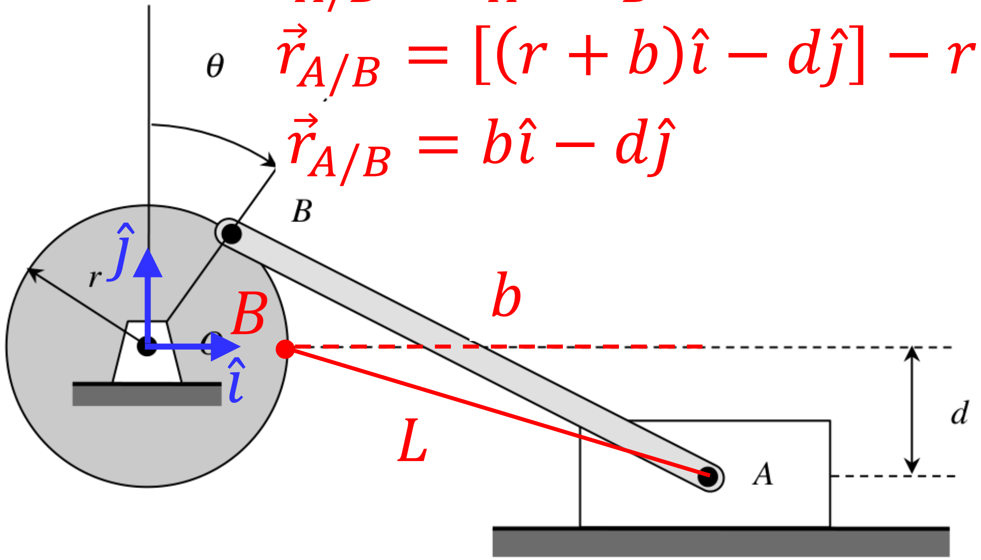
Solution:

Proceed similarly with acceleration:

$$\vec{a}_B = \vec{a}_O + \vec{\alpha}_{BO} \times \vec{r}_{B/O} - \dot{\theta}^2 \vec{r}_{B/O}$$

But $\dot{\theta}$ is constant, so:

$$\vec{a}_B = -\dot{\theta}^2 r \hat{\imath}$$



$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

$$\vec{r}_{A/B} = [(r + b)\hat{\imath} - d\hat{\jmath}] - r\hat{\imath}$$

$$\vec{r}_{A/B} = b\hat{\imath} - d\hat{\jmath}$$

Example 2.A.8

Given: $\dot{\theta} = 10$ rad/s (constant), $\theta = 90°$, $r = 0.1$ m, $d = 0.2$ m, $L = 0.45$ m

Find: (a) $\vec{a}_A$, (b) $\vec{\alpha}_{BA}$

Solution:

In the body BA:

$$\vec{a}_A = \vec{a}_B + \vec{\alpha}_{AB} \times \vec{r}_{A/B} - \omega_{AB}^2 \vec{r}_{A/B}$$

$$\vec{a}_A = -\dot{\theta}^2 r\hat{\imath} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \alpha_{AB} \\ b & -d & 0 \end{vmatrix} - \omega_{AB}^2(b\hat{\imath} - d\hat{\jmath})$$

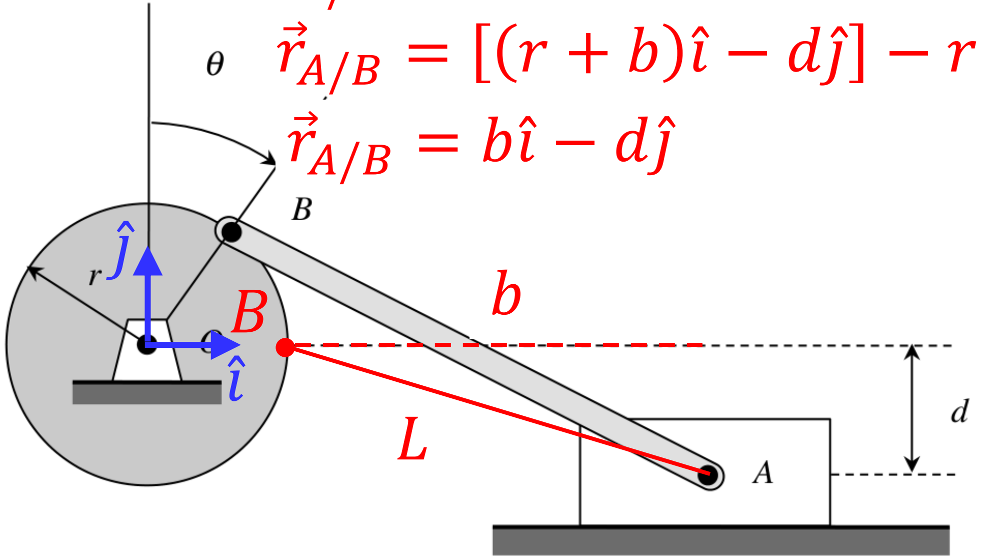
$$\vec{a}_A = -\dot{\theta}^2 r\hat{\imath} + \alpha_{AB} d\hat{\imath} - (-\alpha_{AB} b)\hat{\jmath} - \omega_{AB}^2(b\hat{\imath} - d\hat{\jmath})$$

$$\vec{a}_A = \left(-\dot{\theta}^2 r + \alpha_{AB} d - \omega_{AB}^2 b\right)\hat{\imath} + \left(\alpha_{AB} b + \omega_{AB}^2 d\right)\hat{\jmath}$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

$$\vec{r}_{A/B} = [(r+b)\hat{\imath} - d\hat{\jmath}] - r\hat{\imath}$$

$$\vec{r}_{A/B} = b\hat{\imath} - d\hat{\jmath}$$



Example 2.A.8

Given: $\dot{\theta} = 10$ rad/s (constant), $\theta = 90°$, $r = 0.1$ m, $d = 0.2$ m, $L = 0.45$ m

Find: (a) $\vec{a}_A$, (b) $\vec{\alpha}_{BA}$

Solution:

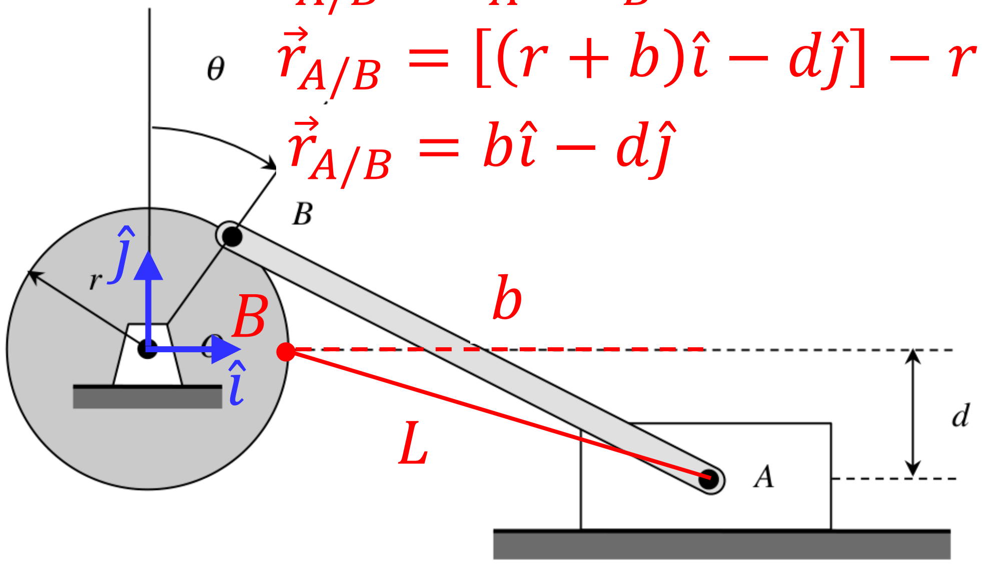
$$\vec{a}_A = \left(-\dot{\theta}^2 r + \alpha_{AB} d - \omega_{AB}^2 b\right)\hat{\imath} + \left(\alpha_{AB} b + \omega_{AB}^2 d\right)\hat{\jmath}$$

Any extra information?

A does not move in Y

$$0 = \alpha_{AB} b + \omega_{AB}^2 d \Rightarrow \vec{\alpha}_{AB} = -\frac{\omega_{AB}^2 d}{b}\hat{k}$$

$$\vec{a}_A = \left(-\dot{\theta}^2 r - \frac{\omega_{AB}^2 d^2}{b} - \omega_{AB}^2 b\right)\hat{\imath}$$



$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

$$\vec{r}_{A/B} = [(r+b)\hat{\imath} - d\hat{\jmath}] - r\hat{\imath}$$

$$\vec{r}_{A/B} = b\hat{\imath} - d\hat{\jmath}$$

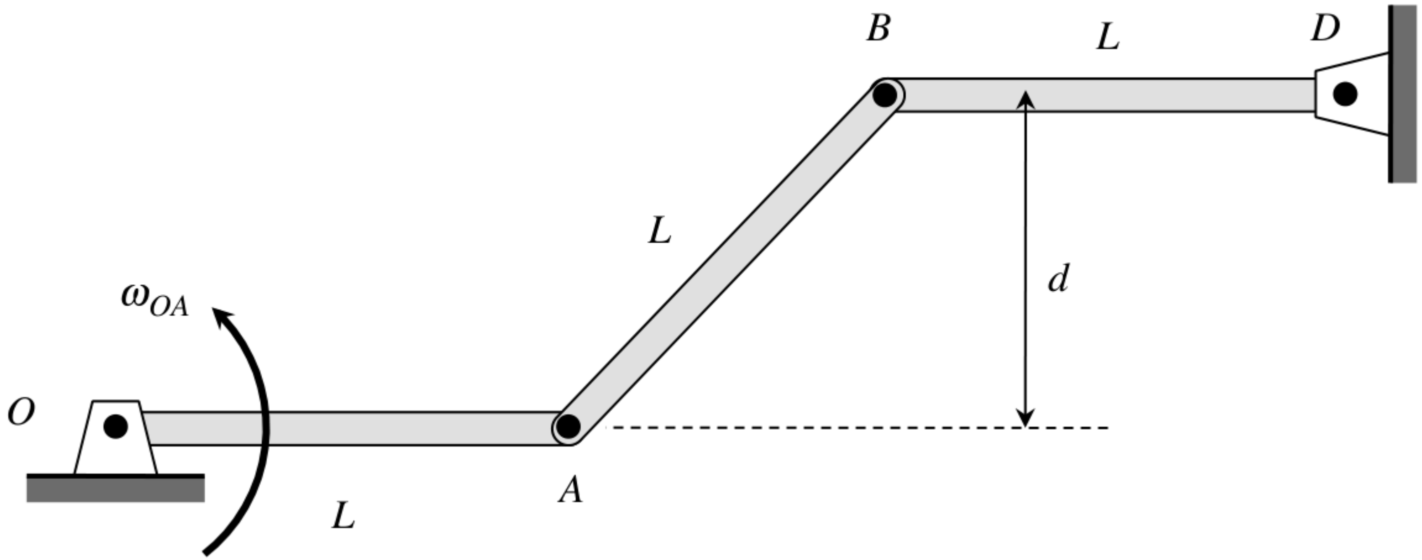
Example 2.A.10

Given: At the instant shown, link OA rotates counterclockwise about pin O with a constant angular speed of ω_{OA}. At the instant shown, links OA and BD are horizontal.

Find: Determine at this instant:

(a) The angular acceleration of link AB; and
(b) The angular acceleration of link BD.

Use the following parameters in your analysis: $\omega_{OA} = 3$ rad/s, $L = 0.5$ m and $d = 0.4$ m. Also, be sure to express your answers as vectors.



Attendance

1. Briefly describe your strategy for solving Example 2.A.10 **before you start solving it**.
2. Before seeing the solution, make sure your answers comply with the following checklist:
   - Are the units right? (velocities in m/s, accelerations in m/s2, etc.)
   - Are the direction of your vectors (e.g., position, angular motion) consistent with your proposed coordinate system?
   - Are all the signs correct? Don't forget the minus in the j-component of the cross-product!
   - Can all your answers be computed numerically with the information given?

Attendance

Step-by-step solution (summary):

1. Velocity at A: $\vec{v}_A = \omega_{OA} L \hat{j}$
2. Knowing $\vec{v}_A$, go to AB: $\vec{v}_B = -\omega_{AB} d \hat{\imath} + (\omega_{OA} L + \omega_{AB} b)\hat{j}$
3. Need more info, go to BD: $\vec{v}_B = -\omega_{BD} L \hat{j}$
4. Equate both $\vec{v}_B$ to get: $\vec{\omega}_{AB} = \vec{0}$ and $\vec{\omega}_{BD} = -\omega_{OA} \hat{k}$
5. Acceleration at A: $\vec{a}_A = -\omega_{OA}^2 L \hat{\imath}$
6. Knowing $\vec{a}_A$, go to AB: $\vec{a}_B = (-\omega_{OA}^2 L - \alpha_{AB} d)\hat{\imath} + \alpha_{AB} b \hat{j}$
7. Need more info, go to BD: $\vec{a}_B = \alpha_{BD} L \hat{j} + \omega_{BD}^2 L \hat{\imath}$
8. Equate both $\vec{a}_B$ to get: $\vec{\alpha}_{AB} = -\frac{2\omega_{OA}^2 L}{d}\hat{k}$ and $\vec{\alpha}_{BD} = \frac{2\omega_{OA}^2 b}{d}\hat{k}$

Attendance

3. Compare your answers with the solution. Briefly describe what worked in your strategy and what you would change next time.
4. Where did you get stuck, and what did you try?