2/16/2026

KINEMATICS OF 3D ROTATING FRAMES

EQUATIONS:

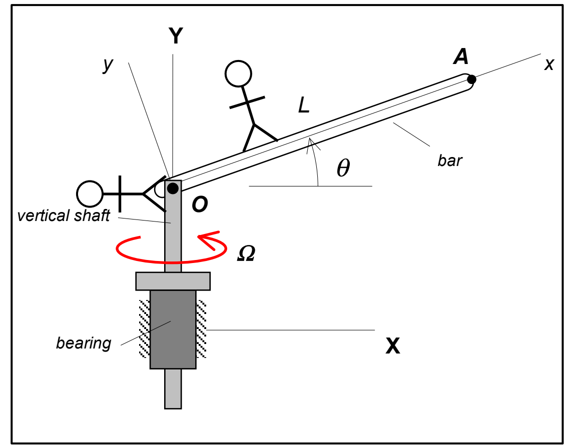
$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega}_{TOT} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha}_{TOT} \times \vec{r}_{B/A} + 2\vec{\omega}_{TOT} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega}_{TOT} \times \left[\vec{\omega}_{TOT} \times \vec{r}_{B/A}\right]$$

$$\dot{\hat{i}} = \vec{\omega}_{TOT} \times \hat{i}; \quad \dot{\hat{j}} = \vec{\omega}_{TOT} \times \hat{j}; \quad \dot{\hat{k}} = \vec{\omega}_{TOT} \times \hat{k}$$

Summary: 3D Moving Reference Frame Kinematics 2

PROBLEM: A person attached to a moving body (reference frame) is observing the motion of point A.



$$\vec{v}_A = \vec{v}_O + \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega} \times \vec{r}_{A/O}$$

$$\vec{a}_A = \vec{a}_O + \left(\vec{a}_{A/O}\right)_{rel} + \vec{\alpha} \times \vec{r}_{A/O} + 2\vec{\omega} \times \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{A/O}\right)$$

CHANGING OBSERVERS: For constant rotation rates,

Observer on vertical shaft:

$$\vec{\omega} = \Omega \hat{J}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \vec{0}$$

$$\left(\vec{v}_{A/O}\right)_{rel} = L\dot{\theta}\hat{j}$$

$$\left(\vec{a}_{A/O}\right)_{rel} = -L\dot{\theta}^2\hat{i}$$

Observer on arm OA:

$$\underline{\vec{\omega}_{TOT} = \Omega \hat{J} + \dot{\theta}\hat{k}}$$

$$\vec{\alpha}_{TOT} = \frac{d\vec{\omega}}{dt} = \dot{\Omega}\hat{J} + \Omega\dot{\hat{J}} + \ddot{\theta}\hat{k} + \dot{\theta}\dot{\hat{k}} = \dot{\theta}\left(\vec{\omega} \times \hat{k}\right)$$

$$\left(\vec{v}_{A/O}\right)_{rel} = \vec{0}$$

$$\left(\vec{a}_{A/O}\right)_{rel} = \vec{0}$$

These give the same result! Try it.

Below is an amusement ride “Mad Tea Party”. If a moving reference is attached to the pink cup that rotates on the pink plate, which is located on the big purple base that rotates at the same time, is this is 2D moving reference frame or a 3D moving reference frame?

(A) 2D
(B) 3D

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Below is the “Spinsanity” amusement ride in Six Flags - Saint Louis. If a moving reference is attached to the plate where people sit on, and the plate can rotate while also moving along a curved rail, is this a case of 2D moving reference frame or 3D moving reference frame?

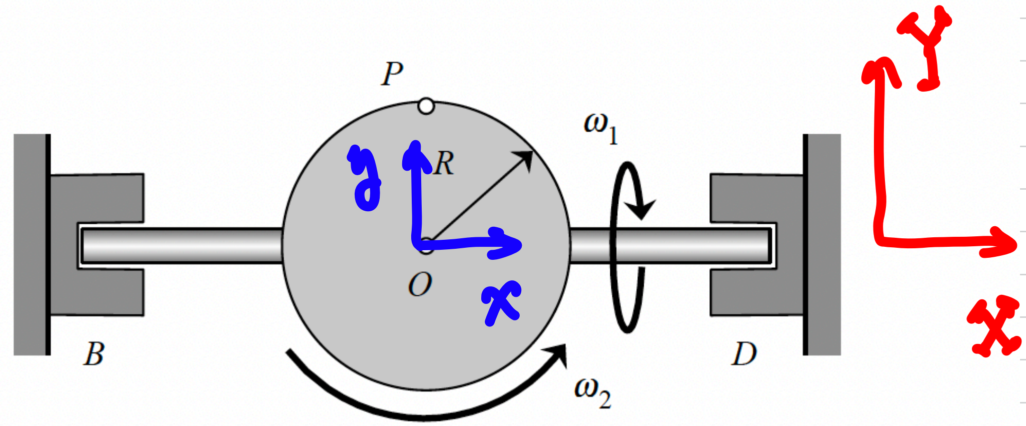
(A) 2D
(B) 3D

Example 3.B.6

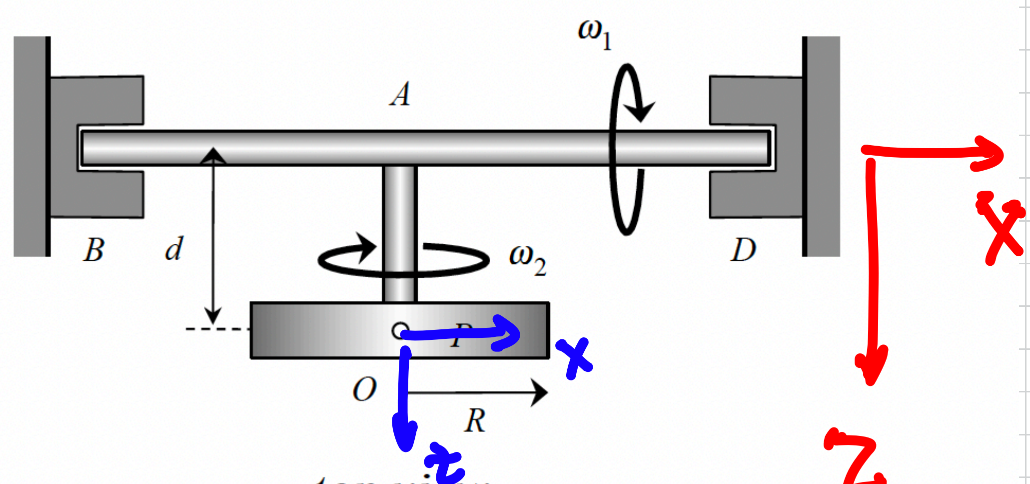
Given: Shaft BD rotates about a fixed axis with a constant rate of ω_1. Shaft OA is rigidly attached to shaft BD with OA being perpendicular to BD. A disk rotates about shaft OA with a constant rate of ω_2 relative to OA.

Find: The acceleration of point P on the edge of the disk for the position shown.

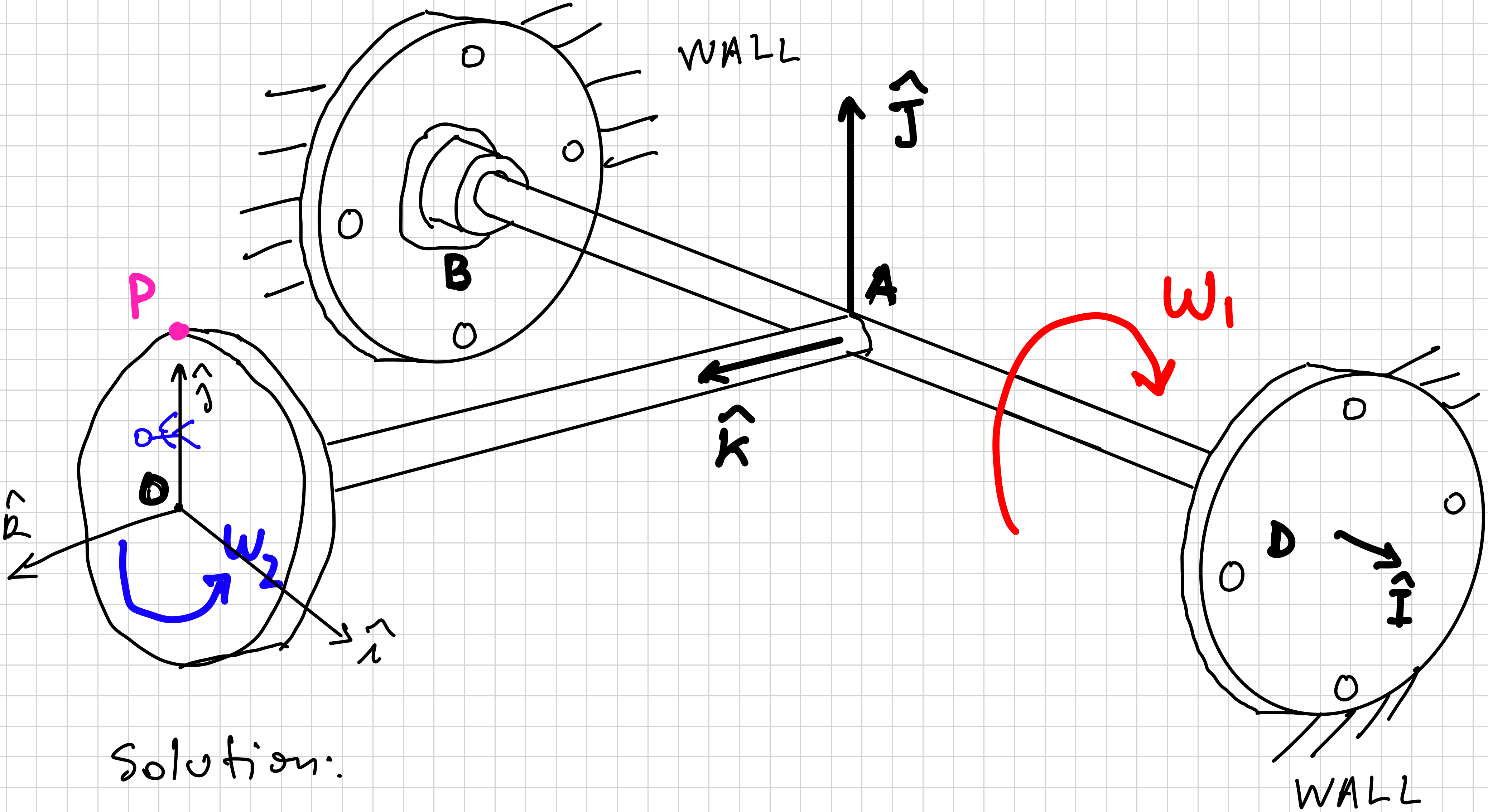
Use the following parameters in your analysis: $\omega_1 = 5$ rad/s, $\omega_2 = 8$ rad/s, $R = 6$ in and $d = 4$ in.



side view



top view



Solution:

$$\vec{\omega}_{TOT} = -\omega_1 \hat{I} + \omega_2 \hat{K}$$

$$\vec{\alpha}_{TOT} = \dot{\vec{\omega}}_{TOT} = -\cancel{\dot{\omega}_1}^{0} \hat{I} - \omega_1 \cancel{\dot{\hat{I}}}^{0} + \cancel{\dot{\omega}_2}^{0} \hat{K} + \omega_2 \dot{\hat{K}}$$

$$\vec{\alpha}_{TOT} = \omega_2 \left[\vec{\omega}_{TOT} \times \hat{K} \right] = \omega_2 \left[(-\omega_1 \hat{I} + \omega_2 \hat{K}) \times \hat{K} \right]$$

$$\vec{\alpha}_{TOT} = \omega_2 \left(\omega_1 \hat{j} \right)$$

$$\vec{\alpha}_{TOT} = \omega_2 \omega_1 \hat{j}$$

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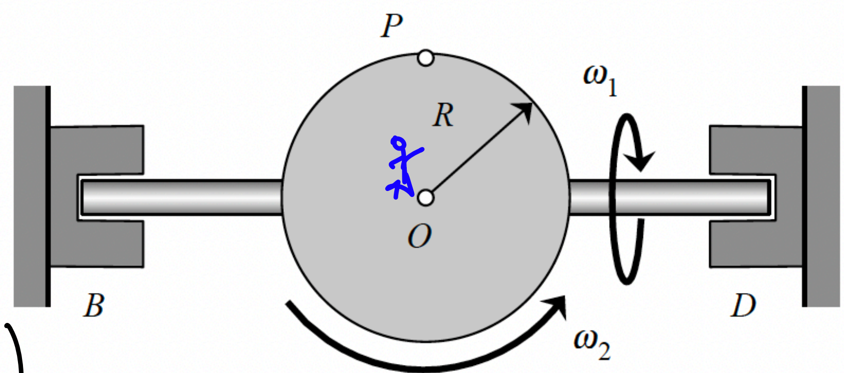
3D Moving Frame

$$\vec{a}_P = \vec{a}_O + \cancel{(\vec{a}_{P/O})_{REL}} + \vec{\alpha}_{TOT} \times \vec{r}_{P/O} + 2\vec{\omega}_{TOT} \times \cancel{(\vec{v}_{P/O})_{REL}} + \vec{\omega}_{TOT} \times (\vec{\omega}_{TOT} \times \vec{r}_{P/O})$$



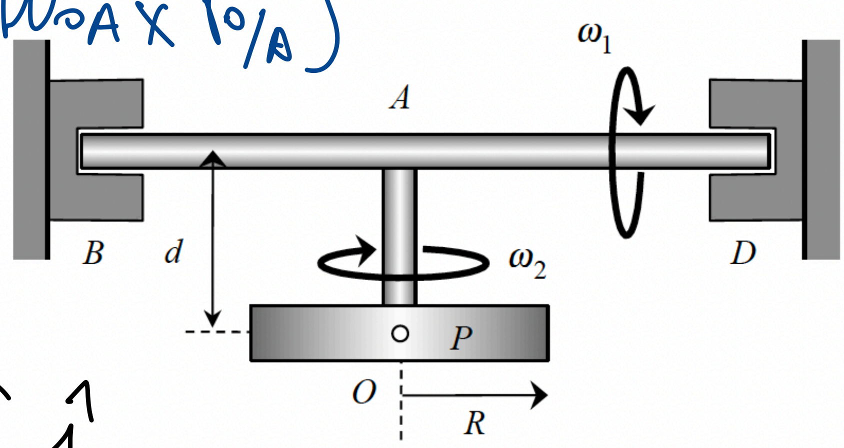
side view

2D Moving Frame

$$\vec{a}_O = \cancel{\vec{a}_A} + \cancel{\vec{\alpha}_{OA}} \times \vec{r}_{O/A} + \vec{\omega}_{OA} \times (\vec{\omega}_{OA} \times \vec{r}_{O/A})$$

$$\vec{a}_O = -\omega_1 \hat{I} \times (-\omega_1 \hat{I} \times -d\hat{K})$$

$$\vec{a}_O = -\omega_1^2 d \hat{K}$$



top view

$$\vec{a}_p = -\omega_1^2 d\,\hat{k} + \cancelto{0}{\omega_2\omega_1\,\hat{j} \times R\,\hat{j}} + (-\omega_1\hat{i} + \omega_2\hat{k}) \times \left[(-\omega_1\hat{i} + \omega_2\hat{k}) \times R\,\hat{j}\right]$$

$$\vec{a}_p = -\omega_1^2 d\,\hat{k} + (-\omega_1\hat{i} + \omega_2\hat{k})(-\omega_1 R\,\hat{k} - \omega_2 R\,\hat{i})$$

$$\boxed{\vec{a}_p = -\omega_1^2 d\,\hat{k} - \omega_1^2 R\,\hat{j} - \omega_2^2 R\,\hat{j}}$$ ← ANSWER

Example 3.B.9

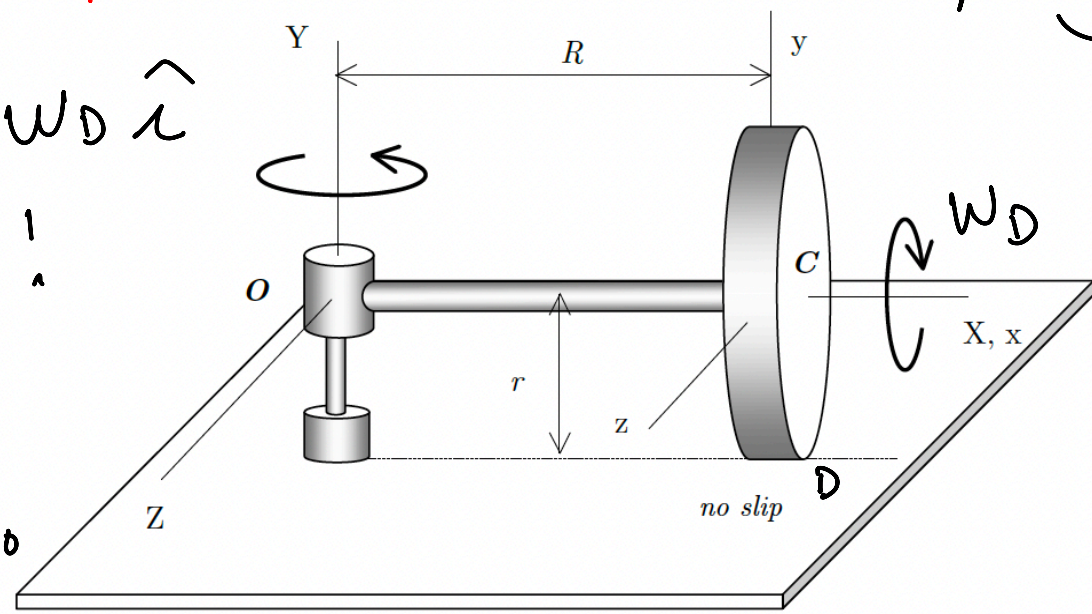
Given: Arm OC rotates about the fixed Y-axis at a constant rate Ω. The disk at C, having a radius of R, is able to rotate about arm OC and rolls without slipping on a fixed horizontal surface. Let the xyz axes be attached to the disk.

Find: Determine the angular acceleration of the disk.

Solution: Need $\vec{\omega}_{TOT}$

$$\bar{\omega}_{TOT} = \Omega \hat{J} - \omega_D \hat{\imath}$$

ω_D is not given!



$\hat{\imath}$ $\hat{K}$ $+$ $\hat{J}$

BAR

$$\bar{v}_C = \cancel{\bar{v}_O}^{\,0} + \bar{\omega}_{OC} \times \bar{r}_{C/O}$$

$$= \Omega \hat{J} \times R \hat{I}$$

$$\bar{v}_C = -\Omega R \hat{K} \quad (1)$$

DISK

$$\vec{v}_C = \cancel{\vec{v}_D}^{\,0} + \bar{\omega}_{TOT} \times \bar{r}_{C/D}$$

$$= (\Omega \hat{J} - \omega_D \hat{\imath}) \times (r \hat{J})$$

$$\hat{i} = \hat{I} \quad , \quad \hat{j} = \hat{J} \quad , \quad \hat{k} = \hat{K}$$

$$v_C = -\omega_D \, r \, \hat{k} \qquad ②$$

$$① = ②$$

$$-\Omega R \hat{k} = -\omega_D \, r \, k$$

$$\omega_D = \Omega \frac{R}{r}$$

$$\vec{\omega}_{TOT} = \Omega \hat{J} - \Omega \frac{R}{r} \hat{i}$$

$$\vec{\alpha}_{TOT} = \dot{\vec{\omega}}_{TOT} = \overset{0}{\cancel{\dot{\Omega}}} \hat{J} + \Omega \overset{0}{\cancel{\dot{\hat{J}}}} - \overset{0}{\cancel{\dot{\Omega}}} \frac{R}{r} \hat{i} - \Omega \frac{R}{r} \dot{\hat{i}}$$

$$\vec{\alpha}_{TOT} = -\Omega \frac{R}{r} \left(\vec{\omega}_{TOT} \times \hat{i} \right)$$

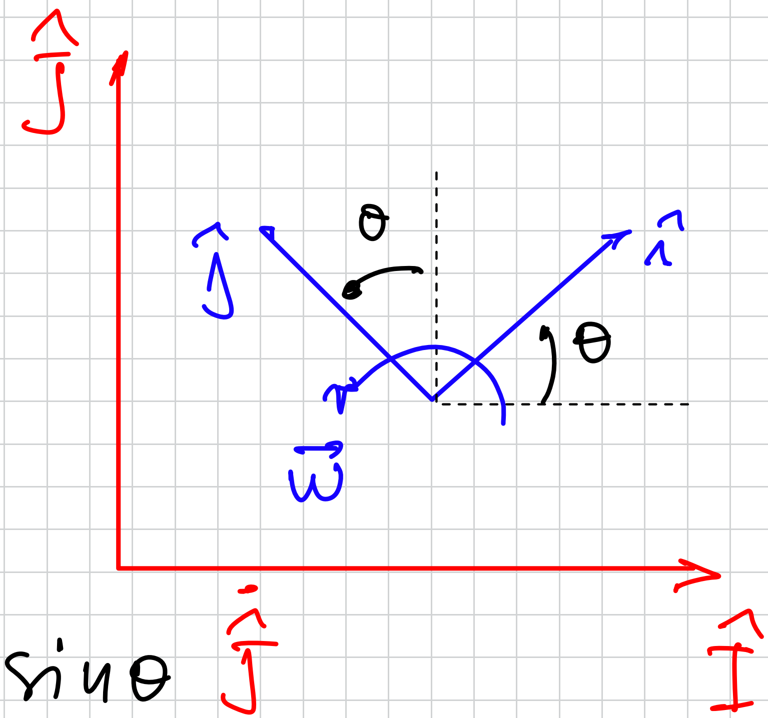
$$= -\Omega \frac{R}{r} \left(\left(\Omega \hat{J} - \Omega \frac{R}{r} \hat{i} \right) \times \hat{i} \right)$$

$$\vec{\alpha}_{ROT} = -\Omega^2 \frac{R}{r} \hat{r}$$ ← ANSW

why $\dot{\hat{i}} = \bar{\omega} \times \hat{i}$

$\dot{\hat{j}} = \bar{\omega} \times \hat{j}$

$\dot{\hat{k}} = \bar{\omega} \times \hat{j}$?



$\hat{i} = \cos\theta \, \hat{I} + \sin\theta \, \hat{J}$

$\hat{j} = -\sin\theta \, \hat{I} + \cos\theta \, \hat{J}$

$$\frac{d}{dt}(\hat{i}) = -\dot{\theta}\cos\theta \, \hat{I} + \sin\theta \, \dot{\hat{I}} + \dot{\theta}\cos\theta \, \hat{J} + \sin\theta \, \dot{\hat{J}}$$

$$\dot{\hat{i}} = \dot{\theta} \underbrace{\left(-\sin\theta \, \hat{I} + \cos\theta \, \hat{J}\right)}_{\hat{j}}$$

$$\dot{\hat{i}} = \dot{\theta}\,\hat{j} = \dot{\theta}\,(\hat{k} \times \hat{i})$$

$$\dot{\hat{i}} = \bar{\omega} \times \hat{i}$$