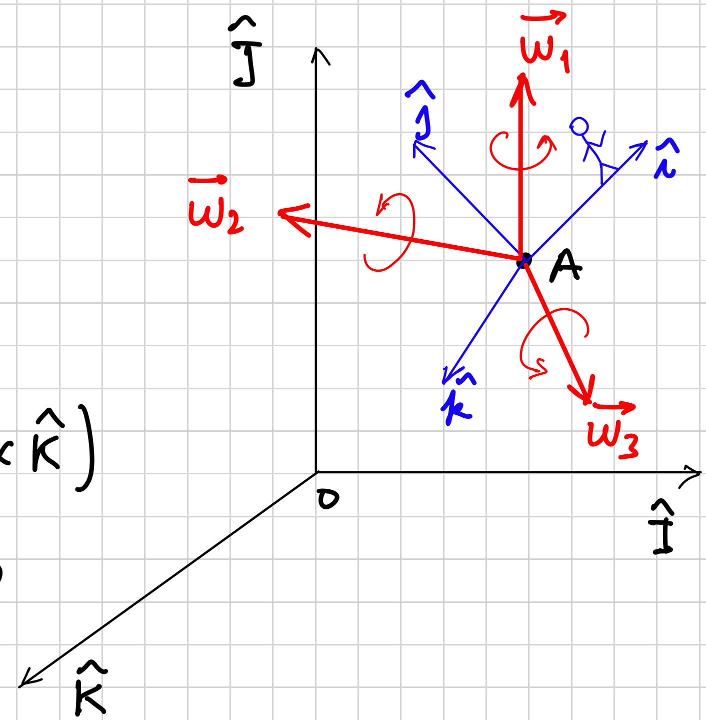
2/12/2026

KINEMATICS OF 3D ROTATING FRAMES

Previously, we analyzed systems in which the angular motion $(\vec{\omega}, \vec{\alpha})$ happened only around the $\hat{k}$, which WAS stationary $(\hat{k} = \hat{K})$

For 3D motion, the axis of rotation for $\vec{\omega}$ will not be stationary



MOVING REFERENCES IN 2D

$$\vec{\omega} = \dot{\theta}\,\hat{k}$$

$$\vec{\alpha} = \ddot{\theta}\,\hat{k} + \dot{\theta}\,\dot{\hat{k}} = \ddot{\theta}\,\hat{k} + \dot{\theta}\,(\vec{\omega} \times \hat{k})$$

$$\text{but } \vec{\omega} \times \hat{k} = \omega\,\hat{k} \times \hat{k} = 0$$

This is not true for 3D!

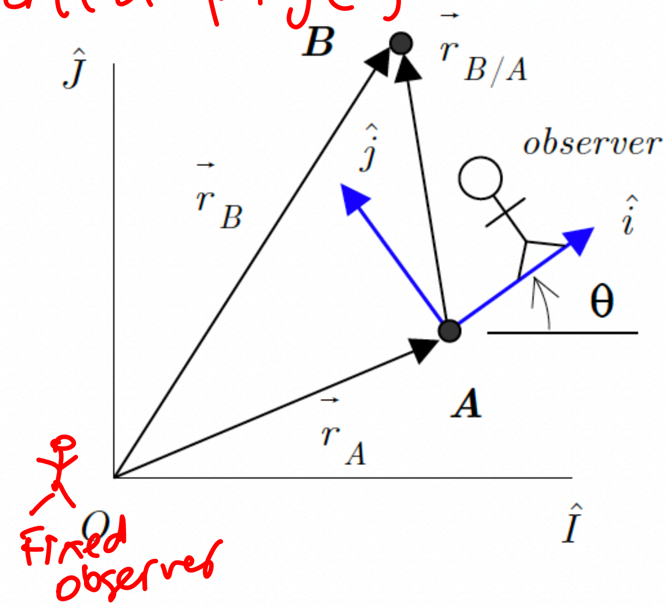
For 3D motion, the angular velocity of the observer will be made up of several components

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3 + \cdots$$

Some of these can be about FIXED axes and some around MOVING axes. It is very important to identify clearly which is which. Example problems here require knowledge of how to write $\vec{\omega}$s and of how to recognize if the rotation is about a fixed or moving axis.

- $X(\hat{I})$, $Y(\hat{J})$, $Z(\hat{K})$ are fixed axes
- $x(\hat{i})$, $y(\hat{j})$, $z(\hat{k})$ are moving/rotating axes

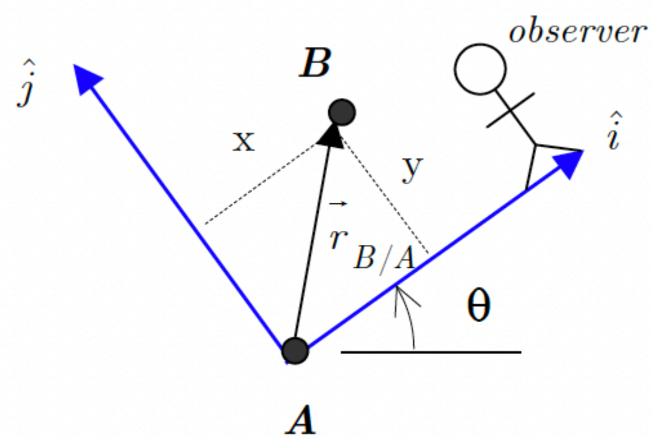
Velocity equation 2D (repeated page)



$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}.$$

But $\vec{r}_{B/A}$ can be written in terms of the observer's xy coordinates: $\vec{r}_{B/A} = x\hat{i} + y\hat{j}$

Differentiating $\vec{r}_B$ with respect to time yields:



$$\frac{d}{dt}\left[\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}\right]$$

$$\vec{v}_B = \vec{v}_A + \dot{x}\hat{i} + x\dot{\hat{i}} + \dot{y}\hat{j} + y\dot{\hat{j}}$$

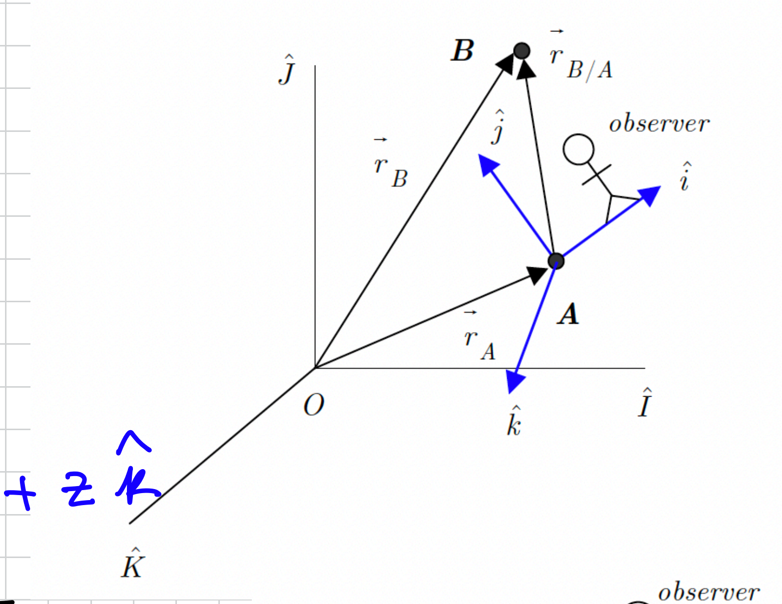
$$\vec{v}_B = \vec{v}_A + \dot{x}\hat{i} + \dot{y}\hat{j} + x(\vec{\omega}\times\hat{i}) + y(\vec{\omega}\times\hat{j})$$

$$= \vec{v}_A + \dot{x}\hat{i} + \dot{y}\hat{j} + \vec{\omega}\times(x\hat{i} + y\hat{j})$$

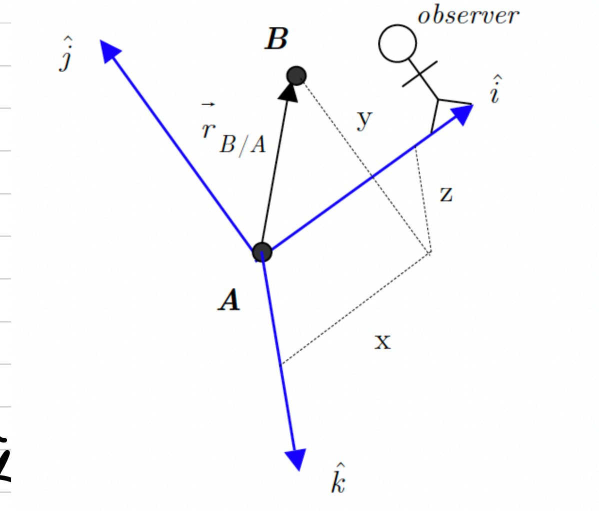
$$\vec{v}_B = \vec{v}_A + (\vec{v}_{B/A})_{rel} + \vec{\omega}\times\vec{r}_{B/A}$$

Velocity equation 3D

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}.$$



But $\vec{r}_{B/A}$ can be written in terms of the observer's xy coordinates : $\vec{r}_{B/A} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$



Differentiating $\vec{r}_B$ with respect to time yields:

$$\frac{d}{dt}\left[\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}\right]$$

$$\vec{v}_B = \vec{v}_A + \dot{x}\hat{\imath} + x\dot{\hat{\imath}} + \dot{y}\hat{\jmath} + y\dot{\hat{\jmath}} + \dot{z}\hat{k} + z\dot{\hat{k}}$$

$$\vec{v}_B = \vec{v}_A + \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} + x(\vec{\omega}\times\hat{\imath}) + y(\vec{\omega}\times\hat{\jmath}) + z(\omega\times\hat{k})$$

$$= \vec{v}_A + \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} \quad \vec{\omega}\times(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

$$\vec{v}_B = \vec{v}_A + (\vec{v}_{B/A})_{rel} + \vec{\omega}\times\vec{r}_{B/A}$$

Notice that now

$$(\vec{v}_{B/A})_{rel} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath} + \dot{z}\,\hat{k}$$

is the relative velocity seen by the observer

Acceleration equation 3D

$$\frac{d}{dt}\left[\vec{v}_B = \vec{v}_A + (\vec{v}_{B/A})_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right]$$

$$\vec{a}_B = \vec{a}_A + \frac{d}{dt}(v_{B/A})_{rel} + \underbrace{\frac{d}{dt}(\vec{\omega} \times \vec{r}_{B/A})}_{\text{PRODUCT RULE}}$$

$$\vec{a}_B = \vec{a}_A + \frac{d}{dt}(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \frac{d}{dt}(\vec{r}_{B/A})$$

from the velocity derivation:

$$\frac{d}{dt}\left[\vec{r}_{B/A}\right] = (\vec{v}_{B/A})_{REL} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \underbrace{(\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k})} + \underbrace{(\dot{x}\dot{\hat{i}} + \dot{y}\dot{\hat{j}} + \dot{z}\dot{\hat{k}})} + \vec{\alpha}\times\vec{r}_{B/A} + \vec{\omega}\times\left[(\vec{v}_{B/A})_{REL} + \vec{\omega}\times\vec{r}_{B/A}\right]$$

$$\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_{REL} + \boxed{\dot{x}(\vec{\omega}\times\hat{i}) + \dot{y}(\vec{\omega}\times\hat{j}) + \dot{z}(\vec{\omega}\times\hat{k})} + \vec{\alpha}\times\vec{r}_{B/A} + \vec{\omega}\times\left[(\vec{v}_{B/A})_{REL} + \vec{\omega}\times\vec{r}_{B/A}\right]$$

$$\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_{REL} + \boxed{\vec{\omega}\times(\dot{x}\hat{i} + y\hat{j} + \dot{z}\hat{k})} + \vec{\alpha}\times\vec{r}_{B/A} + \vec{\omega}\times\left[(\vec{v}_{B/A})_{REL} + \vec{\omega}\times\vec{r}_{B/A}\right]$$

$$\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_{REL} + \vec{\omega}\times(\vec{v}_{B/A})_{REL} + \vec{\alpha}\times\vec{r}_{B/A} + \omega\times(\vec{v}_{B/A})_{REL} + \vec{\omega}\times[\vec{\omega}\times\vec{r}_{B/A}]$$

$$\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_{rel} + \vec{\alpha}\times\vec{r}_{B/A} + 2\vec{\omega}\times(\vec{v}_{B/A})_{rel} + \vec{\omega}\times[\vec{\omega}\times\vec{r}_{B/A}]$$

Relative acceleration of B as seen by the observer.

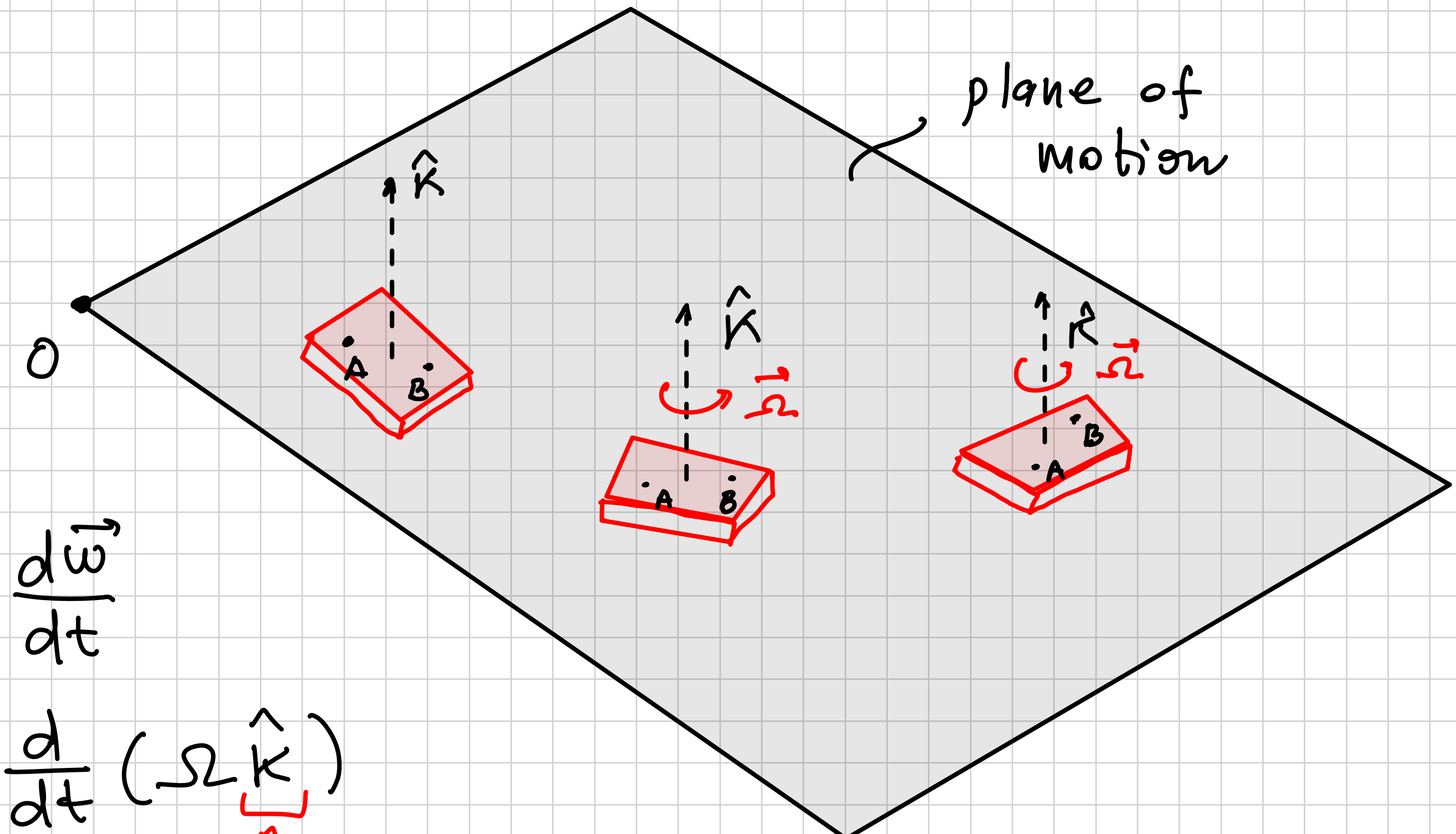
In 3D motion, $\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) \neq -\omega^2 \vec{r}_{B/A}$

In 3D motion, $\vec{\omega} \neq \dot{\theta} \hat{k}$

Angular acceleration of rotating frame

$$\vec{\alpha} = \frac{d}{dt}\vec{\omega}$$

For 2D problems, the moving reference frame is rotating @ a FIXED axis ($\perp$ to the plane of motion).



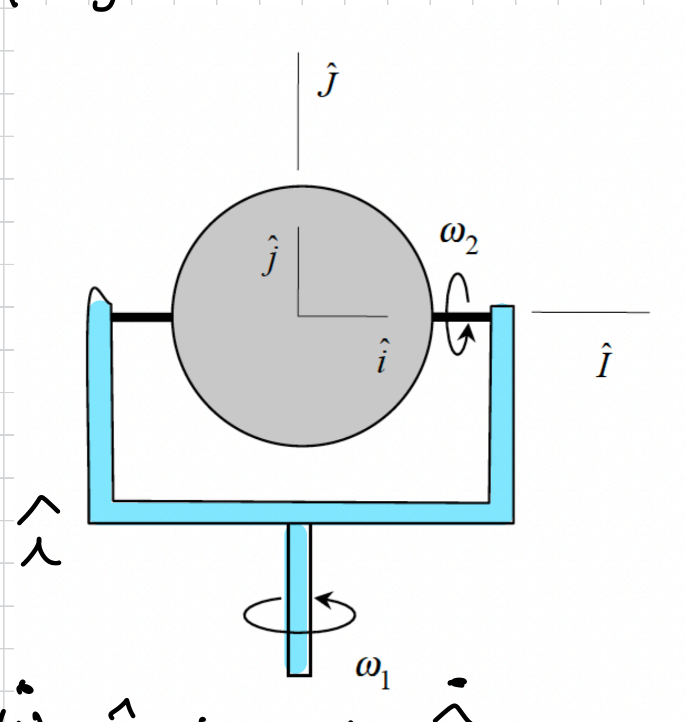
$$\alpha = \frac{d\vec{\omega}}{dt}$$

$$= \frac{d}{dt}(\Omega \underbrace{\hat{K}}_{\text{FIXED}})$$

$$= \dot{\Omega}\hat{K}$$

For 3D Problems $\frac{d\vec{\omega}}{dt}$ is not straightforward

Consider the gimbal below: the blue frame rotates about the fixed $\hat{J}$ axis, and the disk rotates about the moving $\hat{\imath}$ axis.



$$\Rightarrow \quad \vec{\omega}_{TOTAL} = \omega_1 \hat{J} + \vec{\omega}_2 \hat{\imath}$$

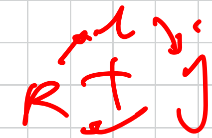
$$\vec{\alpha} = \frac{d}{dt}(\vec{\omega}_{TOT}) = \dot{\omega}_1 \hat{J} + \omega_1 \cancel{\dot{\hat{J}}}^{\,0} + \dot{\omega}_2 \hat{\imath} + \omega_2 \underbrace{\dot{\hat{\imath}}}_{\vec{\omega}_{TOT} \times \hat{\imath}}$$

$$\vec{\alpha} = \dot{\omega}_1 \hat{J} + \dot{\omega}_2 \hat{\imath} + \omega_2 (\vec{\omega}_{TOT} \times \hat{\imath})$$

$$\vec{\alpha} = \dot{\omega}_1 \hat{J} + \dot{\omega}_2 \hat{\imath} + \omega_2 [(\omega_1 \hat{J} + \omega_2 \hat{\imath}) \times \hat{\imath}]$$

$$\vec{\alpha} = \dot{\omega}_1 \hat{J} + \dot{\omega}_2 \hat{\imath} - \omega_2 \omega_1 (\hat{J} \times \hat{\imath}) \qquad (\hat{J} = \hat{j})$$

$$\Rightarrow \vec{\alpha} = \dot{\omega}_1 \hat{j} + \dot{\omega}_2 \hat{\imath} - \omega_2 \omega_1 \hat{k}$$



So, even if one has constant rates of change of the angular velocities, α may not be zero!

$$\vec{\alpha} = -\omega_2 \omega_1 \hat{k}$$

Summary:

$$\vec{v}_B = \vec{v}_A + (\vec{v}_{B/A})_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \vec{r}_{B/A} + \vec{\omega} \times [\vec{\omega} \times r_{B/A}]$$

and $\vec{\alpha}$ may not be zero even if $\dot{\omega}_1 = \dot{\omega}_2 = 0$

and $\omega_{TOT} = \omega_1 + \omega_2 + \cdots$

An excavator is rotating its turret at a constant $\Omega\,\widehat{K}$ and lifting its boom at a constant $\dot{\theta}\,\hat{\imath}$. The magnitude of the total angular acceleration that the load is "feeling" is:

A) Since both angular velocities are constant, obviously the total $|\vec{\alpha}|$ should be zero.

B) Would be the sum of both rates of change of the angular velocities.

C) Proportional to the product of $\dot{\theta}$, and Ω.

D) More information is needed to determine the resulting angular acceleration magnitude.

$$\omega = \Omega \hat{K} + \dot{\theta}\,\hat{\imath}$$

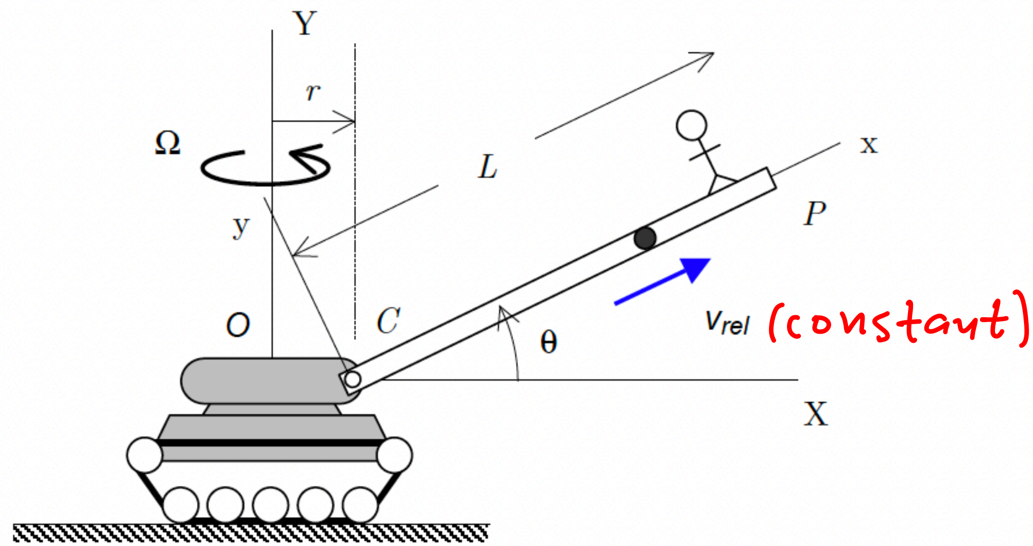
$$\alpha = ?$$

Example 3.B.4

Given: The turret on a tank is rotating about a fixed vertical axis at a constant rate of $\Omega = 0.4$ rad/s. The barrel is being raised at a constant rate of $\dot{\theta} = 0.4$ rad/s . A cannon shell is fired with a constant muzzle speed of $v_{rel} = 200$ ft/s relative to the barrel. The observer and the xyz axes are attached to the barrel, while the XYZ axes are fixed. Here, $r = 3$ ft and $L = 15$ ft.
The tank is not moving at the instant shown

Find:

(a) The angular velocity of the barrel at the instant shown;
(b) The angular acceleration of the barrel at the instant shown; and
(c) The acceleration of the shell as it leaves the barrel at P.



Solution:

$$\vec{\omega}_{TOT} = \Omega \hat{J} + \dot{\theta} \hat{k}$$

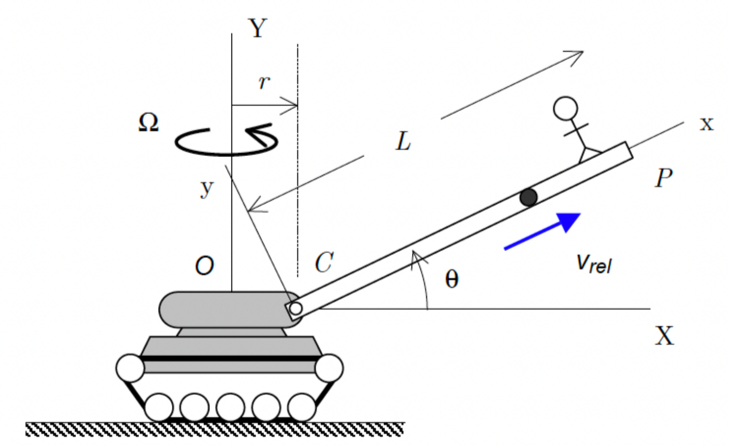
REMEMBER $\dot{\hat{k}} = \vec{\omega}_{TOT} \times \hat{k}$

$$\vec{\alpha} = \frac{d}{dt}(\omega_{TOT}) = \cancel{\dot{\Omega}}^{0} \hat{J} + \Omega \cancel{\dot{\hat{J}}}^{0} + \cancel{\ddot{\theta}}^{0} \hat{k} + \dot{\theta} \dot{\hat{k}} \quad \neq 0$$

$$\vec{\alpha} = \dot{\theta}\left(\vec{\omega}_{TOT} \times \hat{k}\right)$$

$$= \dot{\theta}\left[\left(\Omega \hat{J} + \dot{\theta}\hat{K}\right) \times \hat{K}\right]$$

$$\hat{K} = \hat{k}$$



$$\boxed{\vec{\alpha} = \dot{\theta}\,\Omega\,\hat{I}}$$

Only now we can approach the accel. eqn.

$$\vec{a}_P = \vec{a}_C + \cancel{\left(\vec{a}_{P/C}\right)_{rel}}^{\,0} + \vec{\alpha} \times \vec{r}_{P/C} + 2\vec{\omega}_{TOT} \times \left(\vec{v}_{P/C}\right)_{rel} + \vec{\omega}_T \times \left[\vec{\omega}_T \times \vec{r}_{P/C}\right]$$

$$\vec{a}_C = \cancel{\vec{a}_O}^{\,0} + \cancel{\vec{\alpha}_O}^{\,0} \times \vec{r}_{C/O} + \vec{\omega}_{CO} \times \left(\vec{\omega}_{CO} \times \vec{r}_{C/O}\right)$$

$$\vec{a}_C = \Omega \hat{J} \times \left(\Omega \hat{J} \times r\hat{I}\right)$$

$$\vec{a}_C = -\Omega\, r\, \hat{K}$$

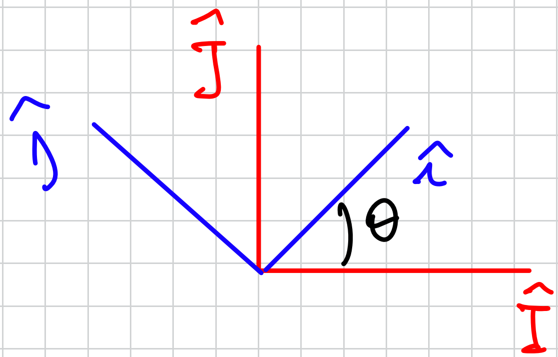
$\hat{I}$ $\hat{K}$ $\hat{J}$

$$\left(\vec{a}_{P/C}\right)_{rel} = \vec{0} \quad \text{(constant muzzle velocity)}$$

$$\vec{r}_{P/C} = L\,\hat{i}$$

$$\left(\vec{v}_{P/C}\right)_{rel} = v_{REL}\,\hat{i}$$

$$\Rightarrow \vec{a}_p = -\Omega^2 r\,\hat{I} + \dot{\theta}\,\Omega\,\hat{I} \times L\,\hat{i} + 2\left(\Omega\hat{J} + \dot{\theta}\hat{K}\right) \times v_{REL}\,\hat{i}$$
$$+ \left(\Omega\hat{J} + \dot{\theta}\hat{K}\right) \times \left[\left(\Omega\hat{J} + \dot{\theta}\hat{K}\right) \times L\hat{i}\right]$$



$$\hat{i} = \cos\theta\,\hat{I} + \sin\theta\,\hat{J}$$

$$\Rightarrow \vec{a}_p = -\Omega^2 r\,\hat{I} + \dot{\theta}\,\Omega\,\hat{I} \times \left(L\cos\theta\,\hat{I} + L\sin\theta\,\hat{J}\right)$$
$$+ 2\left(\Omega\hat{J} + \dot{\theta}\hat{K}\right) \times v_{REL}\left(\cos\theta\,\hat{I} + \sin\theta\,\hat{J}\right)$$
$$+ \left(\Omega\hat{J} + \dot{\theta}\hat{K}\right) \times \left[\left(\Omega\hat{J} + \dot{\theta}\hat{K}\right) \times \left(L\cos\theta\,\hat{I} + L\sin\theta\,\hat{J}\right)\right]$$

$$\Rightarrow \vec{a}_p = -\Omega^2 r\hat{I} + \dot{\theta}\Omega L\sin\theta\hat{K} +$$
$$-2\Omega v_{REL}\cos\theta\hat{K} + 2\dot{\theta}v_{REL}\cos\theta\hat{J} - 2\dot{\theta}v_{REL}\sin\theta\hat{I}$$
$$+(\Omega\hat{J} + \dot{\theta}\hat{K}) \times \left[-\Omega L\cos\theta\hat{K} + \dot{\theta}L\cos\theta\hat{J} - \dot{\theta}L\sin\theta\hat{I}\right]$$

$$\Rightarrow \vec{a}_p = -\Omega^2 r\hat{I} + \dot{\theta}\Omega L\sin\theta\hat{K} + 2\Omega v_{rel}\cos\theta\hat{K}$$
$$+2\dot{\theta}v_{REL}\cos\theta\hat{J} - 2\dot{\theta}v_{REL}\sin\theta$$
$$-\Omega^2 L\cos\theta\hat{I} - \Omega\dot{\theta}L\sin\theta\hat{K} - \dot{\theta}^2 L\cos\theta\hat{I} - \dot{\theta}^2 L\sin\theta\hat{J}$$