MOVING REFERENCE KINEMATICS 2D

2/6/26

Our previous analysis considered specifically the case where points A & B are on the SAME rigid body:

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})$$

However, there are cases where the two points of interest are NOT on the same rigid body, and as such, the above equations are not valid. Typical cases of this situation involve the motion of pins in slots/tracks or telescopic points

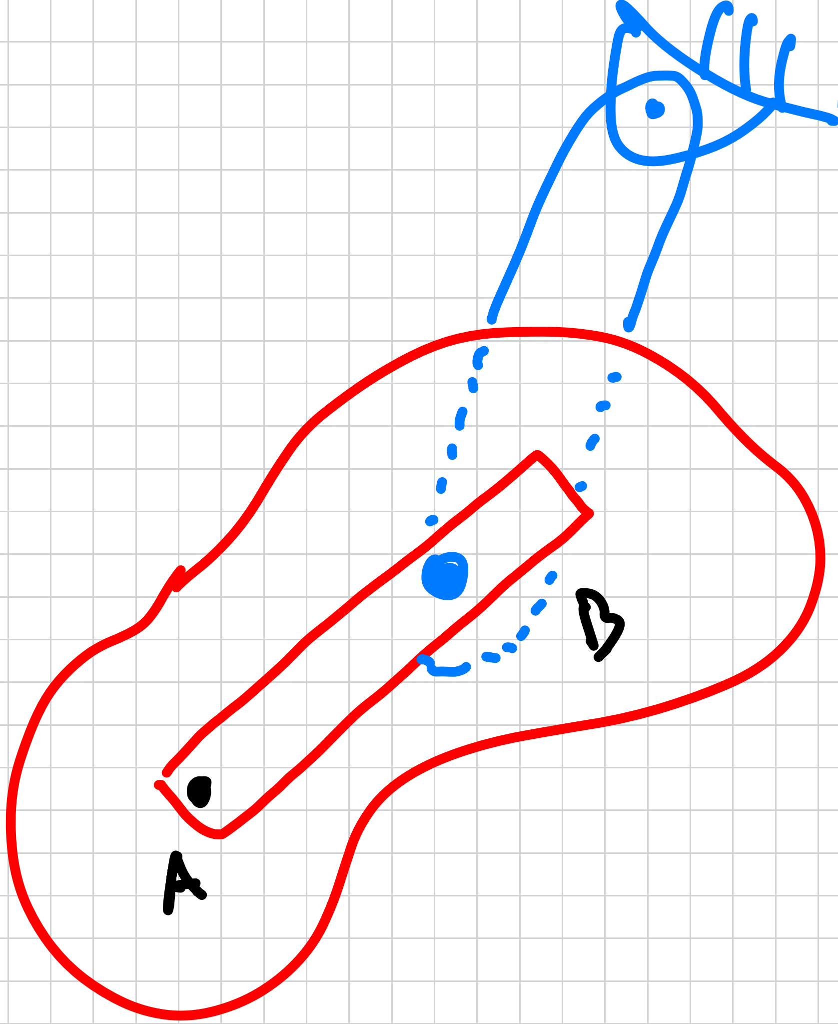
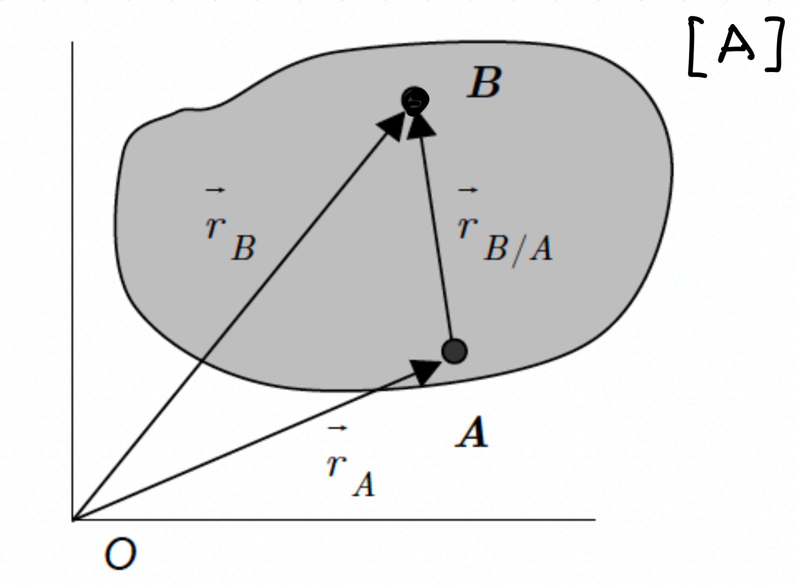
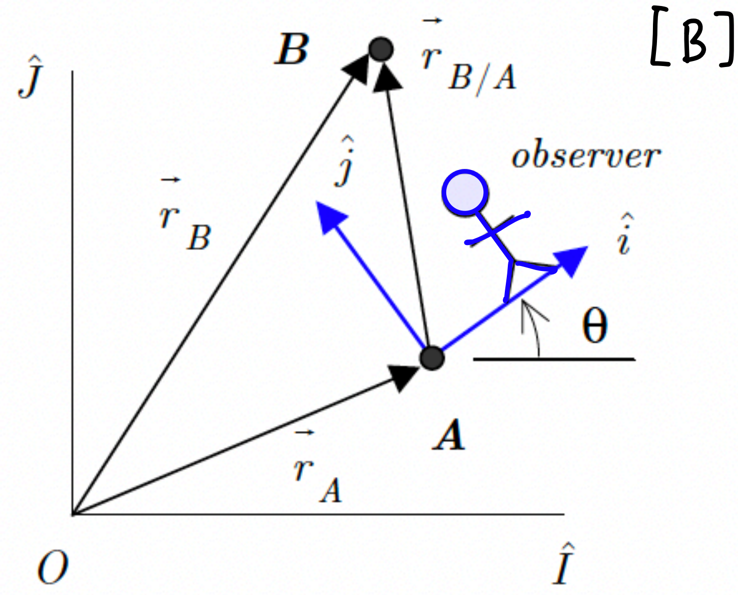

A
B

Figure [A] depicts our previous analysis (a point B, referenced from point A, both in the same rigid body)



Now, what if point B were to move away from A and we wanted to describe its motion? [B]



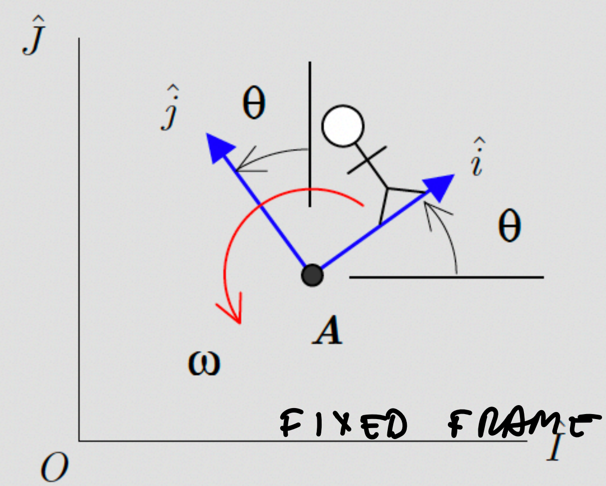
$$\Rightarrow \boxed{\vec{r}_{B/A} = x_{B/A}\,\hat{\imath} + y_{B/A}\,\hat{\jmath}} \longleftarrow$$

Seen by Observer

but $\hat{i}$ and $\hat{j}$ are not fixed (i.e., their directions change with the moving $\hat{i}$-$\hat{j}$ frame.

Let's focus on the observer, who is standing on a moving xyz frame (translating & rotating). Furthermore, we introduce the FIXED reference frame XYZ

The angle of rotation θ is the angle between the X-Y axes and the xy axes.



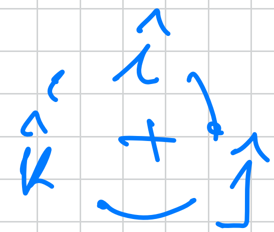
NOT FIXED
$$\begin{cases} \hat{i} = \cos\theta\,\hat{I} + \sin\theta\,\hat{J} \\ \hat{j} = -\sin\theta\,\hat{I} + \cos\theta\,\hat{J} \end{cases} \Rightarrow$$

$$\frac{d\hat{i}}{dt} = -\dot{\theta}\sin\theta\,\hat{I} + \dot{\theta}\cos\theta\,\hat{J} = \dot{\theta}\left(-\sin\theta\,\hat{I} + \cos\theta\,\hat{J}\right)$$
$$= \dot{\theta}\,\hat{j}$$
$$\frac{d\hat{j}}{dt} = -\dot{\theta}\cos\theta\,\hat{I} - \dot{\theta}\sin\theta\,\hat{J}$$
$$= -\dot{\theta}\,\hat{i}$$

Thus, the angular speed of the observer is

$$\frac{d\theta}{dt} = \dot{\theta} \, ,$$

and consequently, the angular velocity of the moving frame is



$$\vec{\omega} = \dot{\theta}\,\hat{k}$$

Let's derive the MOVING FRAME KINEMATIC EQUATIONS. From the figure above, we determined:

$$\frac{d\hat{\imath}}{dt} = \dot{\theta}\,\hat{\jmath} \qquad\qquad \frac{d\hat{\jmath}}{dt} = -\dot{\theta}\,\hat{\imath}$$

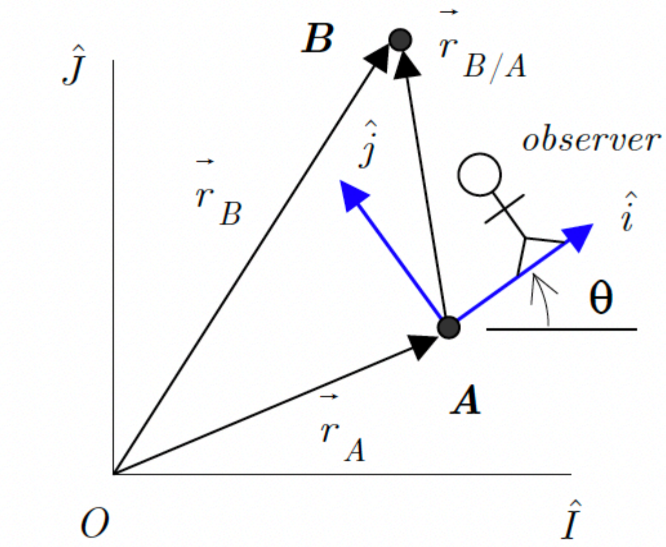
$$\hat{\jmath} = (\hat{k} \times \hat{\imath}) \qquad\qquad \hat{\imath} = -(\hat{k} \times \hat{\jmath})$$

$$\Rightarrow \quad \frac{d\hat{\imath}}{dt} = \underbrace{\dot{\theta}\,(\hat{k}}_{\vec{\omega}} \times \hat{\imath}\,) \qquad\qquad \Rightarrow \quad \frac{d\hat{\jmath}}{dt} = \underbrace{-\dot{\theta}\,[-(\hat{k}}_{\vec{\omega}} \times \hat{\jmath})]$$

$$\boxed{\dot{\hat{\imath}} = \frac{d\hat{\imath}}{dt} = \vec{\omega} \times \hat{\imath}} \qquad\qquad \boxed{\dot{\hat{\jmath}} = \frac{d\hat{\jmath}}{dt} = \hat{\omega} \times \hat{\jmath}}$$

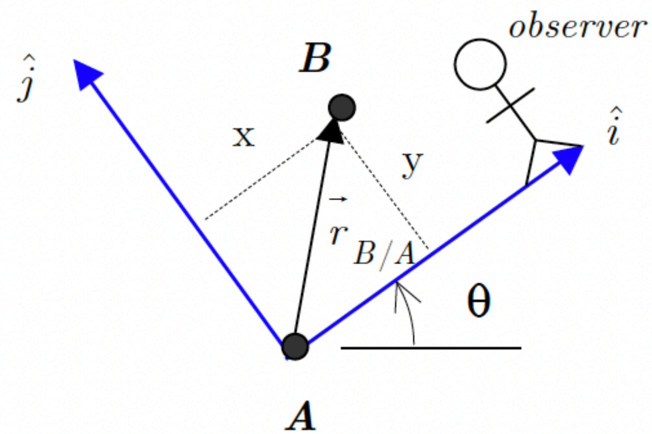
Velocity equation

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A} .$$



But $\vec{r}_{B/A}$ can be written in terms of the observer's xy coordinates: $\vec{r}_{B/A} = x\hat{i} + y\hat{j}$

Differentiating $\vec{r}_B$ with respect to time yields:



$$\frac{d}{dt}\left[\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}\right]$$

$$\vec{v}_B = \vec{v}_A + \dot{x}\hat{i} + x\dot{\hat{i}} + \dot{y}\hat{j} + y\dot{\hat{j}}$$

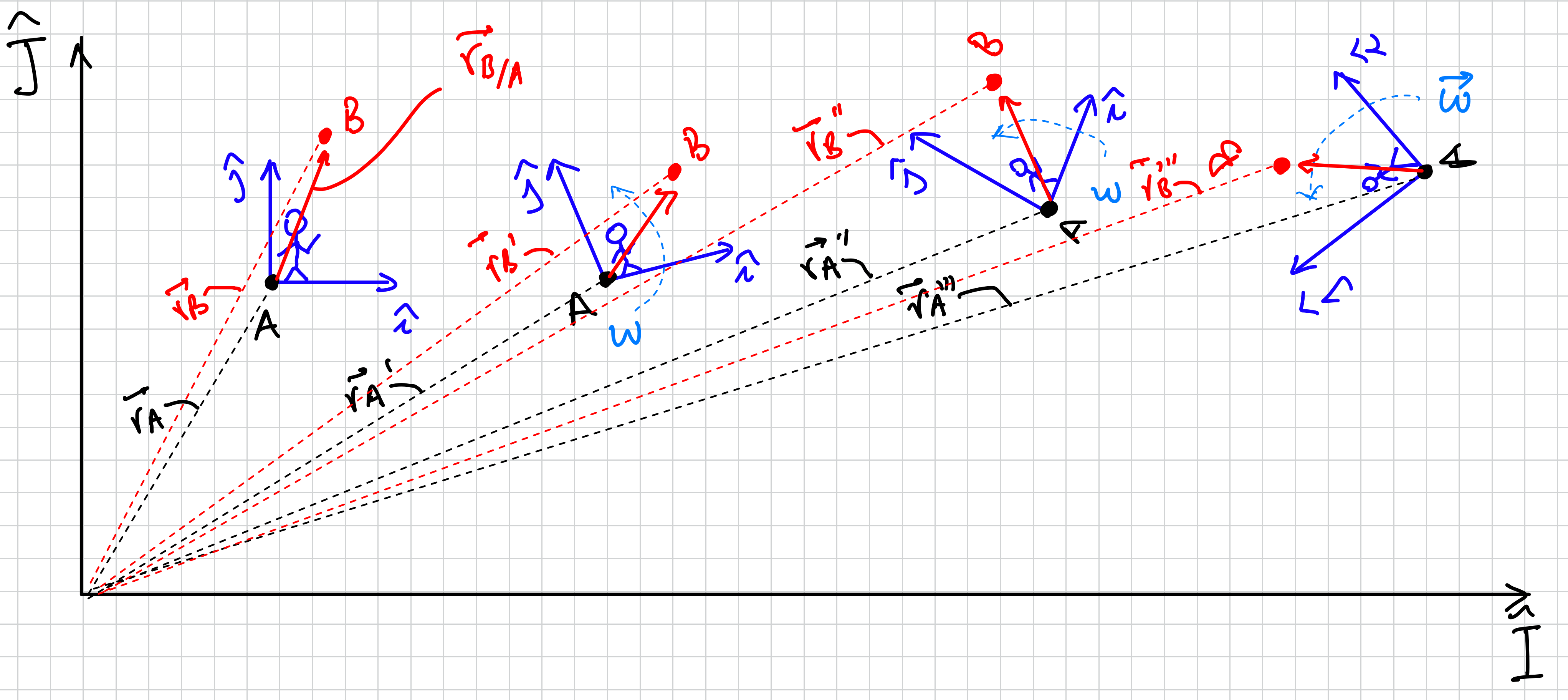
$$\vec{v}_B = \vec{v}_A + \dot{x}\hat{i} + \dot{y}\hat{j} + x(\vec{\omega}\times\hat{i}) + y(\vec{\omega}\times\hat{j})$$

$$= \vec{v}_A + \dot{x}\hat{i} + \dot{y}\hat{j} + \vec{\omega}\times(x\hat{i} + y\hat{j})$$

$$\vec{v}_B = \vec{v}_A + (\vec{v}_{B/A})_{rel} + \vec{\omega}\times\vec{r}_{B/A}$$

The term $\boxed{(\vec{v}_{B/A})_{rel} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}}$ is the velocity of B as seen by the moving observer.

By "moving observer" we mean an observer standing in the moving frame (observer standing, frame moving)



Acceleration equation :

$$\frac{d}{dt}\left[\vec{v}_B = \vec{v}_A + (\vec{v}_{B/A})_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right]$$

$$\vec{a}_B = \vec{a}_A + \frac{d}{dt}(\vec{v}_{B/A})_{rel} + \frac{d}{dt}(\vec{\omega} \times \vec{r}_{B/A})$$

$$\vec{a}_B = \vec{a}_A + \frac{d}{dt}(\vec{v}_{B/A})_{rel} + \frac{d\vec{\omega}}{dt}\vec{r}_{B/A} + \vec{\omega} \times \frac{d}{dt}(\vec{r}_{B/A})$$

$$= \vec{a}_A + \frac{d}{dt}(\dot{x}\hat{i} + \dot{y}\hat{j}) + \frac{d\vec{\omega}}{dt}\vec{r}_{B/A} + \vec{\omega} \times \left[(\vec{v}_{B/A})_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right]$$

$$\vec{a}_B = \vec{a}_A + (\ddot{x}\hat{i} + \ddot{y}\hat{j} + \dot{x}\dot{\hat{i}} + \dot{y}\dot{\hat{j}}) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[(\vec{v}_{B/A})_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right]$$

$$= \vec{a}_A + (\ddot{x}\hat{i} + \ddot{y}\hat{j}) + \dot{x}(\vec{\omega} \times \hat{i}) + \dot{y}(\vec{\omega} \times \hat{j}) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[(\vec{v}_{B/A})_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right]$$

$$= \vec{a}_A + (\ddot{x}\hat{i} + \ddot{y}\hat{j}) + \vec{\omega} \times (\vec{v}_{B/A})_{rel} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[(\vec{v}_{B/A})_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right]$$

$$\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times (\vec{v}_{B/A})_{rel} + \vec{\omega} \times \left[\omega \times \vec{r}_{B/A}\right]$$

where $(\vec{a}_{B/A})_{rel}$ is the acceleration of B as seen by an observer.

$\vec{\alpha}$ is the angular acceleration of the observer:

$$\vec{\alpha} = \frac{dw}{dt}\hat{k}$$

What are all these terms???

$$\vec{v}_B = \vec{v}_A + (\vec{v}_{B/A})_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times (\vec{v}_{B/A})_{rel} + \vec{\omega} \times [\omega \times \vec{r}_{B/A}]$$

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

$\vec{v}_A$ and $\vec{v}_B$: velocities of A and B seen by Fixed observer.

$\vec{a}_A$ and $\vec{a}_B$: accelerations of A and B seen by Fixed observer.

$(\vec{v}_{B/A})_{rel}$: velocity of point B as seen by a moving observer located in $\hat{i}$-$\hat{j}$. Describes how the moving observer "sees" point B.

$(\vec{a}_{B/A})_{rel}$: acceleration of point B as seen by a moving observer located in $\hat{i}$-$\hat{j}$.

$2\vec{\omega} \times (\vec{v}_{B/A})_{rel}$: Coriolis component of acceleration. This is due to an observed velocity of B by the moving observer when the moving observer has a non-zero angular velocity.

What is the advantage of using an “observer” in moving reference kinematics?

A) There are no observers in problems of moving reference kinematics .

B) “What the observer sees” helps visualize the relationship between A and B.

C) “How the observer moves” is useful in establishing the relationship between A and B.

D) There is no advantage. This complicates things more.

The observer him/her/they self is not moving in the $\hat{\imath}$-$\hat{\jmath}$ reference frame what are the differences between $\vec{v}_{B/A}$ & $(\vec{v}_{B/A})_{rel}$, and $\vec{a}_{B/A}$ & $(\vec{a}_{B/A})_{rel}$

$\left.\begin{array}{l}(\vec{v}_{B/A})_{rel}\\ (\vec{a}_{B/A})_{rel}\end{array}\right\}$ are those $\vec{v}$ and $\vec{a}$ SEEN BY THE OBSERVER standing in the MOVING reference frame

$\left.\begin{array}{l}\vec{v}_{B/A}\\ \vec{a}_{B/A}\end{array}\right\}$ are those $\vec{v}$ and $\vec{a}$ when the OBSERVER moving reference frame is translating ONLY, but not rotating $(\vec{\omega}=\vec{\alpha}=0)$ in which case

$$(\vec{v}_{B/A})_{rel} = \vec{v}_{B/A} \quad \text{and} \quad (\vec{a}_{B/A})_{rel} = \vec{a}_{B/A}$$

Procedure for solving MRF problems:

1. Choose your moving reference frame (observer). It is recommended that you draw a stick figure of your observer on this frame to remind yourself of your choice of reference frame. Note that point A must be on this reference frame.

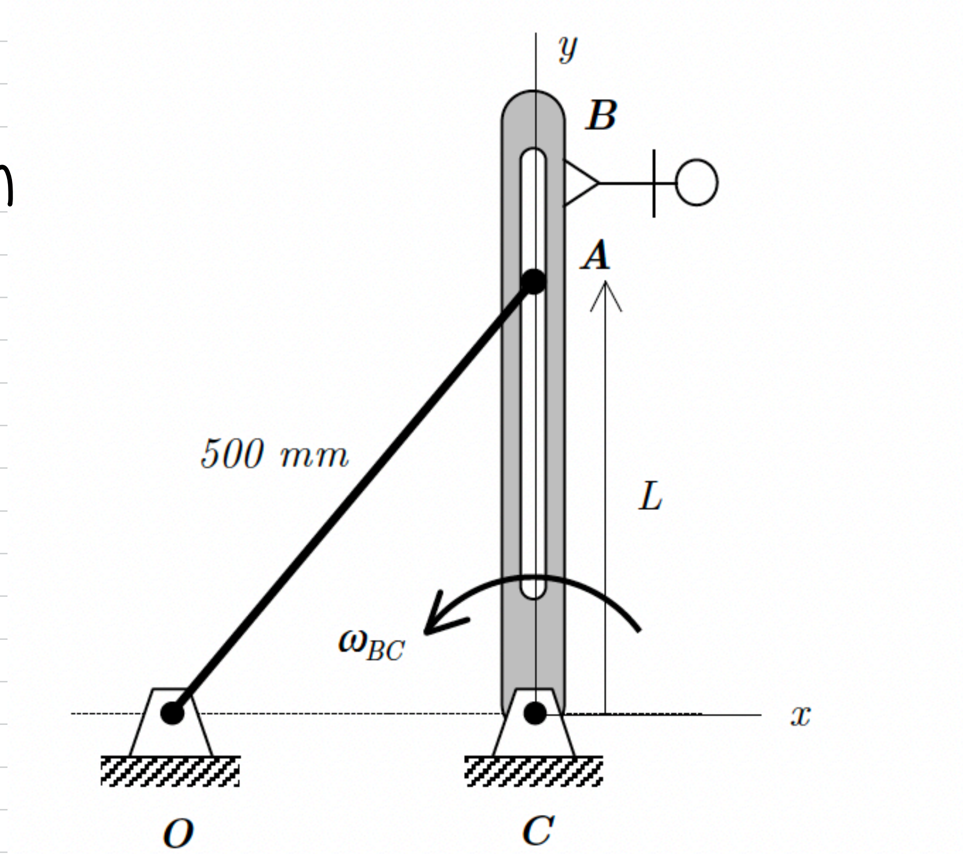
2. Draw your choice of xyz axes for the moving reference frame. State in words to what the xyz axes are attached. Also show your choice of stationary XYZ axes for the problem.

3. Determine the angular velocity $\vec{\omega}$ of the moving reference frame. [Note: this represents the ANGULAR MOTION OF THE OBSERVER, and not what the observer sees.]

4. Determine the angular acceleration $\vec{\alpha}$ of the moving reference frame.

5. Imagine yourself as the observer on the moving reference frame. Answer the question: How do I see point B move if I am that observer? Based on this answer, write down $(\vec{v}_{B/A})_{rel}$ and $(\vec{a}_{B/A})_{rel}$. [Note: this is the MOTION THAT THE OBSERVER SEES, not the motion of the observer.]

When we need to describe the motion of sliding pins (such as A), we cannot use the rigid body kinematics because



pin A is not fixed. So, these should be used:

Link OA (rigid): $\vec{v}_A = \vec{v}_O + \vec{\omega}_{OA} \times \vec{r}_{A/O}$

Link BC (sliding): $\vec{v}_A = \vec{v}_C + (\vec{v}_{A/C})_{rel} + \vec{\omega}_{BC} \times \vec{r}_{A/C}$

Example 3.A.1

Given: The disk shown below rolls without slipping on a horizontal surface. At the instant shown, the center O is moving to the right with a speed of $v_0 = 5$ m/s with this speed decreasing at a rate of 2 m/s^2. Also for this instant, the particle P is at a position of $x_p = 0.2$ m with $\dot{x}_p = 2$ m/s $=$ *constant*, where x_p is measured relative to the xyz coordinate system that is attached to the disk.

Find: Determine:
(a) The velocity of particle P; and
(b) The acceleration of particle P.

Use the following parameters in your analysis: $h = 0.2$ m and $r = 0.6$ m.

$\vec{r}_{P/O} = x_p\hat{i} + h\hat{j}$

$\vec{v}_{P/O} = \dot{x}_p\hat{i} + 0\hat{j}$

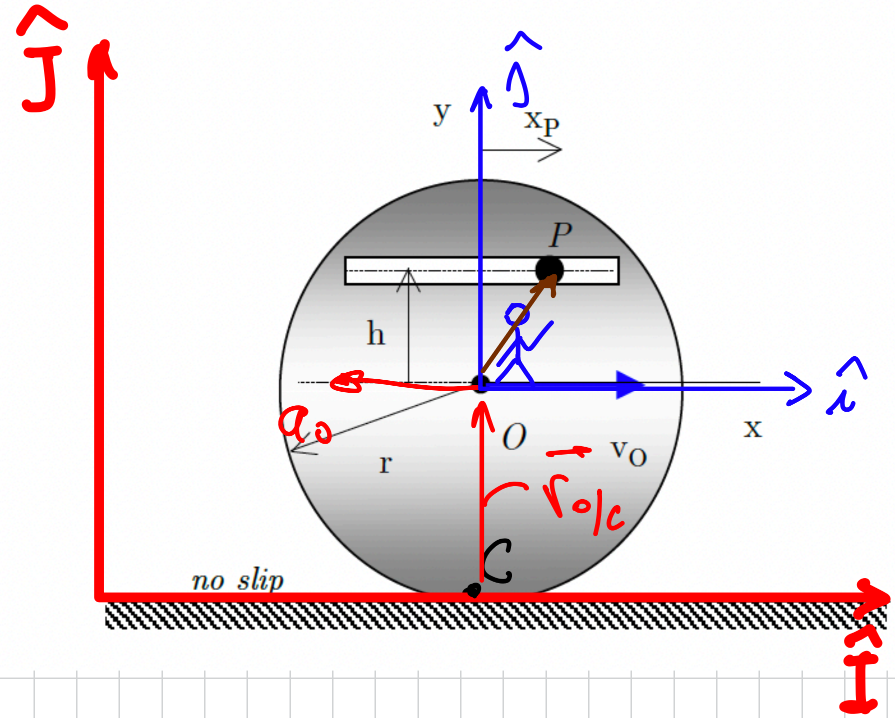
Solution: Find ω, α

$$\vec{v}_O = \cancel{\vec{v}_C}^{\,0} + \vec{\omega} \times \vec{r}_{O/C}$$

$$v_O\hat{I} = \omega\hat{K} \times r\hat{J}$$

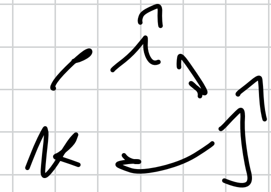
$$v_O\hat{I} = -\omega r\hat{I}$$

$$\hat{I}: \quad \vec{\omega} = -\frac{v_O}{r}\hat{K}$$



$$\vec{a}_O = \vec{a}_C + \vec{\alpha} \times \vec{r}_{O/C} - \omega^2\vec{r}_{O/C}$$

$$a_o \hat{I} = a_c \hat{J} + \alpha \hat{K} \times r \hat{J} - \omega^2 r \hat{J}$$



$$a_o \hat{I} = a_c \hat{J} - \alpha r \hat{I} - \omega^2 r \hat{J}$$

$\hat{I}$: $$a_o = -\alpha r \Rightarrow \vec{\alpha} = -\frac{a_o}{r} \hat{K}$$

Now we do moving ref. frame.

$$\vec{v}_p = v_o + (\vec{v}_{P/O})_{REL} + \omega \times \vec{r}_{P/O}$$

$$= v_o \hat{I} + \dot{x}_p \hat{i} + \left(-\frac{v_o}{r} \hat{k}\right) \times (x_p \hat{i} + h \hat{j})$$

$$= v_o \hat{I} + \dot{x}_p \hat{i} - \frac{v_o}{r} x_p \hat{j} + \frac{v_o}{r} h \hat{i}$$

At instant shown, $\hat{i} = \hat{I}$, $\hat{j} = \hat{J}$

$$\boxed{\vec{v}_p = \left(v_o + \dot{x}_p + \frac{v_o}{r}\right) \hat{I} - \frac{v_o}{r} x_p \hat{J}}$$

ANSWER

(a)

$$\vec{a}_p = a_o \hat{I} + \cancel{(\vec{a}_{p/o})_{REL}} + \underline{\vec{\alpha} \times \vec{r}_{p/o}} + 2\vec{\omega} \times (\vec{v}_{p/o})_{REL} - \omega^2 \vec{r}_{p/o}$$

$$a_o \hat{I} + \alpha \hat{k} \times (x_p \hat{i} + h\hat{j}) + 2\left(-\frac{v_o}{r}\hat{k}\right) \times (\dot{x}_p \hat{i})$$

$$- \omega^2 (x_p \hat{i} + h \hat{j})$$

$$\vec{a}_p = a_o \hat{I} + \left(-\frac{a_o}{r}\hat{k}\right) \times (x_p \hat{i} + h\hat{j}) + 2\left(-\frac{v_o}{r}\hat{k}\right) \times (\dot{x}_p \hat{i}) - \omega^2 (x_p \hat{i} + h\hat{j})$$

$$\vec{a}_p = a_o \hat{I} - \frac{a_o}{r} x_p \hat{j} + \frac{a_o}{r} h \hat{i} - 2\frac{v_o}{r} \dot{x}_p \hat{j} - \frac{v_o^2}{r^2} x_p \hat{i} - \frac{v_o^2}{r^2} h \hat{j}$$

at instant shown, $\hat{I} = \hat{i}$, $\hat{J} = \hat{j}$

$$\therefore \boxed{\vec{a}_p = \left(a_o + \frac{a_o}{r} h + \frac{v_o^2}{r^2} x_p\right)\hat{i} + \left(-\frac{a_o}{r} x_p - \frac{2v_o}{r}\dot{x}_p - \frac{v_o^2}{r^2} h\right)\hat{j}}$$

ANSWER (b)