ME 274: Basic Mechanics II

Lecture 15: Moving Reference Frame Kinematics -3D

PURDUE UNIVERSITY | School of Mechanical Engineering

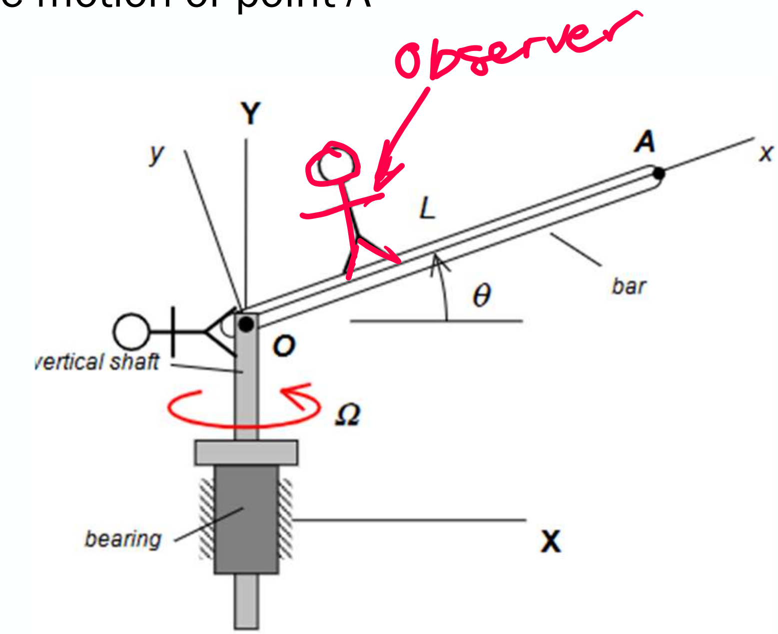
3D Moving Reference Frame Kinematics – Changing observers

Regardless of where you place your observer/ moving reference frame you should arrive at the same final expression for velocity and acceleration.

Problem: A person attached to a moving body is observing the motion of point A

$$\vec{v}_A = \vec{v}_O + \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\vec{r}_{A/O}$$

$$\vec{a}_A = \vec{a}_O + \left(\vec{a}_{A/O}\right)_{rel} + \vec{\alpha}\times\vec{r}_{A/O} + 2\vec{\omega}\times\left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{A/O}\right)$$



Observer on arm OA:

$$\vec{\omega} = \Omega\hat{J} + \dot{\theta}\hat{k}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \dot{\Omega}\hat{J} + \Omega\dot{\hat{J}} + \ddot{\theta}\hat{k} + \dot{\theta}\dot{\hat{k}} = \dot{\theta}\left(\vec{\omega}\times\hat{k}\right)$$

$$\left(\vec{v}_{A/O}\right)_{rel} = \vec{0}$$

$$\left(\vec{a}_{A/O}\right)_{rel} = \vec{0}$$

$\hat{k} = \hat{K}$

$\vec{V}_O = 0$, $\vec{r}_{A/O} = L\hat{i}$

$r_{A/O} = L(\cos\theta\,\hat{I} + \sin\theta\,\hat{J})$

$\vec{V}_A = 0 + 0 + (\Omega\hat{J} + \dot{\theta}\hat{k})\times(L\hat{i})$

$= (\Omega\hat{J} + \dot{\theta}\hat{K})\times L(\cos\theta\,\hat{I} + \sin\theta\,\hat{J})$

$$\boxed{\vec{V}_A = -\Omega L\cos\theta\,\hat{K} + \dot{\theta}L\cos\theta\,\hat{J} - \dot{\theta}L\sin\theta\,\hat{I}}$$

$$\vec{a}_A = \dot\theta(\vec{\omega}\times\hat{K})\times(L\hat{\imath}) \rightarrow (\Omega\hat{J}+\dot\theta\hat{K})\times\left[(\Omega\hat{J}+\dot\theta\hat{K})\times(L\hat{\imath})\right]$$

$$(\vec{\omega}\times\hat{K}) = (\Omega\hat{J}+\dot\theta\hat{K})\times\hat{K} = \Omega\hat{I}$$

$$= \dot\theta\Omega\hat{I}\times L(\cos\theta\hat{I}+\sin\theta\hat{J}) + (\Omega\hat{J}+\dot\theta\hat{K})\times\left[(\Omega\hat{J}+\dot\theta\hat{K})\times L(\cos\theta\hat{I}+\sin\theta\hat{J})\right]$$

$$= \dot\theta\Omega L\sin\theta\hat{K} + (\Omega\hat{J}+\dot\theta\hat{K})\times(-\Omega L\cos\theta\hat{K} + \dot\theta L\cos\theta\hat{J} - \dot\theta L\sin\theta\hat{I})$$

$$= \dot\theta\Omega L\sin\theta\hat{K} - \Omega^2 L\cos\theta\hat{I} + \dot\theta\Omega L\sin\theta\hat{K} - \dot\theta^2 L\cos\theta\hat{I}$$
$$- \dot\theta^2 L\sin\theta\hat{J}$$

$$\vec{a}_A = (-\Omega^2 L\cos\theta - \dot\theta^2 L\cos\theta)\hat{I} - \dot\theta^2 L\sin\theta\hat{J} + 2\dot\theta\Omega L\sin\theta\hat{K}$$

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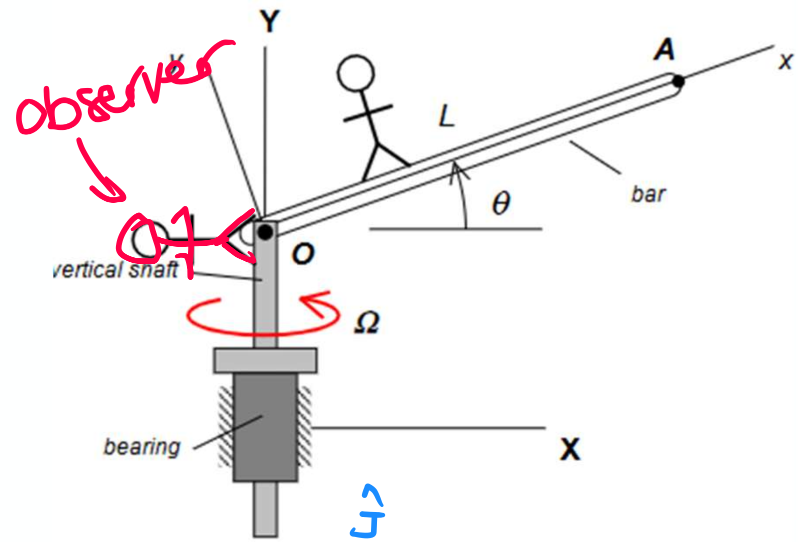
$$\vec{v}_A = \vec{v}_O + \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\vec{r}_{A/O}$$

$$\vec{a}_A = \vec{a}_O + \left(\vec{a}_{A/O}\right)_{rel} + \vec{\alpha}\times\vec{r}_{A/O} + 2\vec{\omega}\times\left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{A/O}\right)$$



Observer on vertical shaft:

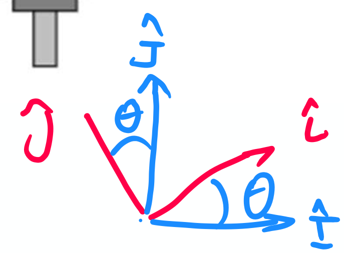
$$\vec{\omega} = \Omega\hat{J}$$

$$\vec{r}_{A/O} = L(\cos\theta\,\hat{I} + \sin\theta\,\hat{J})$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \vec{0}$$

$$\left(\vec{v}_{A/O}\right)_{rel} = L\dot{\theta}\,\hat{j}$$

$$\left(\vec{a}_{A/O}\right)_{rel} = -L\dot{\theta}^2\,\hat{i}$$



$$\vec{V}_A = \vec{0} + L\dot{\theta}\hat{j} + \Omega\hat{J}\times L(\cos\theta\,\hat{I} + \sin\theta\,\hat{J})$$

$$= L\dot{\theta}\hat{j} - \Omega L\cos\theta\,\hat{K} \qquad \hat{j} = -\sin\theta\,\hat{I} + \cos\theta\,\hat{J}$$

$$\vec{V}_A = -L\dot{\theta}\sin\theta\,\hat{I} + L\dot{\theta}\cos\theta\,\hat{J} - \Omega L\cos\theta\,\hat{K}$$

$$\boxed{\vec{V}_A = -\Omega L\cos\theta\,\hat{K} + \dot{\theta}L\cos\theta\,\hat{J} - \dot{\theta}L\sin\theta\,\hat{I}}$$

✓ matches other case

$$\vec{a}_A = 0 - L\dot{\theta}^2 \hat{i} + 0 + 2\Omega\hat{J} \times (L\dot{\theta}\hat{j}) + \Omega\hat{J} \times [\Omega\hat{J} \times (L\cos\theta\,\hat{I} + L\sin\theta\,\hat{J})]$$

$$= -L\dot{\theta}^2(\cos\theta\,\hat{I} + \sin\theta\,\hat{J}) + 2\Omega J \times [L\dot{\theta}(-\sin\theta\,\hat{I} + \cos\theta\,\hat{J})]$$

$$+ \Omega\hat{J} \times [\Omega\hat{J} \times (L\cos\theta\,\hat{I} + L\sin\theta\,\hat{J})]$$

$$= -L\dot{\theta}^2\cos\theta\,\hat{I} - L\dot{\theta}^2\sin\theta\,\hat{J} + 2L\Omega\dot{\theta}\sin\theta\,\hat{K} + \Omega\hat{J} \times (-\Omega L\cos\theta\,\hat{K})$$

$$= -L\dot{\theta}^2\cos\theta\,\hat{I} - L\dot{\theta}^2\sin\theta\,\hat{J} + 2L\Omega\dot{\theta}\sin\theta\,\hat{K} - \Omega^2 L\cos\theta\,\hat{I}$$

$$\vec{a}_A = (-L\dot{\theta}^2\cos\theta - \Omega^2 L\cos\theta)\hat{I} - L\dot{\theta}^2\sin\theta\,\hat{J} + 2L\Omega\dot{\theta}\sin\theta\,\hat{K}$$

$$\vec{a}_A = (-\Omega^2 L\cos\theta - \dot{\theta}^2 L\cos\theta)\hat{I} - \dot{\theta}^2 L\sin\theta\,\hat{J} + 2\dot{\theta}\Omega L\sin\theta\,\hat{K}$$

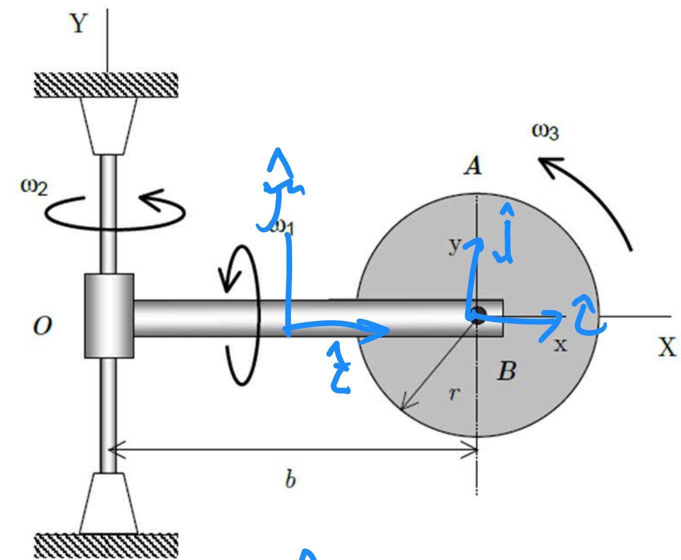
Example 3.B.11

Given: Rotation rates ω_1, ω_2 and ω_3 are all constant. The XYZ axes are fixed, and the xyz axes are attached to the disk. At the instant shown, A is directly above the center B of the disk and the xyz and XYZ axes are aligned.

Find: Determine:

(a) The velocity of point A; and

(b) The acceleration of point A.



$\hat{i}\,\hat{j}\,\hat{k}$ → attached to disk

$\hat{i}\,\hat{j}\,\hat{k}$ → attached to OB

$\hat{I}\,\hat{J}\,\hat{K}$ → fixed

rotation of $\hat{i}\,\hat{j}\,\hat{k}$

$\vec{\omega}_{OB} = \omega_2\hat{J} - \omega_1\hat{i}$

$\vec{\alpha}_{OB} = \cancel{\dot{\omega}_2\hat{J}} + \cancel{\omega_2\dot{\hat{J}}} - \cancel{\dot{\omega}_1\hat{i}} - \omega_1\dot{\hat{i}}$

$\vec{\alpha}_{OB} = -\omega_1(\vec{\omega}_{OB}\times\hat{i})$

rotation of $\hat{i}\,\hat{j}\,\hat{k}$

$\vec{\omega}_{AB} = \omega_2\hat{J} - \omega_1\hat{i} - \omega_3\hat{k}$

$\alpha_{AB} = \cancel{\dot{\omega}_2\hat{J}} + \cancel{\omega_2\dot{\hat{J}}} - \cancel{\dot{\omega}_1\hat{i}} - \omega_1\dot{\hat{i}} - \cancel{\dot{\omega}_3\hat{k}} - \omega_3\dot{\hat{k}}$

$= \omega_1(\vec{\omega}_{OB}\times\hat{i}) + \omega_3(\vec{\omega}_{AB}\times\hat{k})$

$= \omega_1[(\omega_2\hat{J} - \omega_1\hat{i})\times\hat{i}] + \omega_3[(\omega_2\hat{J} + \omega_1\hat{i} - \omega_3\hat{k})\times\hat{k}]$

Example 3.B.11

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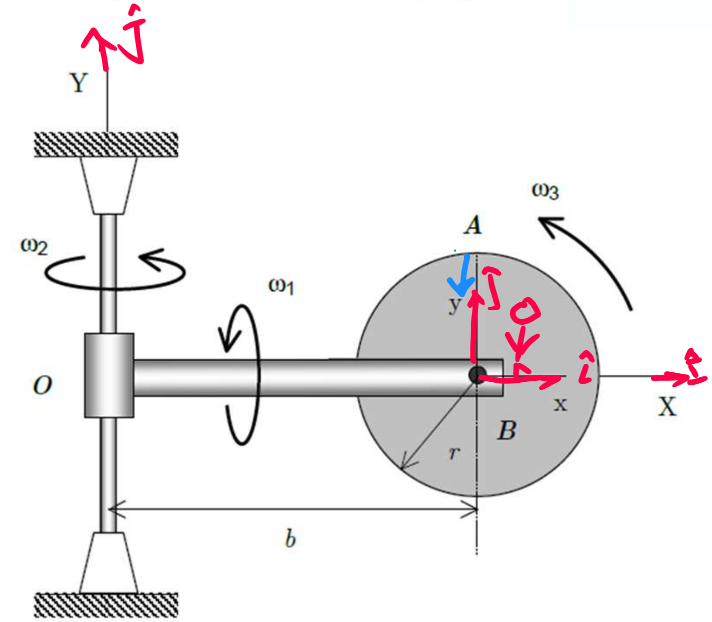
arm OB



Find: Determine:

(a) The velocity of point A; and

(b) The acceleration of point A.

$\vec{\omega}$ & $\vec{\alpha}$

at this instant $\hat{i} = \hat{I}$

$$\vec{\omega} = \omega_2 \hat{j} - \omega_1 \hat{i}$$
$$= \omega_2 \hat{J} - \omega_1 \hat{I}$$
$$\vec{\alpha} = \cancel{\dot{\omega}_2 \hat{j} + \omega_2 \dot{\hat{j}} - \dot{\omega}_1 \hat{i}} - \omega_1 \dot{\hat{i}}$$
$$= -\omega_1 (\vec{\omega} \times \hat{i}) = -\omega_1 [(\omega_2 \hat{j} - \omega_1 \hat{i}) \times \hat{i}]$$
$$= \omega_1 \omega_2 \hat{K}$$

a) velocity of A

$$\vec{V}_A = \vec{V}_B + (\vec{V}_{A/B})_{rel} + \vec{\omega} \times \vec{r}_{AB}$$

rigid Body for V_B → $\vec{V}_B = \cancel{\vec{V}_O} + \vec{\omega} \times \vec{r}_{B/O}$

$$= \omega_2 \hat{J} \times b \hat{I} = -\omega_2 b \hat{K}$$

$$\vec{v}_A = -\omega_2 b\hat{K} + (-r\omega_3\hat{i}) + (\omega_2\hat{J} - \omega_1\hat{i}) \times r\hat{j}$$

$$= -\omega_2 b\hat{K} - r\omega_3\hat{I} - \omega_1 r\hat{K}$$

$$\boxed{= -r\omega_3\hat{I} + (-\omega_1 r - \omega_2 b)\hat{K}}$$

b) acceleration of A

$$\vec{a}_A = \vec{a}_B + (\vec{a}_{A/B})_{rel} + \vec{\alpha} \times \vec{r}_{A/B} + 2\vec{\omega} \times (\vec{v}_{A/B})_{rel} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/B})$$

rigid body for $\vec{a}_B$

$$\vec{a}_B = \vec{a}_0 + \vec{\alpha}_{BO} \times \vec{r}_{B/O} + \vec{\omega}_{BO} \times (\vec{\omega}_{BO} \times r_{B/O})$$

$$= \vec{0} + \vec{0} + \omega_2\hat{J} \times (\omega_2\hat{J} \times b\hat{i}) = -\omega_2^2 b\hat{I}$$

$$\vec{a}_A = -\omega_2^2 b\hat{I} - \omega_3^2 r\hat{j} + \omega_1\omega_2\hat{K} \times r\hat{j} + 2(\omega_2\hat{J} - \omega_1\hat{i}) \times (-r\omega_3\hat{i})$$

$$+ (\omega_2\hat{J} - \omega_1\hat{i}) \times [(\omega_2\hat{J} - \omega_1\hat{i}) \times r\hat{j}]$$

$$= -\omega_2^2 b \hat{I} - \omega_3^2 r \hat{J} - \omega_1 \omega_2 r \hat{I} + 2 \omega_2 \omega_3 r \hat{K}$$

$$+ (\omega_2 \hat{J} - \omega_1 \hat{I}) \times [-\omega_1 r \hat{K}]$$

$$= -\omega_2^2 b \hat{I} - \omega_3^2 r \hat{J} - \omega_1 \omega_2 r \hat{I} + 2 \omega_2 \omega_3 r \hat{K} - \omega_1 \omega_2 r \hat{I} - \omega_1^2 r \hat{J}$$

$$\boxed{\vec{a}_A = (-\omega_2^2 b - 2\omega_1 \omega_2 r)\hat{I} + (-\omega_3^2 r - \omega_1^2 r)\hat{J} + 2\omega_2 \omega_3 r \hat{K}}$$