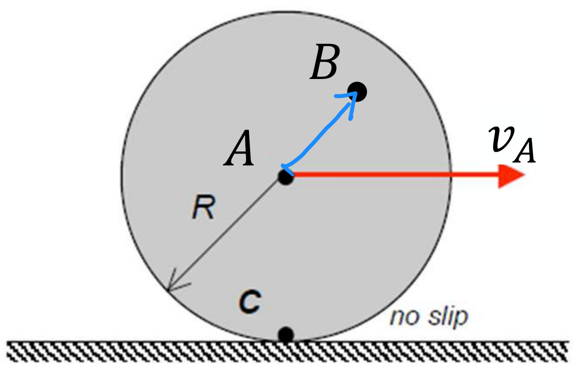
ME 274: Basic Mechanics II

Lecture 11: Moving Reference Frame Kinematics

PURDUE UNIVERSITY | School of Mechanical Engineering

Previously: Rigid body kinematics



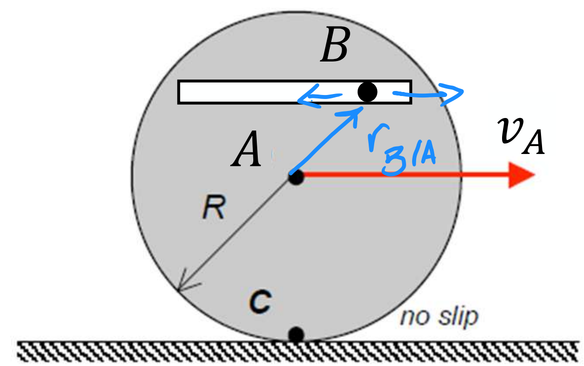
- Points A and B are on the same body
- $r = |\vec{r}_{B/A}|$=CONSTANT and $\dot{r} = 0$

Velocity and acceleration of point B with respect to point A:

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\omega} \times [\vec{\omega} \times \vec{r}_{B/A}] + \vec{\alpha} \times \vec{r}_{B/A}$$

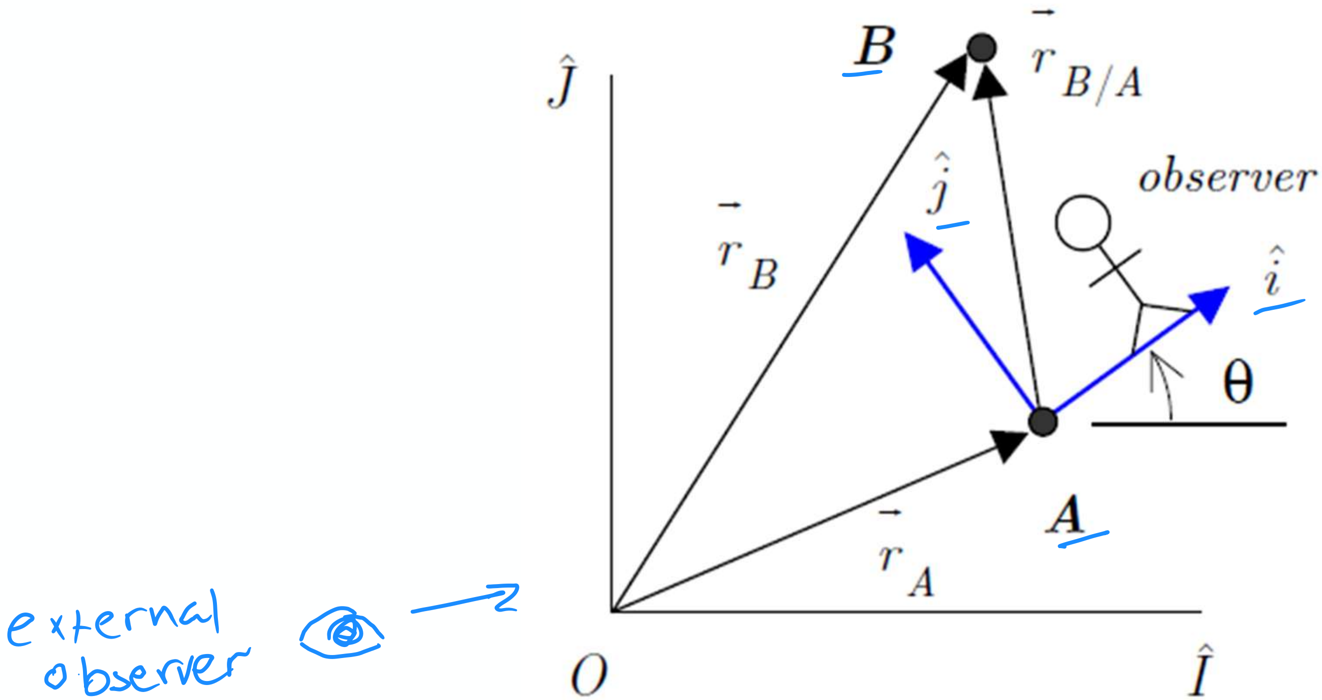
Today: Moving reference frame kinematics

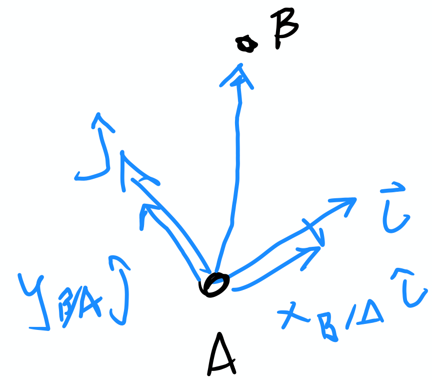


- Points A and B are NOT on the same body
- $r = |\vec{r}_{B/A}| \neq$ CONSTANT and $\dot{r} \neq 0$

How do we relate the motion of the two bodies?

Moving Reference Frames





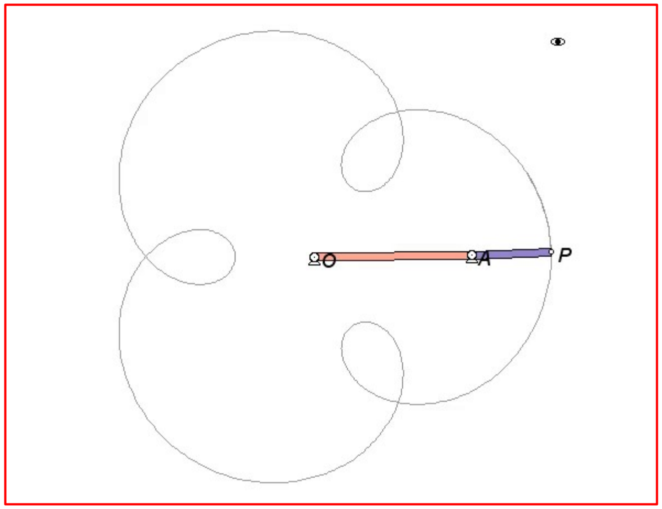
- Stationary axis ⟶ XYZ axis ⟶ $\hat{I}, \hat{J}, \hat{K}$ unit vectors
- Moving reference frame (translating and rotating) ⟶ xyz axis ⟶ $\hat{\imath}, \hat{\jmath}, \hat{k}$ unit vectors
- The observer at point A describes the position of point B as:

$$\vec{r}_{B/A} = x_{B/A}\hat{i} + y_{B/A}\hat{j}$$

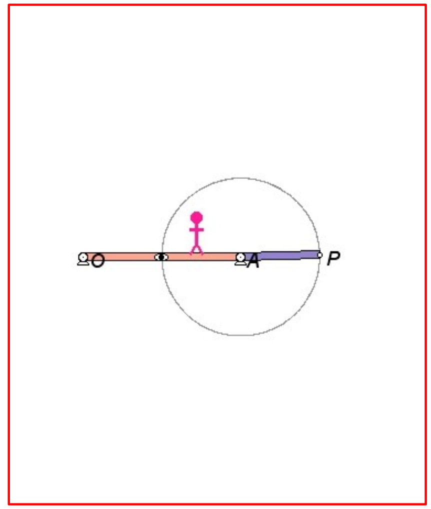
- NOTE: the vectors $\hat{\imath}, \hat{\jmath}$ rotate with the observer – their orientations change with time!

Animation: Views from different reference frames

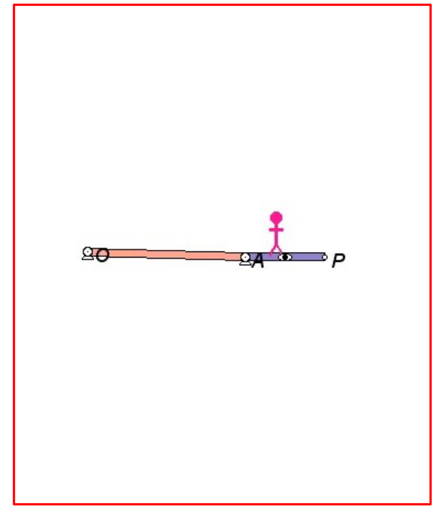
External Observer



Observer on OA



Observer on AP



→ too complex

→ simplified

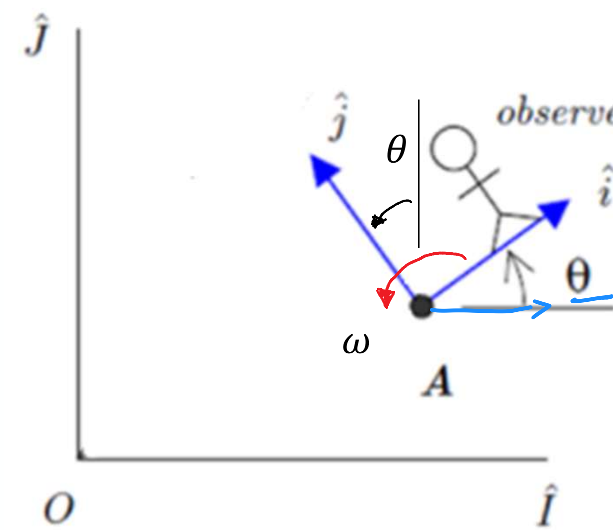
→ no motion

Question: Which observer do you think would be most useful in calculating the velocity and acceleration of point P?

Angular Velocity of the Rotating Reference Frame

- θ is angle between the x and X axis and the y and Y axis.
- The angular speed of the observer is given by the time rate of change of the angle θ

$$\frac{d\theta}{dt} = \dot{\theta}, \quad \vec{\omega} = \dot{\theta}\vec{k}$$



- $\hat{\imath}, \hat{\jmath}$ unit vectors are a constant length, but change direction ⟶ time derivatives $\neq \mathbf{0}$

  Writing $\hat{\imath}$ and $\hat{\jmath}$ in terms of their components in the $\hat{I}$ and $\hat{J}$ direction:

$$\hat{i} = \cos\theta\,\hat{I} + \sin\theta\,\hat{J}$$

$$\hat{j} = -\sin\theta\,\hat{I} + \cos\theta\,\hat{J}$$

$$\frac{d\hat{i}}{dt} = -\dot{\theta}\sin\theta\,\hat{I} + \dot{\theta}\cos\theta\,\hat{J} = \dot{\theta}\hat{j}$$

$$\frac{d\hat{j}}{dt} = -\dot{\theta}\cos\theta\,\hat{I} - \dot{\theta}\sin\theta\,\hat{J} = -\dot{\theta}\hat{i}$$

- Using the right-hand rule we know $\hat{\imath} = -\hat{k} \times \hat{\jmath}$, $\hat{\jmath} = \hat{k} \times \hat{\imath}$

$$\frac{d\hat{i}}{dt} = \dot{\theta}(\hat{k} \times \hat{i}) = \vec{\omega} \times \hat{i}$$

$$\frac{d\hat{j}}{dt} = -\dot{\theta}(-\hat{k} \times \hat{j}) = \vec{\omega} \times \hat{j}$$

Velocity Equation – 2D

The position of point B written with respect to point A:

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

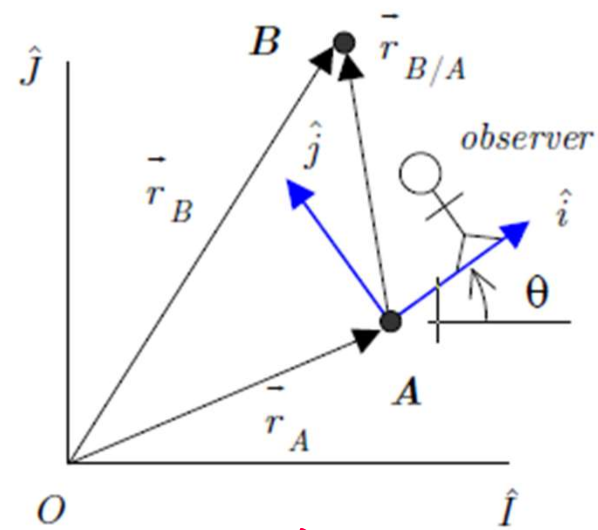
The vector $\vec{r}_{B/A}$ written in terms of the observer's xy coordinates:

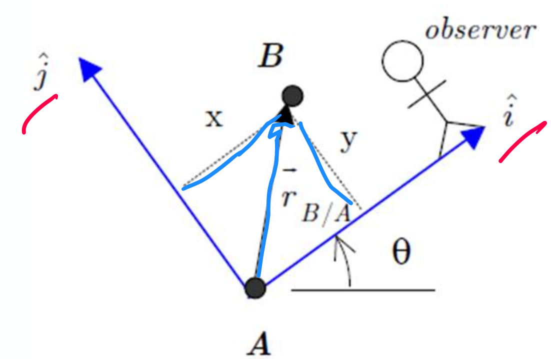
$$\vec{r}_{B/A} = x\hat{i} + y\hat{j}$$

Differentiating with respect to time (remember $\hat{\imath}, \hat{\jmath}$ vary with time):

$$\frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\vec{r}_{B/A}}{dt} \qquad \Longrightarrow$$

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \frac{d}{dt}\left(x\hat{i} + y\hat{j}\right) \\
&= \vec{v}_A + \frac{dx}{dt}\hat{i} + x\frac{d\hat{i}}{dt} + \frac{dy}{dt}\hat{j} + y\frac{d\hat{j}}{dt} \\
&= \vec{v}_A + \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + x\left(\vec{\omega} \times \hat{i}\right) + y\left(\vec{\omega} \times \hat{j}\right) \\
&= \vec{v}_A + \dot{x}\hat{i} + \dot{y}\hat{j} + \vec{\omega} \times \left(x\hat{i} + y\hat{j}\right) \\
&= \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}
\end{aligned}$$





where $\vec{\omega}$ is the angular velocity of the observer at A and $\left(\vec{v}_{B/A}\right)_{rel}$ is the "velocity of B as seen by the observer at A".

Acceleration Equation – 2D

Starting with our equation for the velocity of point B:

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

And differentiating with respect to time:

product rule

$$\frac{d\vec{v}_B}{dt} = \frac{d\vec{v}_A}{dt} + \frac{d}{dt}\left(\vec{v}_{B/A}\right)_{rel} + \frac{d}{dt}\left(\vec{\omega} \times \vec{r}_{B/A}\right) \implies$$

previous slide

$$\vec{a}_B = \vec{a}_A + \frac{d}{dt}\left(\vec{v}_{B/A}\right)_{rel} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \frac{d\vec{r}_{B/A}}{dt}$$

$$= \vec{a}_A + \frac{d}{dt}\left(\dot{x}\hat{i} + \dot{y}\hat{j}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \omega \times \vec{r}_{B/A}\right]$$

product rule

$$= \vec{a}_A + \left(\ddot{x}\hat{i} + \ddot{y}\hat{j}\right) + \left(\dot{x}\frac{d\hat{i}}{dt} + \dot{y}\frac{d\hat{j}}{dt}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right]$$

$$= \vec{a}_A + \left(\ddot{x}\hat{i} + \ddot{y}\hat{j}\right) + \dot{x}\left(\vec{\omega} \times \hat{i}\right) + \dot{y}\left(\vec{\omega} \times \hat{j}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \omega \times \vec{r}_{B/A}\right]$$

distributive prop.

$$\vec{\omega} \times \left(\dot{x}\hat{i} + \dot{y}\hat{j}\right)$$

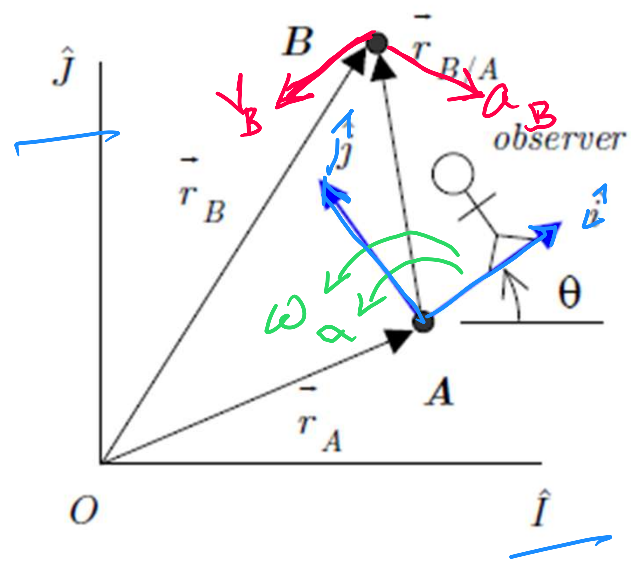
$$= \vec{a}_A + \left(\ddot{x}\hat{i} + \ddot{y}\hat{j}\right) + \vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \omega \times \vec{r}_{B/A}\right]$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$$

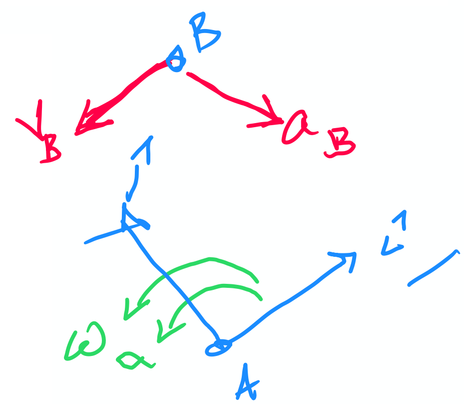
What do each of these terms represent?

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$
$$= \vec{v}_A + \vec{v}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$$
$$= \vec{a}_A + \vec{a}_{B/A}$$



- $\vec{v}_A$ and $\vec{v}_B$ are the velocities of points B and A as seen by a fixed observer.
- $\vec{a}_A$ and $\vec{a}_B$ are the accelerations of points B and A as seen by a fixed observer.
- $\vec{\omega}$ is angular velocity of the moving observer.
- $\vec{\alpha}$ is the angular acceleration of the moving observer.
- $\left(\vec{v}_{B/A}\right)_{rel}$ is the "velocity of point B as seen by the moving observer at A". In order to write down this term, you must clearly understand how the moving observer views the motion of B.
- $\left(\vec{a}_{B/A}\right)_{rel}$ is the "acceleration of point B as seen by the moving observer at A". In order to write down this term, you must clearly understand how the moving observer views the motion of B.
- The term $2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel}$ is known as the "Coriolis" component of acceleration. The term arises due to an observed velocity of B by the moving observer, when the observer has a non-zero angular velocity. We will introduce a number of physical examples that demonstrate the significance of this term during lecture.



$-\vec{V}_B \hat{i} = (V_{B/A})_{rel}$

$a_{Bx}\hat{i} + a_{By}\hat{j} = (a_{B/A})_{rel}$

CHALLENGE QUESTIONS: What are the differences between $\vec{v}_{B/A}$ and $\left(\vec{v}_{B/A}\right)_{rel}$? And between $\vec{a}_{B/A}$ and $\left(\vec{a}_{B/A}\right)_{rel}$? Are they ever the same?

Starting with the equations we derived earlier:

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$$

$$\left(\vec{V}_{B/A}\right)_{rel} = \underbrace{\vec{V}_B - \vec{V}_A}_{} - \vec{\omega} \times \vec{r}_{B/A}$$

$$\left(\vec{V}_{B/A}\right)_{rel} = \vec{V}_{B/A} + \cancel{\vec{\omega} \times \vec{r}_{B/A}}^{\,0}$$

Discussion – The "How To" with Moving Reference Frame Kinematics Equations

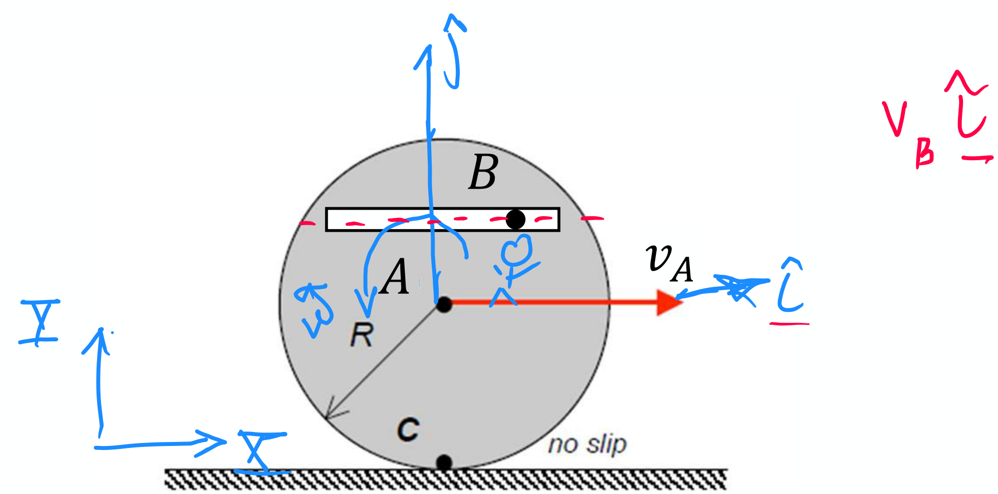
1. Choose your moving reference frame (observer). It is recommended that you draw a stick figure of your observer on this frame to remind yourself of your choice of reference frame. Note that point A must be on this reference frame.

2. Draw your choice of xyz axes for the moving reference frame. State in words to what the xyz axes are attached. Also show your choice of stationary XYZ axes for the problem.

3. Determine the angular velocity $\vec{\omega}$ of the moving reference frame. [Note: this represents the ANGULAR MOTION OF THE OBSERVER, and not what the observer sees.]

4. Determine the angular acceleration $\vec{\alpha}$ of the moving reference frame.

5. Imagine yourself as the observer on the moving reference frame. Answer the question: How do I see point B move if I am that observer? Based on this answer, write down $(\vec{v}_{B/A})_{rel}$ and $(\vec{a}_{B/A})_{rel}$. [Note: this is the MOTION THAT THE OBSERVER SEES, not the motion of the observer.]



Example 3.A.1

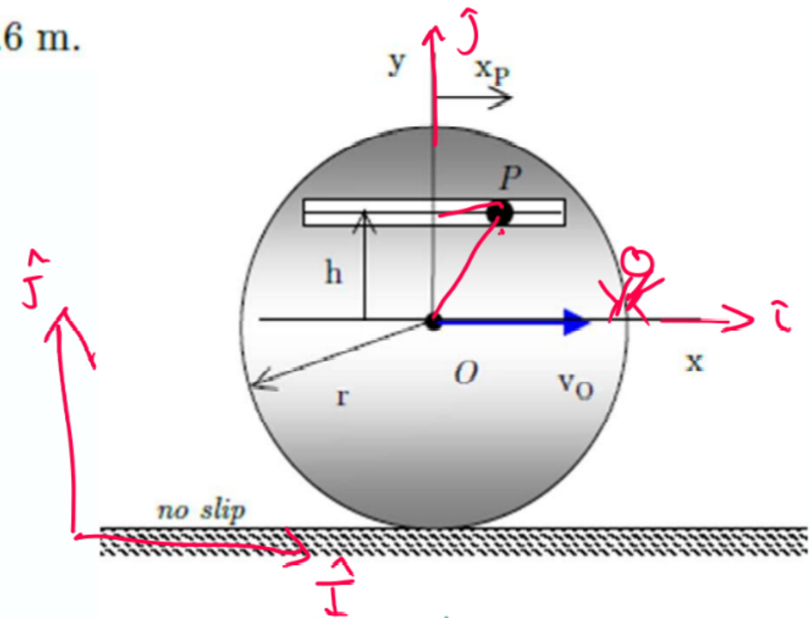
Given: The disk shown below rolls without slipping on a horizontal surface. At the instant shown, the center O is moving to the right with a speed of $v_0 = 5$ m/s with this speed decreasing at a rate of 2 m/s^2. Also for this instant, the particle P is at a position of $x_p = 0.2$ m with $\dot{x}_p = 2$ m/s $=$ *constant*, where x_p is measured relative to the xyz coordinate system that is attached to the disk.

$a_0 = -2$
$\ddot{x}_p = 0$

Use the following parameters in your analysis: $h = 0.2$ m and $r = 0.6$ m.

Find: Determine:

(a) The velocity of particle P; and
(b) The acceleration of particle P.



$\vec{v}_P = \vec{v}_O + (\vec{v}_{P/O})_{rel} + \vec{\omega} \times r_{P/O}$

ang vel of observer

$\vec{\omega} = -\frac{v_0}{r}\hat{k}$ ← rolling w/o slip

$r_{P/O} = x_p\hat{i} + h\hat{j}$

$(v_{P/O})_{rel} = \dot{x}_p\hat{i}$

$v_P = v_0\hat{I} + \dot{x}_p\hat{i} + \left(-\frac{v_0}{r}\hat{k}\right) \times (x_p\hat{i} + h\hat{j})$

$= v_0\hat{I} + \dot{x}_p\hat{i} - \frac{v_0}{r}x_p\hat{j} + \frac{v_0 h}{r}\hat{i}$

convert to one coordinate set → $\hat{i} = \hat{I}$ $\quad \hat{j} = \hat{J}$

$v_P = \left(v_0 + \dot{x}_p + \frac{v_0 h}{r}\right)\hat{i} - \frac{v_0 x_p}{r}\hat{j}$

solving for α:

$a_C = a_0 + \alpha \times r_{C/O} - \omega^2 r_{C/O}$

$= a_0\hat{i} + \alpha\hat{k} \times (-r\hat{j}) - \left(\frac{v_0}{r}\right)^2(-r\hat{j})$

$\hat{i}: \; 0 = a_0 + r\alpha \rightarrow \alpha = -\frac{a_0}{r}$

acceleration: $\dot{x}_p = \text{const}$

$$\vec{a}_p = \vec{a}_o + \cancel{(\vec{a}_{p/o})_{rel}}^{\,0} + \vec{\alpha} \times \vec{r}_{p/o} + 2\vec{\omega} \times (\vec{V}_{p/o})_{rel} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{p/o})$$

$$= a_o\hat{i} + \left(-\frac{a_o}{r}\hat{k}\right) \times (x_p\hat{i} + h\hat{j}) + 2\left(-\frac{V_o}{r}\hat{k}\right) \times (\dot{x}\hat{i}) - \left(\frac{V_o}{r}\right)^2 (x_p\hat{i} + h\hat{j})$$

$$= a_o\hat{i} - \frac{a_o x_p}{r}\hat{j} + \frac{a_o h}{r}\hat{i} - \frac{2V_o\dot{x}}{r}\hat{j} - \frac{V_o^2 x_p}{r^2}\hat{i} - \frac{V_o^2 h}{r^2}\hat{j}$$

$$a_p = \left(a_o + \frac{a_o h}{r} - \frac{V_o^2 x_p}{r^2}\right)\hat{i} + \left(-\frac{a_o x_p}{r} - \frac{2V_o \dot{x}}{r} - \frac{V_o^2 h}{r^2}\right)\hat{j}$$

Example 3.A.3

Given: The fire truck moves forward at a constant speed of 50 ft/s. The ladder is being raised at a constant rate of $\dot{\theta} = 0.3$ rad/s. In addition, the ladder is being extended at a constant rate of $\dot{b} = 2$ ft/s.

Find: The acceleration of end P of the ladder when $b = 25$ ft and $\theta = 30°$.

$$\vec{a}_P = \vec{a}_O + (\vec{a}_{P/O})_{rel} + \vec{\alpha}\times\vec{r}_{P/O} + 2\vec{\omega}\times(\vec{v}_{P/O})_{rel} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{P/O})$$

$$= 0 + 0 + 0 + 2\dot{\theta}\hat{k}\times(\dot{b}\,\hat{i}) - \dot{\theta}^2 b\,\hat{i}$$

$$\vec{a}_P = 2\dot{\theta}\dot{b}\,\hat{j} - \dot{\theta}^2 b\,\hat{i}$$