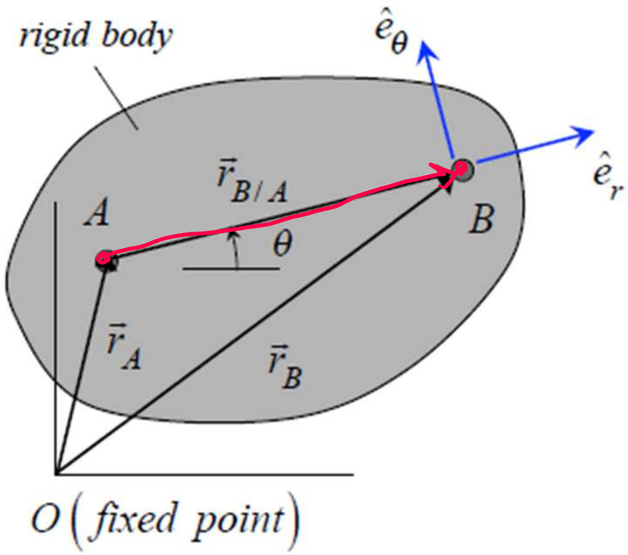
ME 274: Basic Mechanics II

Lecture 7: Rigid Body Motion: Rolling without Slipping

PURDUE UNIVERSITY
School of Mechanical Engineering

Review: Rigid Body Motion

If we know the motion of one point and the rotation of the body, we can determine the motion of any point on the body



Rigid body:

$$r = \left| \vec{r}_{B/A} \right| = \text{CONSTANT and } \dot{r} = 0$$

Velocity and acceleration of point B with respect to point A:

$$\vec{v}_B = \vec{v}_A + \underbrace{\vec{\omega}}_{\text{ang. velocity}} \times \vec{r}_{B/A}$$

→ 2D & 3D

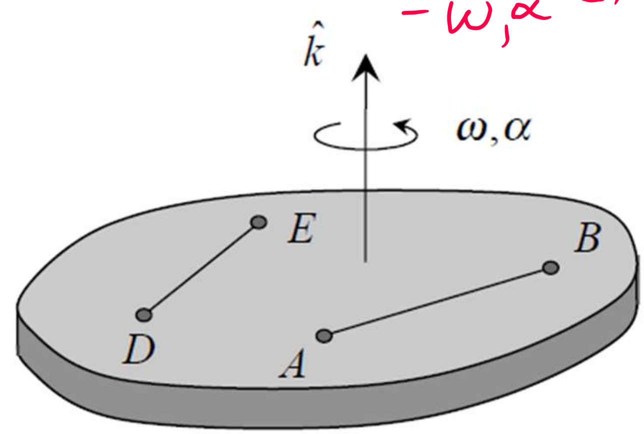
$$\vec{a}_B = \vec{a}_A + \vec{\omega} \times [\vec{\omega} \times \vec{r}_{B/A}] + \underbrace{\vec{\alpha}}_{\text{ang accel.}} \times \vec{r}_{B/A}$$

→ 3D

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}.$$

→ 2D

+ω,α → ccw
−ω,α → cw

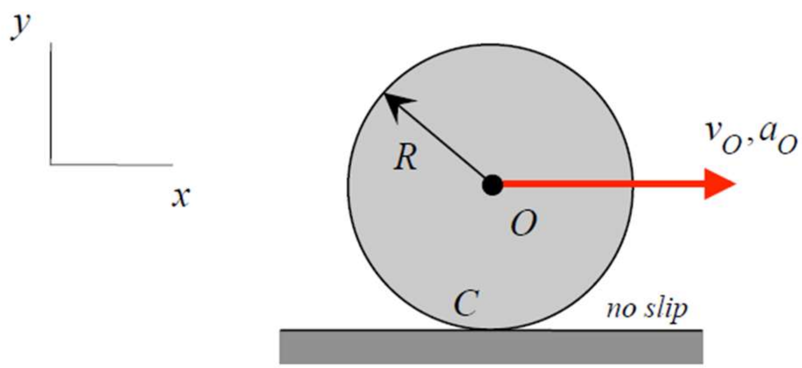


Angular velocity ω, and angular acceleration α are properties of the body

- $\vec{\omega} = \omega\hat{k} = \dot{\theta}\hat{k}, \quad \vec{\alpha} = \alpha\hat{k} = \ddot{\theta}\hat{k}$
- Sign of ω, α determines direction of rotation (using right hand rule)

Special case: Rolling without Slipping

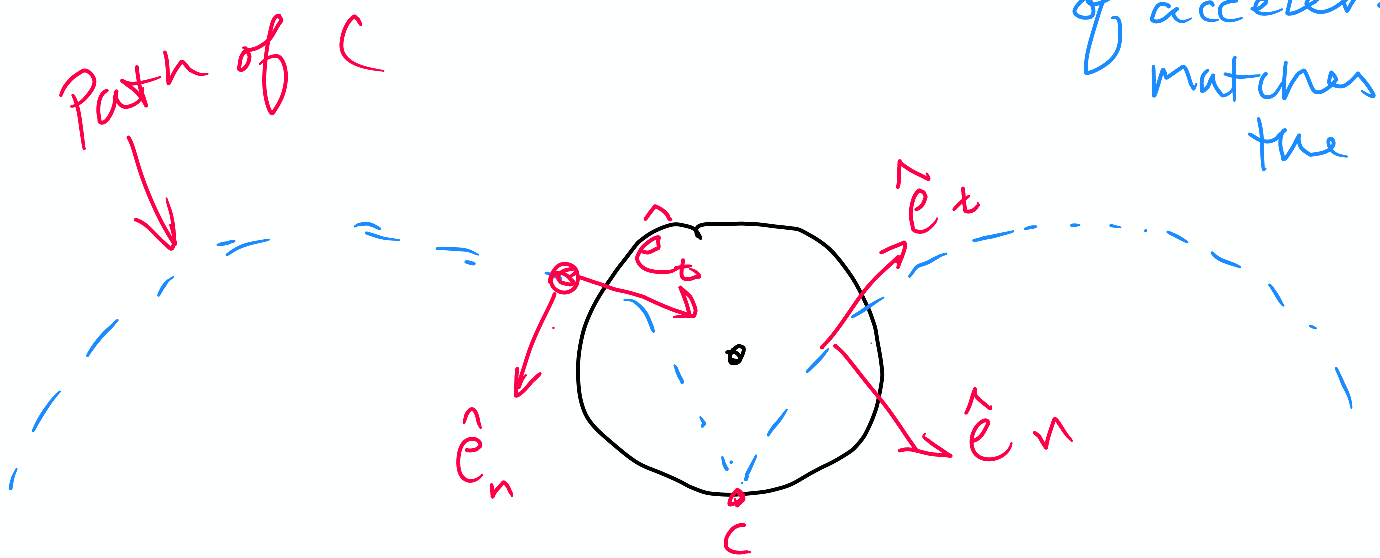
The wheel rolls on a rough, stationary surface. The center of the wheel O has a velocity and acceleration v_O, a_O . It is assumed that sufficient friction acts between the wheel and surface such that the contact point, C, does not slip



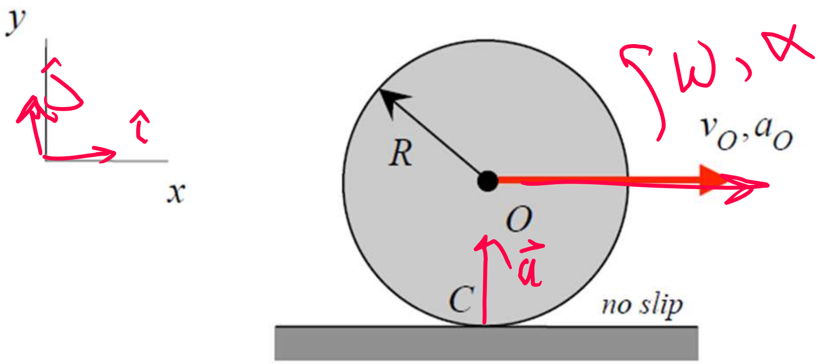
"No slip" condition:

$$v_{cx} = 0, \qquad a_{cx} = 0$$

tangential component of acceleration & velocity matches that of the contact surface



Question: If C is a no-slip point, what are the y-components for the velocity and acceleration of C?



Known information:

- O moves on a straight, horizontal path

  $\vec{V}_o = V_o\hat{i}$, $\vec{a}_o = a_o\hat{i}$

- "No slip" condition

  $V_{cx} = 0\hat{i}$, $\vec{a}_{cx} = 0\hat{i}$

Using our rigid body equation for velocity: $\vec{v}_C = \vec{v}_O + \vec{\omega} \times \underline{\vec{r}_{C/O}}$

$$\cancel{\vec{V}_{cx}} + \vec{V}_{cy} = V_o\hat{i} + \omega\hat{k} \times (-R\hat{j})$$

$$V_{cy}\hat{j} = V_o\hat{i} + \omega R\hat{i}$$

$\hat{j}: \; V_{cy} = 0$

$\hat{i}: \; 0 = V_o + \omega R \rightarrow \underline{V_o = -\omega R}$

rotating CW

$\omega = -\frac{V_o}{R}$

Using our rigid body equation for acceleration: $\vec{a}_c = \vec{a}_O + \vec{\alpha} \times \vec{r}_{C/O} - \omega^2\vec{r}_{C/O}$

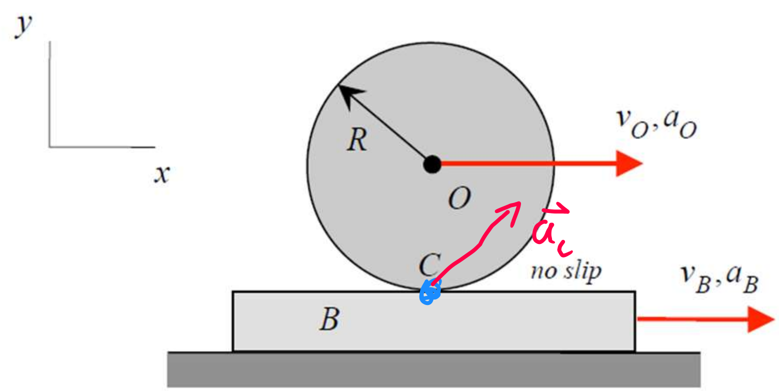
$$\cancel{a_{cx}\hat{i}}^{\,0} + a_{cy}\hat{j} = a_o\hat{i} + \alpha\hat{k} \times (-R\hat{j}) - \omega^2(-R\hat{j})$$

$$a_{cy}\hat{j} = a_o\hat{i} + \alpha R\hat{i} + \underline{\omega^2 R\hat{j}}$$

$i: \; 0 = a_o + \alpha R \rightarrow \alpha = -\frac{a_o}{R}$

$j: \; a_{cy} = \omega^2 R\hat{j}$

Question: What if C is a no-slip point, but the surface it is on is moving?



Known information:

- O moves on a straight, horizontal path $\vec{V}_o = V_o\hat{i}$, $\vec{a}_o = a_o\hat{i}$
- B is translating in the x-direction $\vec{V}_B = V_B\hat{i}$ $\vec{a}_B = a_B\hat{i}$
- "No slip" condition $V_{cx} = V_B$ $a_{cx} = a_B$

Using our rigid body equation for velocity: $\vec{v}_C = \vec{v}_O + \vec{\omega} \times \vec{r}_{C/O}$

$$V_{cx}\hat{i} + V_{cy}\hat{j} = V_o\hat{i} + \omega\hat{k} \times (-R\hat{j})$$

$$V_B\hat{i} + V_{cy}\hat{j} = V_o\hat{i} + \omega R\hat{i}$$

$$\hat{i}: \; V_B = V_o + \omega R \rightarrow \omega = \frac{V_B - V_o}{R}$$

$$\hat{j}: \; V_{cy} = 0$$

Using our rigid body equation for acceleration: $\vec{a}_c = \vec{a}_O + \vec{\alpha} \times \vec{r}_{C/O} - \omega^2\vec{r}_{C/O}$

$$a_{cx}\hat{i} + a_{cy}\hat{j} = a_o\hat{i} + \alpha\hat{k} \times (-R\hat{j}) - \omega^2(-R\hat{j})$$

$$a_B\hat{i} + a_{cy}\hat{j} = a_o\hat{i} + \alpha R\hat{i} + \omega^2 R\hat{j}$$

$$\hat{i}: \; a_B = a_o + \alpha R \longrightarrow \alpha = \frac{a_B - a_o}{R}$$

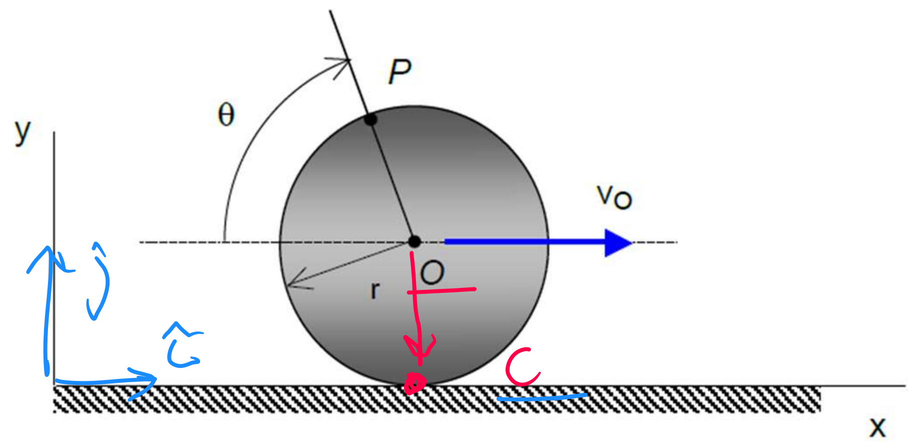
$$\hat{j}: \; a_{cy} = \omega^2 R$$

Example 2.A.5

Given: A wheel rolls without slipping on a rough horizontal surface. At one instant, when $\theta = 90°$, the center of the wheel O is moving to the right with a speed of $v_O = 5$ ft/s with this speed decreasing at a rate of 3 ft/s^2. $a_0 = -3\ ft/s^2$

Find: Determine the acceleration of point P on the circumference of the wheel at this instant, if $r = 2$ ft. Make a sketch of this acceleration vector at P.

1) draw unit vectors
2) relate pts. with the most info



pt O: $\vec{V}_0 = V_0\hat{i}$

pt C: $\vec{V}_{cx} = 0$, $\vec{a}_{cx} = 0$

$\vec{V}_{cy} = 0$

3) rigid body motion → velocity

$$\vec{V}_c = \vec{V}_0 + \vec{\omega} \times r_{c/0}$$

$$V_{cx}\hat{i} + V_{cy}\hat{j} = \vec{V}_0\hat{i} + \omega\hat{k} \times (-r\hat{j})$$

$$= V_0\hat{i} + \omega r\hat{i}$$

$\hat{i}$: $V_{cx} = V_0 + \omega r \rightarrow \omega = \frac{\cancel{V_{cx}}^{0} - V_0}{r}$ $\quad \omega = \frac{-V_0}{r} = -\frac{5}{2} = -2.5\ \frac{rad}{s}\hat{k}$

$\hat{j}$: $V_{cy} = 0$

4) rigid body motion → acceleration

$$\vec{a}_c = \vec{a}_o + \vec{\alpha} \times r_{c/o} - \omega^2 r_{c/o}$$

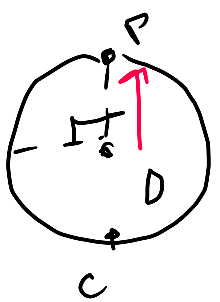
$$a_{cx}\hat{i} + a_{cy}\hat{j} = a_o\hat{i} + \alpha\hat{k} \times (-r\hat{j}) + \omega^2 r\hat{j}$$

$$a_{cx}\hat{i} + a_{cy}\hat{j} = a_o\hat{i} + \alpha r\hat{i} + \omega^2 r\hat{j}$$

$$\hat{i}: \cancel{a_{cx}}^{0} = a_o + \alpha r \rightarrow \alpha = -\frac{a_o}{r} = -\frac{3}{2} = -1.5 \text{ rad/s}^2\hat{k}$$

$$\hat{j}\quad a_{cy} = \omega^2 r = \left(\frac{-5}{2}\right)^2 2 = 12.5 \text{ ft/s}^2$$

5) motion of pt. P



$$\vec{a}_p = \vec{a}_o + \vec{\alpha} \times r_{p/o} - \omega^2 r_{p/o}$$

$$\vec{a}_p = a_o\hat{i} + \alpha\hat{k} \times (r\hat{j}) - \omega^2 r\hat{j}$$

$$= a_o\hat{i} - \alpha r\hat{i} - \omega^2 r\hat{j}$$

$$\vec{a}_p = [3 - (-1.5 \cdot 2)]\hat{i} - \left(\left(-\frac{5}{2}\right)(2)\right) =$$

$$\vec{a}_p = 0\hat{i} - 12.5\hat{j} \text{ ft/s}^2$$

Example 2.A.4

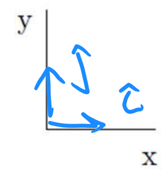
Given: Rack A moves to the right with a speed of v_A and an acceleration of a_A. Rack B moves to the left with a constant speed of v_B. Assume $v_A = 0.8$ m/s, $a_A = 2$ m/s^2, $v_B = 0.6$ m/s, $r = 0.1$ m and $R = 0.16$ m.

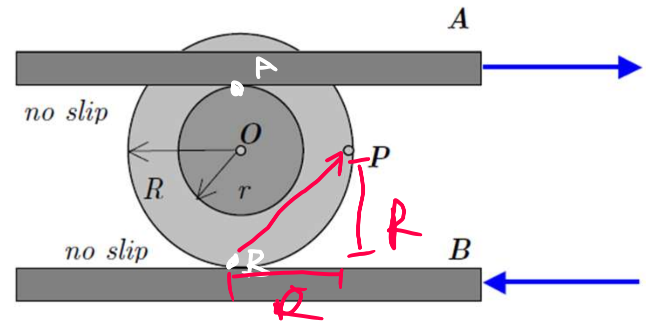
$a_{Bx} = 0$

Find: Determine:

(a) The velocity of point P on the outer rim of the gear; and

(b) The acceleration of point P on the outer rim of the gear.





1) unit vectors

2) known info

pt. A: $\vec{V}_A = V_A \hat{i}$, $\vec{a}_A = a_{Ax}\hat{i} + a_{Ay}\hat{j}$

pt. B: $V_B = -V_B \hat{i}$, $\vec{a}_B = 0\hat{i} + a_{By}\hat{j}$

3) find ω using pt A & B

$$\vec{V}_A = \vec{V}_B + \vec{\omega} \times \vec{r}_{A/B}$$

$$V_A \hat{i} = -V_B \hat{i} - \omega \hat{k} \times (R + r\hat{j})$$

$$V_A \hat{i} = -V_B \hat{i} - \omega (R + r)\hat{i}$$

$$-\frac{V_A + V_B}{(R + r)} \vec{k} = \vec{\omega}$$

4) find α using pt A & B

$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}_{A/B} - \omega^2 \vec{r}_{A/B}$$

$$a_{Ax}\hat{i} + a_{Ay}\hat{j} = \cancel{a_{Bx}}^{0}\hat{i} + a_{By}\hat{j} + \alpha\hat{k} \times (R+r)\hat{j} - \omega^2 (R+r)\hat{j}$$

$$= a_{By}\hat{j} - \alpha(R+r)\hat{i} - \omega^2 (R+r)\hat{j}$$

$$\hat{i}: \; a_{Ax} = -\alpha(R+r) \rightarrow \alpha = -\frac{a_{Ax}}{R+r}$$

5) use ω, α to find $\vec{V}_P$

$$\vec{V}_P = \vec{V}_B + \vec{\omega} \times \vec{r}_{P/B}$$

$$= V_B\hat{i} + \omega\hat{k} \times (R\hat{i} + R\hat{j})$$

$$= V_B\hat{i} + \omega R\hat{j} - \omega R\hat{i}$$

$$\vec{V}_P = (V_B - \omega R)\hat{i} + \omega R\hat{j} = \left(V_B - \frac{(V_A + V_B)}{(R+r)} R\right)\hat{i} + \omega R\hat{j}$$

(sub in & solve)

6) use ω, α to find $\vec{a}_p$

$$a_p = \vec{a}_B + \vec{\alpha} \times \vec{r}_{P/B} - \omega^2 r_{P/B}$$

$$= a_{By}\hat{j} + \alpha\vec{k} \times (R\hat{i} + R\hat{j}) - \omega^2(R\hat{i} + R\hat{j})$$

$$= a_{By}\hat{j} + \alpha R\hat{j} - \alpha R\hat{i} - \omega^2 R\hat{i} - \underline{\omega^2 R}\hat{j}$$

$$\underline{\omega^2 R}\hat{j} \Leftarrow \text{from earlier}$$

$$\boxed{\vec{a}_p = -R(\alpha + \omega^2)\hat{i} + \alpha R\hat{j}}$$