ME 274: Basic Mechanics II

Lecture 14: Moving Reference Frame Kinematics -3D

PURDUE UNIVERSITY | School of Mechanical Engineering

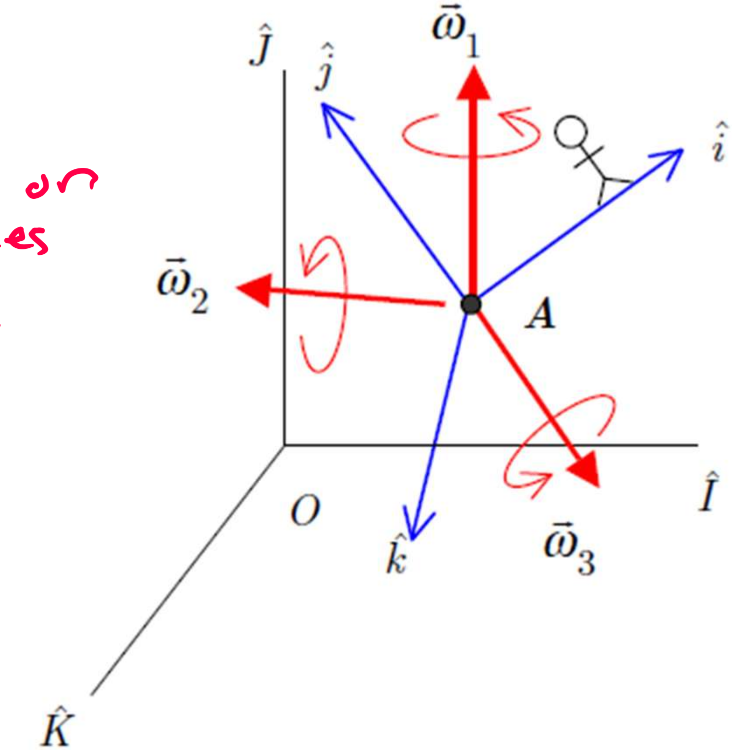
Announcements

- HW 2.E, 2.F **due tonight**
- **Reminder to use correct HW formatting!**
- Office hours ME 2008A: T 9:30-11, W 3:30-4:20, Th 12-1:30

3D Rotating Reference Frames

- 2D rotation about fixed axis ($\hat{k}$) $\rightarrow$ 3D rotating around a moving axis $\rightarrow$ not aligned w/ $\hat{i}\,\hat{j}\,\hat{k}$
- Multiple components each representing a different axis of rotation, $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \ldots$ $\rightarrow$ can include fixed or moving axes
- The total angular velocity of the observer is found by vector addition of all the components: can have 3+ vectors
$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3 + \cdots$$
- Rotation can be about either a fixed or moving axis.
- Unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ change with time:

$$\frac{d\hat{\imath}}{dt} = \vec{\omega} \times \hat{\imath}, \qquad \frac{d\hat{\jmath}}{dt} = \vec{\omega} \times \hat{\jmath}, \qquad \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}$$



- Same velocity and acceleration equations as 2D!

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$
$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$$

So what is different between 2D and 3D problems?

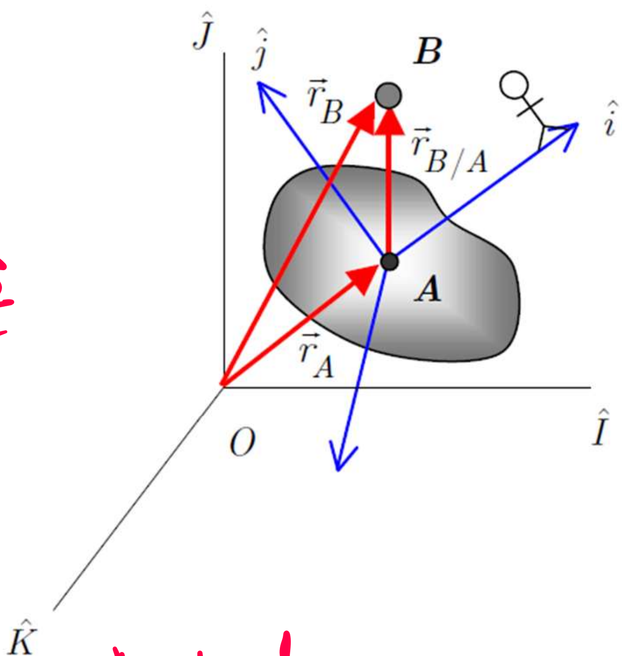
how α & ω are defined!
$\omega \neq \dot{\theta}\hat{k}$, $\alpha \neq \ddot{\theta}\hat{k}$

ω = constant $\rightarrow \vec{\alpha} \neq \vec{0}$

Question C3.6

Consider an observer who is riding along on a moving (translating and rotating) rigid body. We wish to use the observation of this person in describing the motion of some point B, which is not fixed to the body, in the following moving reference frame acceleration equation.

Answer the following questions in words:



- What is the meaning of $\vec{\omega}$? total angular velocity of rotating reference
- What is the meaning of $\vec{\alpha}$?
  angular accel. of rotating ref. $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$
- What is the meaning of $(\vec{v}_{B/A})_{rel}$?
  velocity of B as observed from A
- What is the meaning of $(\vec{a}_{B/A})_{rel}$?
  accel. of B as observed from A.
- What restrictions, if any, are on the choice of point A?
  must be on the body that rot. ref. frame attached
- What is the difference in meaning between $\vec{a}_{B/A}$ and $(\vec{a}_{B/A})_{rel}$?

$\vec{a}_{B/A}$ → rel. acceleration as seen from the fixed ref frame

$(a_{B/A})_{rel}$ → rel. acceleration as seen from moving reference

Question C3.4

The vertical shaft OA rotates about a fixed axis with a constant rate of $\Omega = 8$ rad/s. The arm AB is pinned to OA and is being raised at a constant rate of $\dot{\theta} = 10$ rad/s. An observer and xyz axes are attached to AB. The XYZ axes are stationary. What is the angular acceleration vector for arm AB when $\theta = 90°$?

$$\vec{\omega} = -\Omega \hat{J} + \dot{\theta} \hat{k}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \cancel{-\dot{\Omega}\hat{J}} \cancel{-\Omega\dot{\hat{J}}} + \cancel{\ddot{\theta}\hat{k}} + \dot{\theta}\dot{\hat{k}}$$

$$= \dot{\theta}\dot{\hat{k}} = \dot{\theta}(\vec{\omega} \times \hat{k})$$

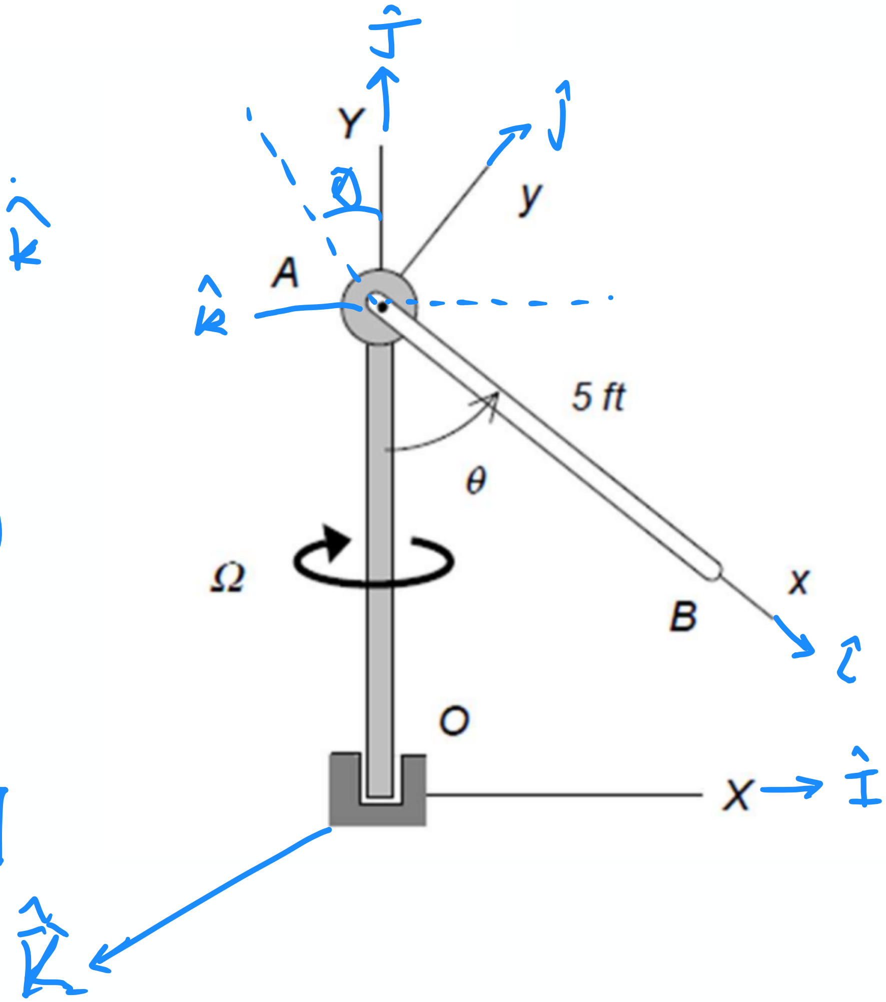
$$= \dot{\theta}\left[(-\Omega\hat{J} + \dot{\theta}\hat{k}) \times \hat{k}\right]$$

$$= \dot{\theta}(-\Omega\hat{J} \times \hat{k})$$

$$\hat{J} = -\cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\vec{\alpha} = -\dot{\theta}\Omega\left[(-\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) \times \hat{k}\right]$$

$$= -\dot{\theta}\Omega(\cos\theta\,\hat{j} + \sin\theta\,\hat{i})$$



Example 3.B.5

Given: The disk of a gyroscope rotates about its own axis at a constant rate of $\omega_2 = 600$ rev/min. The gimbal support is rotating at a constant rate of $\omega_1 = 10$ rad/s about a fixed vertical axis. The observer and the xyz axes are attached to the disk. The XYZ axes are fixed in space.

Find: Determine:

(a) The angular velocity of the observer at the instant shown; and

(b) The angular acceleration of the observer at the instant shown.

a) $\vec{\omega} = \omega_1 \hat{J} + \omega_2 \hat{i}$

$= \omega_1 \hat{j} + \omega_2 \hat{i}$

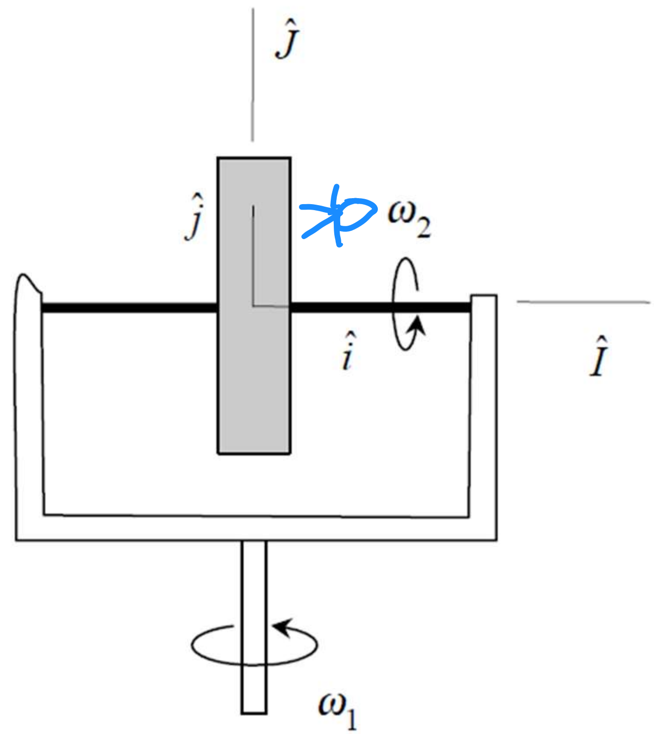
b) $\vec{\alpha} = \dot{\vec{\omega}} = \cancel{\dot{\omega}_1 \hat{J}} + \cancel{\omega_1 \dot{\hat{J}} + \dot{\omega}_2 \hat{i}}^{\,0} + \omega_2 \dot{\hat{i}}$

$\dot{\hat{i}} = \vec{\omega} \times \hat{i} = (\omega_1 \hat{J} + \omega_2 \hat{i}) \times \hat{i}$

$= -\omega_1 \hat{k}$

$\vec{\alpha} = -\omega_2 \omega_1 \hat{k}$

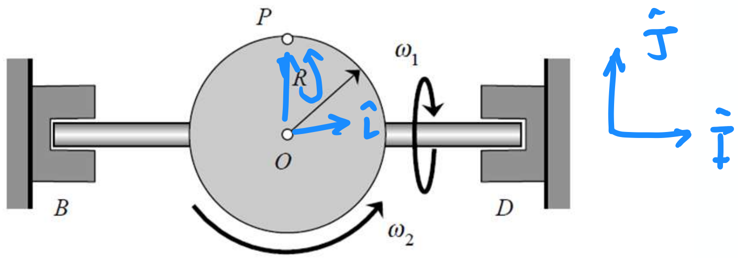
$\hat{J} = \hat{j}$



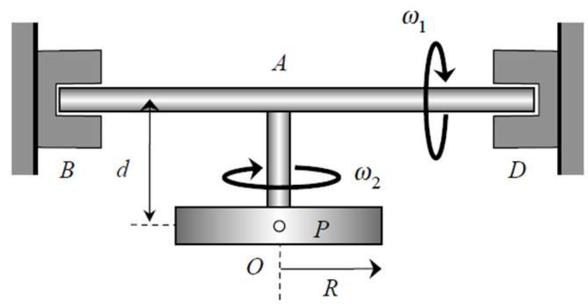
Example 3.B.6

Given: Shaft BD rotates about a fixed axis with a constant rate of ω_1. Shaft OA is rigidly attached to shaft BD with OA being perpendicular to BD. A disk rotates about shaft OA with a constant rate of ω_2 relative to OA.

Find: The acceleration of point P on the edge of the disk for the position shown.



side view



top view

$$\vec{\omega} = -\omega_1 \hat{I} + \omega_2 \hat{k}$$

$$\vec{\alpha} = -\cancel{\dot{\omega}_1 \hat{I}} - \cancel{\omega_1 \dot{\hat{I}}} + \cancel{\dot{\omega}_2 \hat{k}} + \omega_2 \dot{\hat{k}}$$

$$\vec{\alpha} = \omega_2(\omega \times \hat{k})$$

$$= \omega_2[(-\omega_1 \hat{I} + \omega_2 \hat{k}) \times \hat{k}]$$

$$= +\omega_1 \omega_2 \hat{j}$$

$$\vec{a}_P = \vec{a}_O + \cancel{(\vec{a}_{P/O})_{rel}} + \vec{\alpha} \times \vec{r}_{P/O} + \cancel{2\vec{\omega} \times (\vec{v}_{P/O})_{rel}}$$

$$+ \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O})$$

$$\vec{a}_O = \cancel{\vec{a}_A} + \cancel{\vec{\alpha}_{OA} \times \vec{r}_{O/A}} + \vec{\omega}_{OA} \times (\vec{\omega}_{OA} \times \vec{r}_{O/A})$$

$$= -\omega_1 \hat{i} \times (-\omega_1 \hat{i} \times d\hat{K}) = \underline{-\omega_1^2 d\hat{K}}$$

$$\vec{a}_P = -\omega_1^2 d\hat{K} + \omega_1 \omega_2 \hat{j} \times R\hat{j}$$

$$+ (-\omega_1 \hat{i} + \omega_2 \hat{k}) \times [(-\omega_1 \hat{i} + \omega_2 \hat{k}) \times R\hat{j}]$$

$$= -\omega_1^2 d\hat{k} + (-\omega_1\hat{i} + \omega_2\hat{k}) \times (-\omega_1 R\hat{k} + \omega_2 R\hat{i})$$

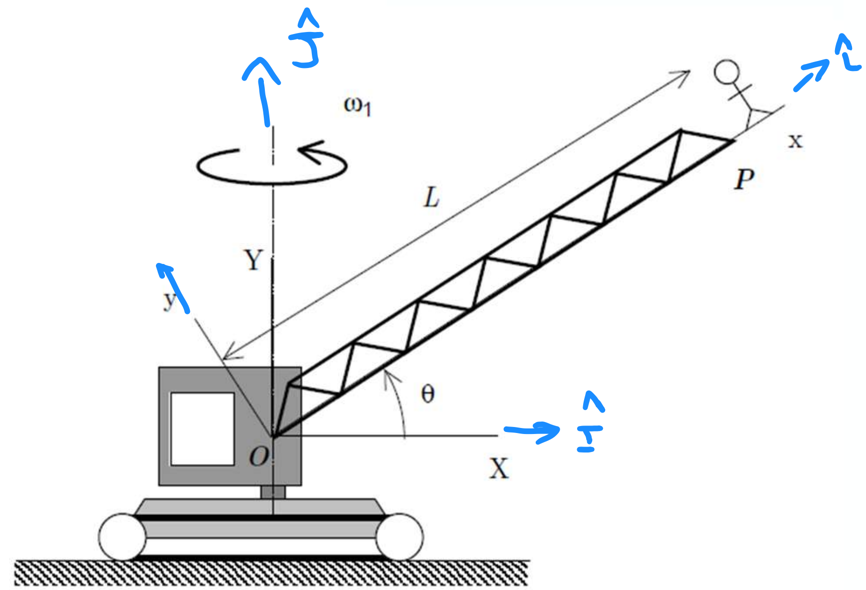
$$\vec{a}_p = -\omega_1^2 d\hat{k} - \omega_1^2 R\hat{j} - \omega_2^2 R\hat{j}$$

Example 3.B.7

Given: $\omega_1 = 0.30$ rad/s $= constant$, $\dot{\theta} = 0.5$ rad/s $= constant$ and $L = 12$ m.

Find: When $\theta = 30°$, determine:

(a) The angular velocity of boom OP;
(b) The angular acceleration of boom OP;
(c) The velocity of end P of the boom; and
(d) The acceleration of end P of the boom.



$$\vec{\omega} = \omega_1 \hat{J} + \dot{\theta}\hat{k}$$

$$\omega_1 \hat{J} + \dot{\theta}\hat{K}$$

$$\vec{\alpha} = \cancel{\dot{\omega}_1 \hat{J} + \omega_1 \dot{\hat{J}} + \ddot{\theta}\hat{k}} + \dot{\theta}\dot{\hat{k}}$$

$$= \dot{\theta}(\vec{\omega} \times \hat{k}) = \dot{\theta}[(\omega_1 \hat{J} + \dot{\theta}\hat{k}) \times \hat{k}]$$

$$= \dot{\theta}\omega_1(\hat{J} \times \hat{k})$$

$$= \dot{\theta}\omega_1 \hat{I}$$

$$\hat{k} = \hat{K}$$

$$\hat{i} = \cos\theta \hat{I} + \sin\theta \hat{J}$$

$$\vec{V}_P = \vec{V}_O + (\vec{V}_{P/O})_{rel} + \vec{\omega} \times \vec{r}_{P/O}$$

$$0 + 0 + (\omega_1 \hat{J} + \dot{\theta}\hat{k}) \times L\hat{i} = L(\omega_1 \hat{J} + \dot{\theta}\hat{K}) \times (\cos\theta \hat{I} + \sin\theta \hat{J})$$

$$= -L\omega_1 \cos\theta \hat{K} + L\dot{\theta}\cos\theta \hat{J} - L\dot{\theta}\sin\theta \hat{I}$$

$$\vec{a}_p = \cancel{\vec{a}_o} + \cancel{(\vec{a}_{p/o})_{rel}} + \vec{\alpha}\times\vec{r}_{p/o} \cancel{+ 2\vec{\omega}\times(\vec{v}_{p/o})_{rel}} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{p/o})$$

$$= \dot{\theta}\omega_1 \hat{I} \times L\hat{i} + (\omega_1\hat{J} + \dot{\theta}\hat{K})\times[(\omega_1 \hat{J} + \dot{\theta}\hat{K})\times L\hat{i}]$$

$$= \dot{\theta}\omega_1 \hat{I} \times L(\cos\theta \hat{I} + \sin\theta \hat{J}) + (\omega_1 \hat{J} + \dot{\theta}\hat{K})[(\omega_1\hat{J}+\dot{\theta} \hat{K})\times L(\cos\theta\hat{I} + \sin\theta \hat{J})]$$

$$= \dot{\theta}\omega_1 L\sin\theta\hat{K} + (\omega_1\hat{J} + \dot{\theta}\hat{K})(-\omega_1 L\cos\theta\hat{K} + \dot{\theta} L\cos\theta\hat{J} - \dot{\theta} L\sin\theta\hat{I})$$

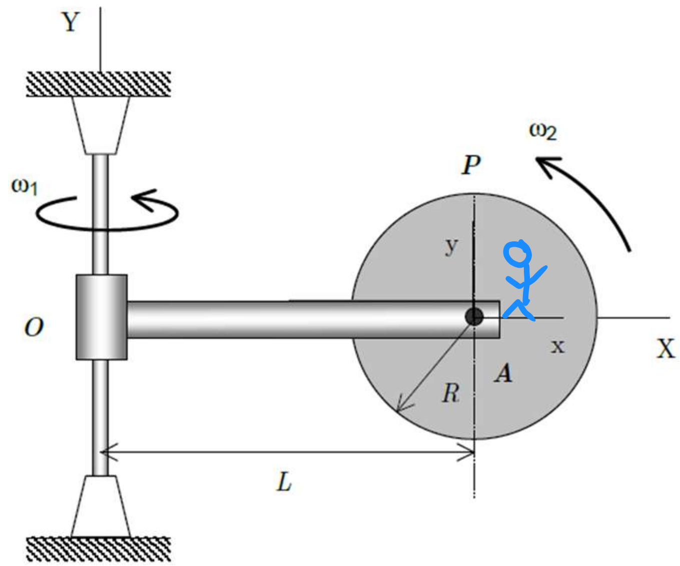
$$= \dot{\theta}\omega_1 L\sin\theta\hat{K} - \omega_1^2 L\cos\theta\hat{I} + \dot{\theta}\omega_1 L\sin\theta\hat{K} - \dot{\theta}^2 L\cos\theta\hat{I} - \dot{\theta}^2 L\sin\theta\hat{J}$$

Example 3.B.8

Find: Determine:

(a) The velocity of point P on the disk at the instant when P is directly above the center A of the disk; and

(b) The acceleration of point P at the same instant.



$\hat{K} = \hat{k}$

$\vec{\omega} = \omega_1 \hat{J} + \omega_2 \hat{k}$

$\omega_1 \hat{J} + \omega_2 \hat{k}$

$\vec{\alpha} = \cancel{\dot{\omega}_1 \hat{J}} + \cancel{\omega_1 \dot{\hat{J}}} + \cancel{\dot{\omega}_2 \hat{k}} + \omega_2 \dot{\hat{k}}$

$= \omega_2 (\vec{\omega} \times \hat{k})$

$= \omega_2 [(\omega_1 \hat{J} + \omega_2 \hat{k}) \times \hat{k}] = \omega_1 \omega_2 \hat{i}$

velocity:

$\vec{V}_P = \vec{V}_A + (\vec{V}_{P/A})_{rel} + \vec{\omega} \times \vec{r}_{P/A}$

$= -\omega_1 L \hat{k} + (\omega_1 \hat{J} + \omega_2 \hat{k}) \times R \hat{J}$

$= -\omega_1 L \hat{k} - \omega_2 R \hat{i}$

rigid body eqn for V_A

$V_A = V_O + \omega_{arm} \times r_{A/O}$

$= \omega_1 \hat{J} \times L \hat{i} = -\omega_1 L \hat{K}$

$\hat{K} = \hat{k}$

accel:

$$\vec{a}_p = \vec{a}_A + \cancel{(\vec{a}_{p/A})_{rel}} + \vec{\alpha} \times \vec{r}_{p/A} + 2\vec{\omega} \times \cancel{(\vec{v}_{p/A})_{rel}} + \omega \times (\omega \times r_{p/A})$$

rigid body for a_A

$\omega_1 \hat{j}$ $\quad L\hat{i}$

$$a_A = \cancel{a_o} + \cancel{\alpha_{OA}} \times r_{A/O} + \vec{\omega}_{AO} \times (\vec{\omega}_{AO} \times \vec{r}_{A/O}) = -\omega_1^2 L \hat{i}$$

$$\vec{a}_p = -\omega_1^2 L\hat{i} + \omega_1 \omega_2 \hat{i} \times R\hat{j} + (\omega_1 \hat{j} + \omega_2 \hat{k}) \times [(\omega_1 \hat{j} + \omega_2 \hat{k}) \times R\hat{j}]$$

$$= -\omega_1^2 L\hat{i} + \omega_1 \omega_2 R\hat{k} + \omega_1 \omega_2 R\hat{k} - \omega_2^2 R\hat{j}$$