ME 274: Basic Mechanics II

Lecture 13: Moving Reference Frame Kinematics -3D

PURDUE UNIVERSITY
School of Mechanical Engineering

Announcements

Exam 1 info:

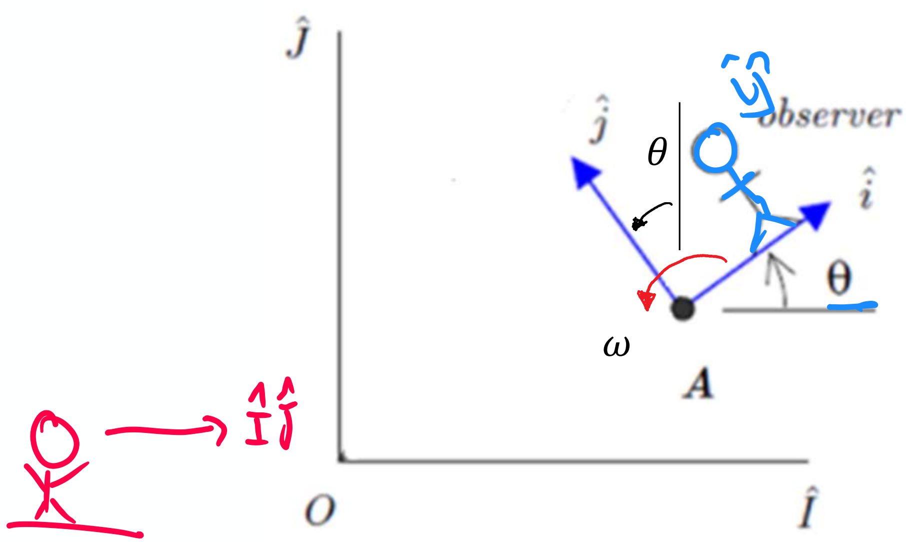
- Exam 1, Thurs. 2/12, 8 – 9:30, MA 175
- Arrive 15 minutes early so we can start on time
- You will have time after the exam to scan and submit.

Homework and class updates:

- No class Friday, 2/13 to account for evening exam
- HW released Monday (2.C, 2.D) **due Friday 2/13**
- HW released on Wednesday (2.E, 2.F) **due Monday 2/16**
- **Reminder to use correct HW formatting!**
- Office hrs change: Th 12-1:30

How do our expressions for velocity and acceleration change when our system is moving in 3 dimensions?

2D Rotating Reference Frames



Fixed XYZ
moving xyz

- Angular velocity measured around a single axis (usually $\hat{k}$)

$$\vec{\omega} = \dot{\theta}\hat{k}$$

- Unit vectors $\hat{\imath}, \hat{\jmath}$ change with time

$$\frac{d\hat{\imath}}{dt} = \vec{\omega} \times \hat{\imath}, \qquad \frac{d\hat{\jmath}}{dt} = \vec{\omega} \times \hat{\jmath}$$

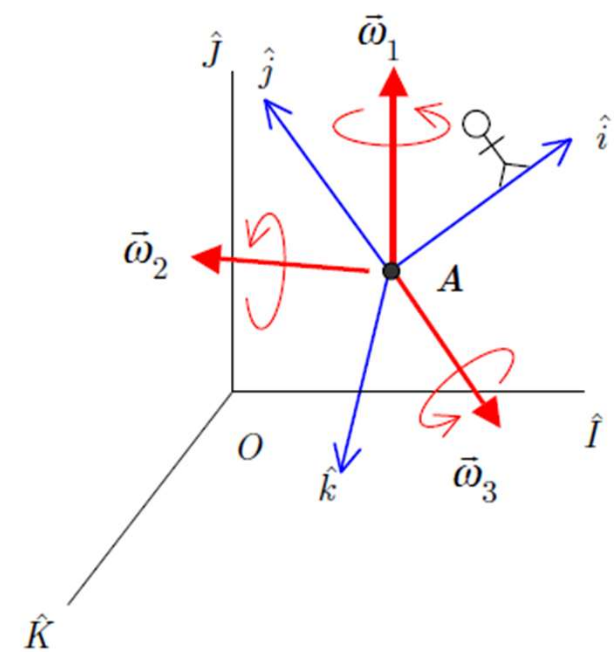
- Velocity: $\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$

$(V_{B/A})_{rel} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$

- Acceleration: $\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$

$(a_{B/A})_{rel} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

3D Rotating Reference Frames



- Angular velocity around a moving axis → not aligned with $\hat\imath, \hat\jmath, \hat k$
- Multiple components each representing a different axis of rotation, $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \ldots$
- The total angular velocity of the observer is found by vector addition of all the components:

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3 + \cdots$$

- Unit vectors $\hat\imath, \hat\jmath, \hat k$ change with time

$$\frac{d\hat\imath}{dt} = \omega \times \hat\imath, \qquad \frac{d\hat\jmath}{dt} = \omega \times \hat\jmath, \qquad \frac{d\hat k}{dt} = \omega \times \hat k$$

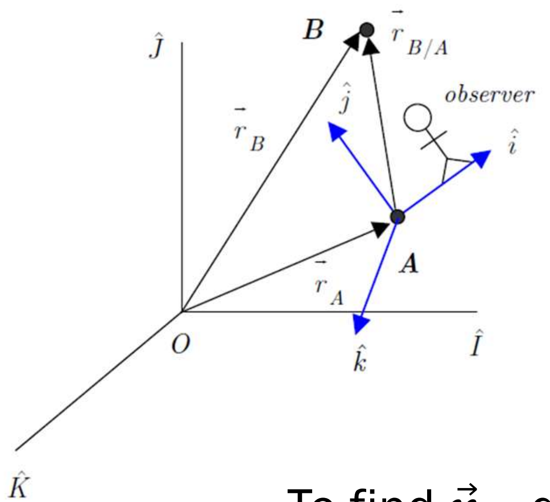
Some rotations about fixed Axis → $\omega_1 \hat K$, $\omega_2 \hat I$, $\omega \hat J$

some about movind Axis ⇒ $\omega_1 \hat k$, $\omega_2 \hat\imath$, $\omega_3 \hat\jmath$

Velocity Equation – 3D

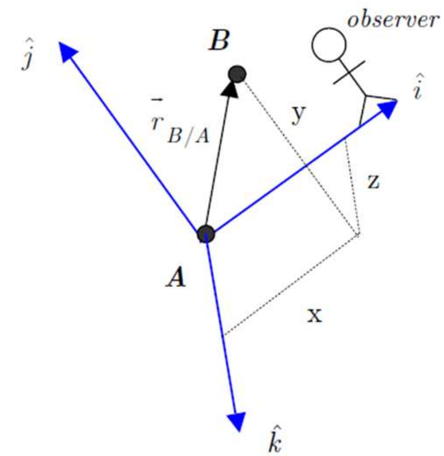
In the fixed reference frame, the position of pt. B is related to the position of pt. A using the relative position vector:

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$



In the moving reference frame, $\vec{r}_{B/A}$ can be described using xyz coordinates:

$$\vec{r}_{B/A} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$



To find $\vec{v}_B$, differentiate $\vec{r}_B$ with respect to time:

$$\frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\vec{r}_{B/A}}{dt} \implies$$

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) \\
&= \vec{v}_A + \frac{dx}{dt}\hat{i} + x\frac{d\hat{i}}{dt} + \frac{dy}{dt}\hat{j} + y\frac{d\hat{j}}{dt} + \frac{dz}{dt}\hat{k} + z\frac{d\hat{k}}{dt} \\
&= \vec{v}_A + \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} + x\left(\vec{\omega} \times \hat{i}\right) + y\left(\vec{\omega} \times \hat{j}\right) + z\left(\vec{\omega} \times \hat{k}\right) \\
&= \vec{v}_A + \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} + \vec{\omega} \times \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \\
\vec{V}_B &= \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}
\end{aligned}$$

← same as 2D!

$\omega_1 \hat{I} + \omega_2 \hat{J} + \dots$

$\vec{\omega} = \omega_1 + \omega_2 + \omega_3 \dots$

$\neq \dot{\theta}\hat{K}$

$(V_{B/A})_{rel} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$

Acceleration Equation – 3D

Acceleration of point B is found by taking the time derivative of the velocity equation

$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\frac{d\vec{v}_B}{dt} = \frac{d\vec{v}_A}{dt} + \frac{d}{dt}\left(\vec{v}_{B/A}\right)_{rel} + \frac{d}{dt}\left(\vec{\omega} \times \vec{r}_{B/A}\right) \qquad \Longrightarrow$$

$$\begin{aligned}
\vec{a}_B &= \vec{a}_A + \frac{d}{dt}\left(\vec{v}_{B/A}\right)_{rel} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \frac{d\vec{r}_{B/A}}{dt} \\
&= \vec{a}_A + \frac{d}{dt}\left(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \omega \times \vec{r}_{B/A}\right] \\
&= \vec{a}_A + \left(\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}\right) + \left(\dot{x}\underline{\frac{d\hat{i}}{dt}} + \dot{y}\underline{\frac{d\hat{j}}{dt}} + \dot{z}\underline{\frac{d\hat{k}}{dt}}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right] \\
&= \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \dot{x}\left(\vec{\omega} \times \hat{i}\right) + \dot{y}\left(\vec{\omega} \times \hat{j}\right) + \dot{z}\left(\vec{\omega} \times \hat{k}\right) + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right] \\
&= \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \underline{\vec{\omega}} \times \left(\vec{v}_{B/A}\right)_{rel} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{B/A} + \vec{\omega} \times \left[\left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}\right]
\end{aligned}$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \underline{\vec{\alpha}} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \underbrace{\vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]}_{2D \rightarrow -\vec{\omega}^2 r \neq 3D} \quad \leftarrow \text{same as 2D}$$

$\left(\vec{a}_{B/A}\right)_{rel} = \underline{\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}}$ → the acceleration of B as seen by the moving observer

$\vec{\alpha} = \dfrac{d\vec{\omega}}{dt}$ → angular acceleration of the observer → change is how α, ω defined

$\neq \ddot{\theta}$

Considerations for determining angular acceleration of the rotating reference frame:

By definition, $\vec{\boldsymbol{\alpha}} = \frac{d\vec{\boldsymbol{\omega}}}{dt}$

In 2D problems, the moving reference frame rotates about a fixed axis (generally $\hat{K}$).

The time derivative of the unit vector therefore is:

$$\frac{d\hat{k}}{dt} = \vec{0}$$

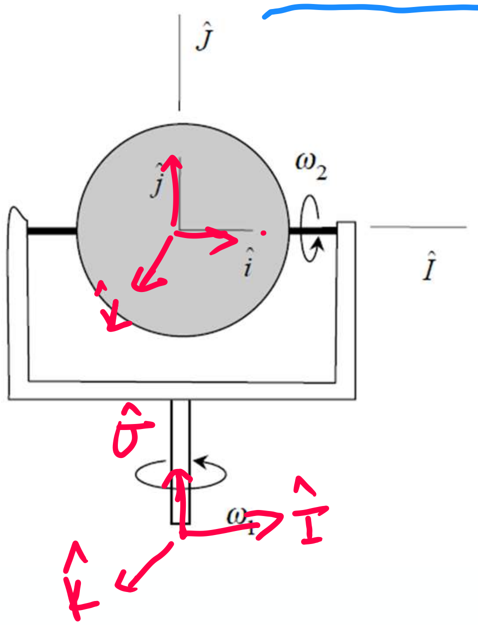
If the angular velocity of the moving reference frame is given by $\vec{\omega} = \Omega\hat{K}$ angular acceleration is found as:

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \dot{\Omega}\hat{K} + \cancel{\Omega\frac{d\hat{K}}{dt}} \nearrow 0 \quad \Rightarrow \vec{\alpha} = \dot{\Omega}\hat{K}$$

always for 2D

In 3D problems, the moving reference frame rotates about both fixed and moving axes for example:

$$\vec{\omega} = \omega_1\hat{J} + \omega_2\hat{i}$$



$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

$$\frac{d\vec{\omega}}{dt} = \dot{\omega}_1\hat{J} + \cancel{\omega_1\frac{d\hat{J}}{dt}} + \dot{\omega}_2\hat{i} + \omega_2\frac{d\hat{i}}{dt} \quad *$$

$\frac{d\hat{J}}{dt} \rightarrow$ fixed $= 0$ $\qquad \frac{d\hat{i}}{dt} = \vec{\omega}\times\hat{i}$

$\hat{J} = \hat{j}$

$$\vec{\omega} \times \hat{i} = (\omega_1 \hat{j} + \omega_2 \hat{i}) \hat{i} = \omega_1 (\hat{j} \times \hat{i}) + \cancel{\omega_2 (\hat{i} \times \hat{i})}^{0}$$

$$= \omega_1 (\hat{j} \times \hat{i}) = -\omega_1 \hat{k} \; *$$

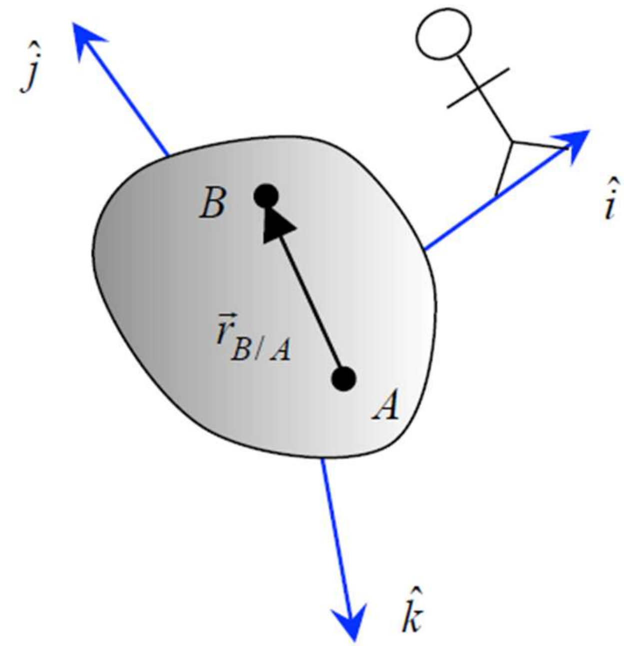
$$\vec{\alpha} = \dot{\omega}_1 \hat{j} + \dot{\omega}_2 \hat{i} - \omega_2 \omega_1 \hat{k}$$

Challenge: in 2D if $\vec{\omega} = \text{const}$, $\vec{\alpha} = 0$

true in 3D?

No! → in ex. if $\dot{\omega}_1, \dot{\omega}_2 = 0$ $\alpha = -\omega_2 \omega_1 \hat{k}$

3D Rigid Body Kinematics Equations



Remember on rigid bodies $|\vec{r}_{B/A}|=const.$ $\rightarrow$

$$(\vec{V}_{B/A})_{rel} = (\vec{a}_{B/A})_{rel} = \vec{0}$$

We can reduce our moving reference frame expressions for velocity and acceleration:

$$\vec{v}_B = \vec{v}_A + \cancel{(\vec{v}_{B/A})_{rel}}^{\,0} + \vec{\omega} \times \vec{r}_{B/A} \qquad \vec{V}_B = \vec{V}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \cancel{(\vec{a}_{B/A})_{rel}}^{\,0} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \cancel{(\vec{v}_{B/A})_{rel}}^{\,0} + \vec{\omega} \times [\vec{\omega} \times \vec{r}_{B/A}] \qquad \vec{a}_B = \vec{a}_A + \alpha \times r_{B/A} + \vec{\omega} \times \vec{\omega} \times r_{B/A}$$

eqns. same as 2D $\rightarrow$ change in now $\vec{\omega}$, $\vec{\alpha}$

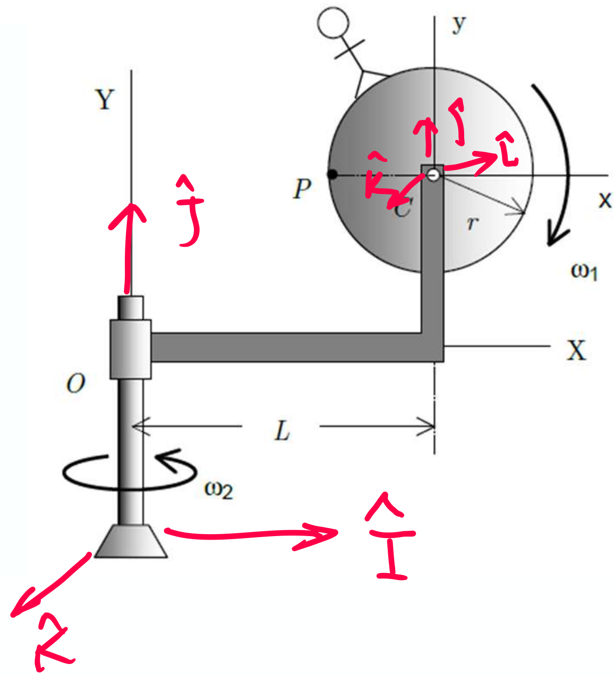
$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \dots \qquad \vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

Example 3.B.3

Given: A disk rotates with a constant rate of $\omega_1 = 20$ rad/s with respect to the arm OC as the arm OC rotates about a fixed vertical axis with a constant rate of $\omega_2 = 5$ rad/s. The observer and the xyz axes are attached to the disk, while the XYZ axes are fixed. At this instant, the XYZ and xyz axes are aligned. → $\hat{i} = \hat{I}$ $\quad \hat{j} = \hat{J}$ $\quad \hat{k} = \hat{K}$

Find: Determine:

(a) The angular velocity of the observer at the instant shown; and
(b) The angular acceleration of the observer at the instant shown.



$$\vec{\omega} = \omega_1 \hat{k} + \omega_2 \hat{J}$$

$$= \omega_1 \hat{K} + \omega_2 \hat{J}$$

c

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \cancel{\dot{\omega}_1 \hat{k}} + \omega_1 \frac{d\hat{k}}{dt} + \cancel{\dot{\omega}_2 \hat{J}} + \cancel{\omega_2 \frac{d\hat{J}}{dt}}$$

0 0

$$\vec{\alpha} = \omega_1 (\omega \times \hat{k})$$

$$= \omega_1 (\omega_1 K + \omega_2 J) \times \hat{k}$$

$$\hat{K} \times \hat{K} = 0$$

$$\hat{J} = \hat{j}$$

$$\vec{\alpha} = -\omega_1 \omega_2 \hat{i}$$

Example 3.B.4

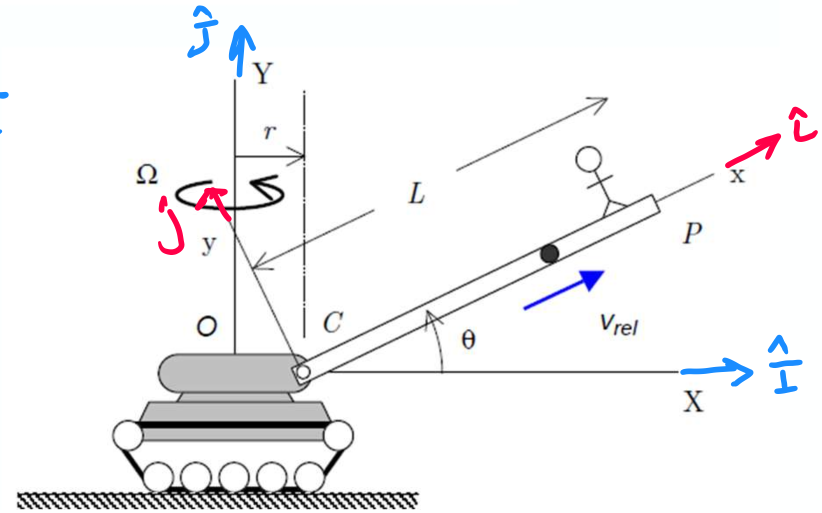
Given: The turret on a tank is rotating about a fixed vertical axis at a constant rate of $\Omega = 0.4$ rad/s. The barrel is being raised at a constant rate of $\dot{\theta} = 0.4$ rad/s . A cannon shell is fired with a constant muzzle speed of $v_{rel} = 200$ ft/s relative to the barrel. The observer and the xyz axes are attached to the barrel, while the XYZ axes are fixed. Here, $r = 3$ ft and $L = 15$ ft.

Find:

(a) The angular velocity of the barrel at the instant shown; $\vec{\omega}$

(b) The angular acceleration of the barrel at the instant shown; and $\vec{\alpha}$

(c) The acceleration of the shell as it leaves the barrel at P. $\vec{a}_P$



a) $\vec{\omega} = \Omega \hat{J} + \dot{\theta} \hat{k}$

$= \Omega \hat{J} + \dot{\theta} \hat{K}$

b) $\vec{\alpha} = \cancel{\dot{\Omega} \hat{J}} + \cancel{\Omega \frac{d\hat{J}}{dt}} + \cancel{\ddot{\theta} \hat{k}} + \dot{\theta} \frac{d\hat{k}}{dt}$

$= \dot{\theta} (\vec{\omega} \times \vec{k})$

$= \dot{\theta} [(\Omega \hat{J} + \dot{\theta} \hat{k}) \times \hat{k}] = \dot{\theta} \Omega \hat{J} \times \hat{K} = \dot{\theta} \Omega \hat{I}$

Example 3.B.4

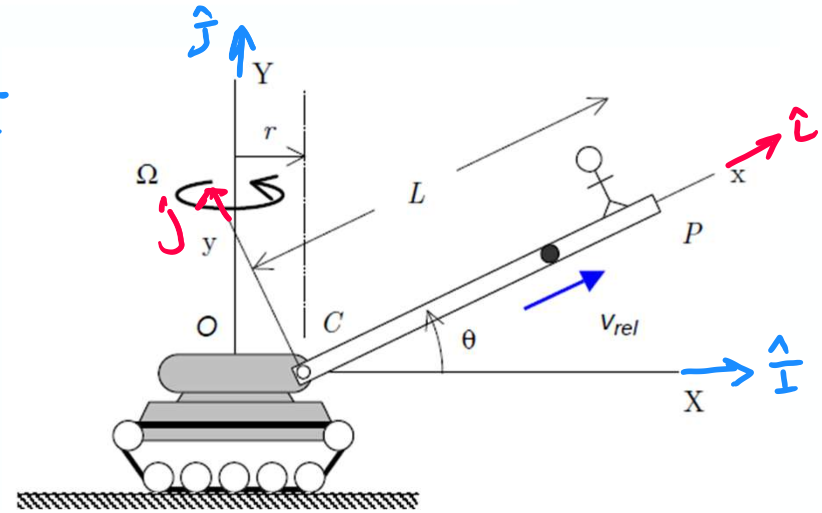
Given: The turret on a tank is rotating about a fixed vertical axis at a constant rate of $\Omega = 0.4$ rad/s. The barrel is being raised at a constant rate of $\dot{\theta} = 0.4$ rad/s . A cannon shell is fired with a constant muzzle speed of $v_{rel} = 200$ ft/s relative to the barrel. The observer and the xyz axes are attached to the barrel, while the XYZ axes are fixed. Here, $r = 3$ ft and $L = 15$ ft.

Find:

(a) The angular velocity of the barrel at the instant shown; $\vec{\omega}$

(b) The angular acceleration of the barrel at the instant shown; and $\vec{\alpha}$

(c) The acceleration of the shell as it leaves the barrel at P. $\vec{a}_P$



$$\vec{a}_P = \vec{a}_C + \cancel{(\vec{a}_{P/C})_{rel}}^{\,0} + \vec{\alpha}\times\vec{r}_{P/C} + 2\vec{\omega}\times\overset{v_{rel}\hat{i}}{\underline{(\vec{v}_{P/C})_{rel}}} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{P/C})$$

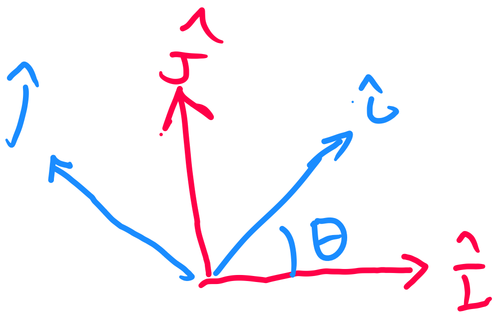
Rigid body OC:

$$\vec{a}_C = \cancel{\vec{a}_O} + \cancel{\vec{\alpha}_{OC}}\times\vec{r}_{C/O} + \omega_{CO}\times(\omega_{CO}\times r_{C/O})$$

$$= \Omega\hat{J}\times(\Omega\hat{J}\times r\hat{I}) = \Omega\hat{J}\times -\Omega r\hat{K} = -\Omega^2 r\hat{I}$$

$$\vec{a}_p = -\Omega^2 r \hat{I} + \dot{\theta}\Omega \hat{I} \times L\hat{i} + 2(\Omega \hat{J} + \dot{\theta}\hat{k}) \times v_{rel}\hat{i}$$
$$+ (\Omega\hat{J} + \dot{\theta}\hat{k}) \times [(\Omega\hat{J} + \dot{\theta}\hat{K}) \times L\hat{i}]$$

Convert to consistent vector system



$$\hat{i} = \cos\theta\hat{I} + \sin\theta\hat{J}$$
$$\hat{k} = \hat{K}$$

$$\vec{a}_p = -\Omega^2 r\hat{I} + \dot{\theta}\Omega\hat{I} \times L(\cos\theta\hat{I} + \sin\theta\hat{J})$$
$$+ 2(\Omega\hat{J} + \dot{\theta}\hat{K}) \times v_{rel}(\cos\theta\hat{I} + \sin\theta\hat{J})$$
$$+ (\Omega\hat{J} + \dot{\theta}\hat{K}) \times (\Omega\hat{J} + \dot{\theta}\hat{K}) \times (L\cos\theta\hat{I} + L\sin\theta\hat{J})$$

$$= -\Omega^2 r\hat{I} + \dot{\theta}\Omega L\sin\theta\hat{K} - 2\Omega v_{rel}\cos\theta\hat{K} + 2\dot{\theta}v_{rel}\cos\theta\hat{J}$$
$$- 2\dot{\theta}\sin\theta\hat{I} - (\Omega\hat{J} + \dot{\theta}\hat{K}) \times (-\Omega L\cos\theta\hat{K} + \dot{\theta}L\cos\theta\hat{J} - \dot{\theta}L\sin\theta\hat{I})$$

$$= -\Omega^2 r\,\hat{I} + \dot{\theta}\Omega L\sin\theta\,\hat{K} - 2\Omega v_{rel}\cos\theta\,\hat{K} + 2\dot{\theta} v_{rel}\cos\theta\,\hat{J}$$

$$-2\dot{\theta} v_{rel}\sin\theta\,\hat{I} - \Omega^2 L\cos\theta\,\hat{I} + \Omega\dot{\theta} L\sin\theta\,\hat{K} - \dot{\theta}^2 L\cos\theta\,\hat{I}$$

$$-\dot{\theta} L\sin\theta\,\hat{J}$$