ME 274 Lecture 15

Moving Reference Kinematics: 3D Part 3

Eugenio “Henny” Frias-Miranda

2/18/26

Housekeeping/Announcements

***Reminder for Henny to wear a mic during the lecture.

1. **HW 14 (3.G and 3.H) due tonight!**
2. **Office hours are changing to ME2008B…**
    - **Second floor of renovated side of ME**

Eugenio Frias Miranda Office Hours 274

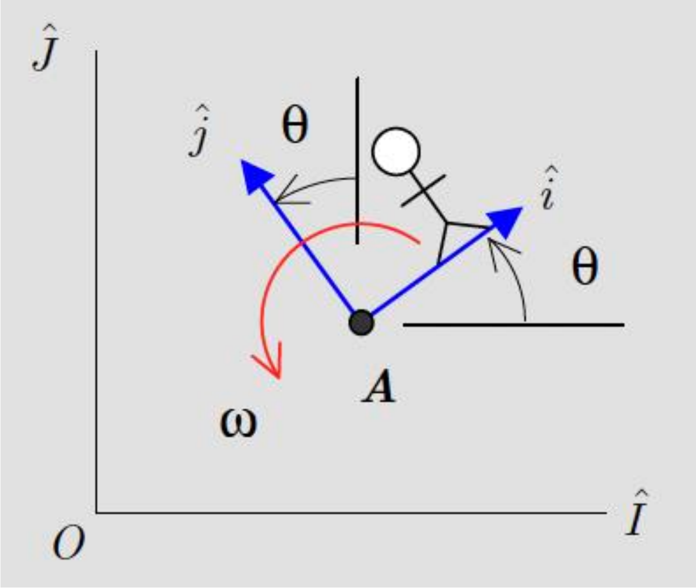
Description:	Approved by Beth Hess
Confirmation status:	Confirmed
Room:	ME 2008 Hotel Offices - ME 2008B Private Office
Start time:	01:30:00PM - Monday 16 February 2026
Duration:	1 hours
End time:	02:30:00PM - Monday 16 February 2026
Type:	Internal
Created by:	ewolters
Modified by:	ewolters
Last updated:	03:56:07PM - Friday 13 February 2026
Repeat type:	Weekly
Repeat every:	1 week
Repeat day:	Monday Wednesday Friday
Repeat end date:	Friday 08 May 2026

(3) Course Evals, bonus points on hwk !

(4) Sending Email w/ above (2) & (3)

(5) Exams tell you where observer is

Chapter 3: Moving Reference Frame Kinematics



$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]$$

- $\vec{v}_A$ and $\vec{v}_B$ are the velocities seen by a **fixed observer** [XYZ]
- $\vec{a}_A$ and $\vec{a}_B$ are the accelerations seen by **fixed observer** [XYZ]
- $\vec{\omega}$ is angular velocity of the **moving observer** [xyz]
- $\vec{\alpha}$ is angular acceleration of the **moving observer** [xyz]

How does observer move?

- $\left(\vec{v}_{B/A}\right)_{rel}$ is the "velocity of point B as seen by the **moving observer** at A"
- $\left(\vec{a}_{B/A}\right)_{rel}$ is the "acceleration of point B as seen by the **moving observer** at A"
- $2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel}$ is known as the **"Coriolis" component** of acceleration.
  - Arises when observer has a non-zero angular velocity

What does the observer see?

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3D Rotating Reference Frames

- For a 3D Rotating Reference Frame system/problem, we will use the same Moving Reference Frame equations as before, except we now have a 'k' term.
  - For derivation, look at pg. 158-159.

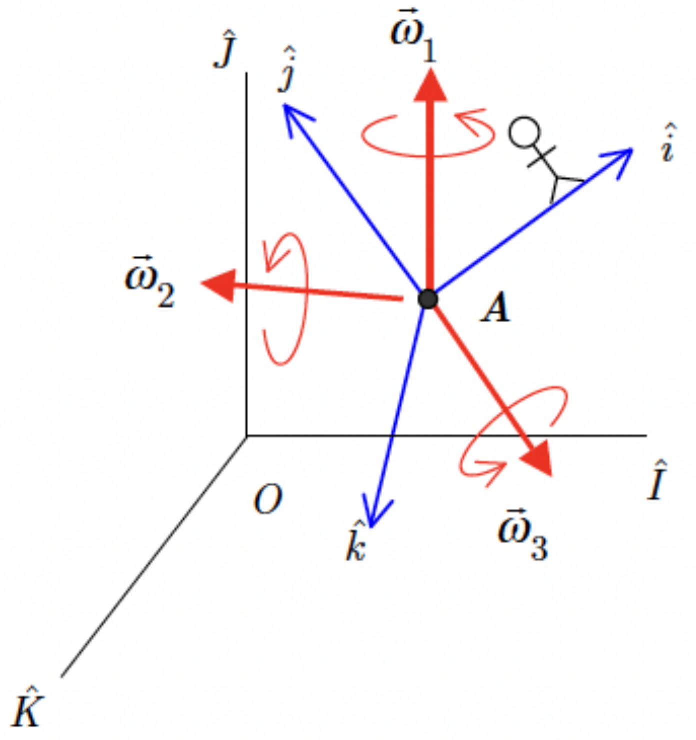
- **Angular Velocity** in 3D motion will usually be made up of several components (omega's). With each component being about a different axis, as shown in the figure.

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3 + \ldots$$

- For **Angular Acceleration**, it is important to note distinction between a **fixed axis** and a **rotating/moving** axis.

- Recalling derivation in pg. 144. **For a rotating axis we see that:**
  - **Straightforward to remember:**

$$\frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i} \qquad \frac{d\hat{j}}{dt} = \vec{\omega} \times \hat{j} \qquad \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}$$

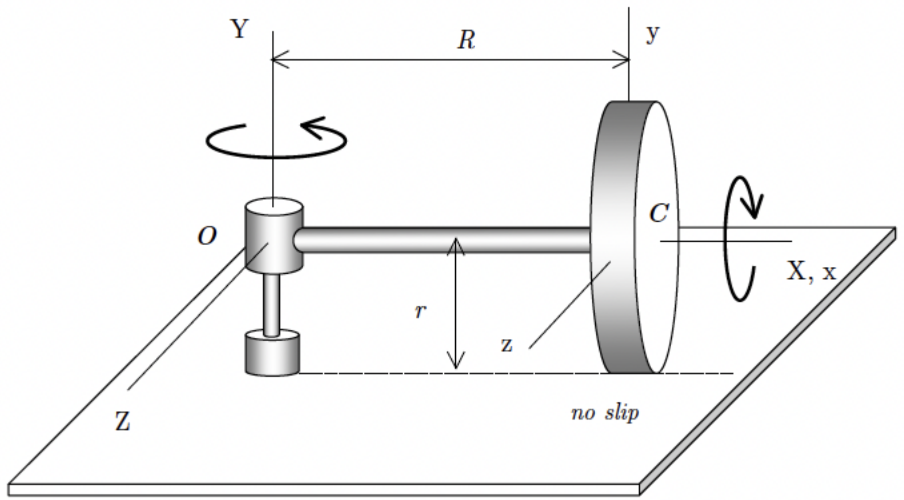


[pg. 157 content]

Example 3.B.9

Given: Arm OC rotates about the fixed Y-axis at a constant rate Ω. The disk at C, having a radius of R, is able to rotate about arm OC and rolls without slipping on a fixed horizontal surface. Let the xyz axes be attached to the disk.

Find: Determine the angular acceleration of the disk.

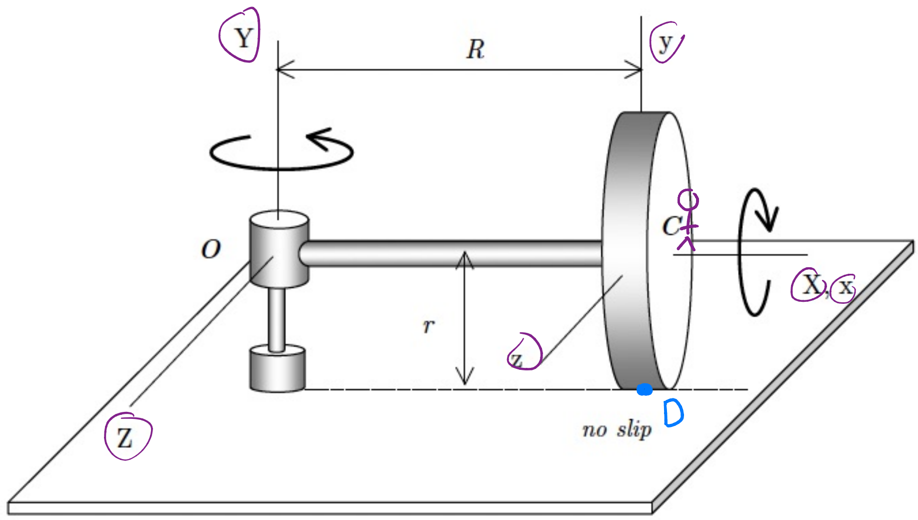


[pg. 171]

Example 3.B.9 p.171

Given: Arm OC rotates about the fixed Y-axis at a constant rate Ω. The disk at C, having a radius of R, is able to rotate about arm OC and rolls without slipping on a fixed horizontal surface. Let the xyz axes be attached to the disk.

Find: Determine the angular acceleration of the disk. α?



① Axes defined. Place observer

② Angular Velocity. How is obs. moving?

$$\vec{\omega} = \Omega \hat{j} + -\omega_{disk}\hat{i}$$

③ Velocity @ point C. Using O & C

$$\vec{v}_C = \vec{v}_O + \vec{\omega}_{OC} \times \vec{r}_{C/O}$$
$$= \vec{0} + \Omega\hat{j} \times R\hat{i}$$
$$= -\Omega R\hat{k}$$

④ Velocity @ point C. Using D & C

$$\vec{v}_C = \vec{v}_D + \vec{\omega}_{CD} \times \vec{r}_{C/D}$$
$$= \vec{0} + -\omega_{disk}\hat{i} \times r\hat{j}$$
$$= -\omega_{disk} r\hat{k}$$

⑤ Equate ③ & ④. Solve for ω_{disk}

$$\Rightarrow -\Omega R = -\omega_{disk} r$$
$$\Rightarrow \omega_{disk} = \frac{\Omega R}{r}$$

⑥ Now w/ $\vec{\omega}_{disk}$ we can plug into ①

$$\vec{\omega} = \Omega\hat{j} - \frac{\Omega R}{r}\hat{i}$$

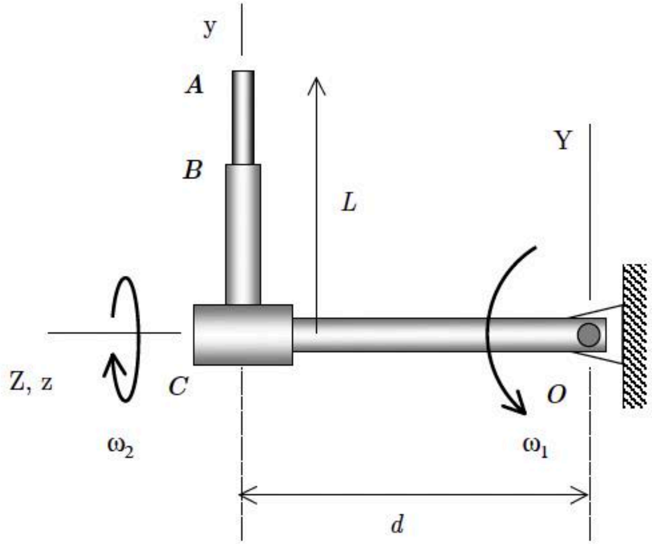
⑦ Take derivative of ω. Solve for α

$$\vec{\alpha} = \dot{\Omega}\hat{j} + \Omega\dot{\hat{j}} - \frac{\dot{\Omega} R}{r}\hat{i} - \frac{\Omega R}{r}\dot{\hat{i}}$$
$$= -\frac{\Omega R}{r}[\vec{\omega} \times \hat{i}]$$
$$= -\frac{\Omega R}{r}[(\Omega\hat{j} - \frac{\Omega R}{r}\hat{i}) \times \hat{i}]$$
$$= +\frac{\Omega^2 R}{r}\hat{k}$$

Example 3.B.10

Given: $\dot{L} = 0.06$ m/s $=$ *constant*, $\omega_1 = 1.2$ rad/s $=$ *constant* and $\omega_2 = 1.5$ rad/s $=$ *constant*. At the position shown, AC is aligned with the fixed Y-axis, $L = 0.12$ m, and $d = 0.02$ m.

Find: Determine:
(a) The velocity of end A of the telescoping rod AC; and
(b) The acceleration of the same point.



[pg. 172]

Example 3.B.10 p.172

Given: $\dot{L} = 0.06$ m/s $=$ *constant*, $\omega_1 = 1.2$ rad/s $=$ *constant* and $\omega_2 = 1.5$ rad/s $=$ *constant*. At the position shown, AC is aligned with the fixed Y-axis, $L = 0.12$ m, and $d = 0.02$ m.

Find: Determine:

(a) The velocity of end A of the telescoping rod AC; and $\vec{v}_A$?

(b) The acceleration of the same point. $\vec{a}_A$?

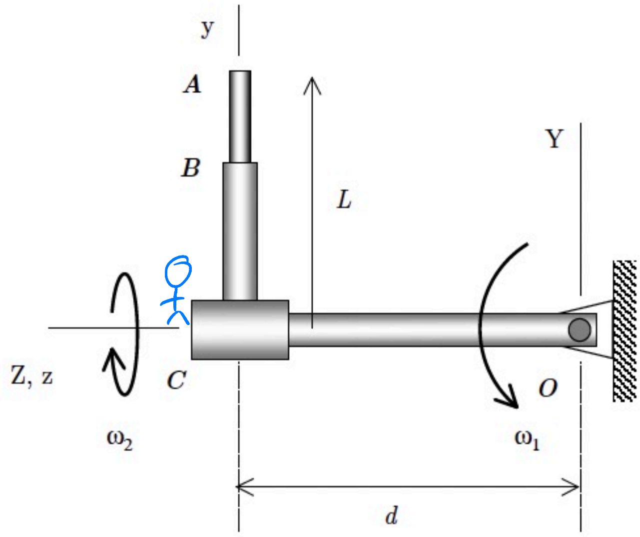
Given:

$L = 0.12m$; $\dot{L} = 0.06$ m/s ; $\ddot{L} = 0$

$\omega_1 = 1.2$ rad/s ; $\dot{\omega}_1 = 0$

$\omega_2 = 1.5$ rad/s ; $\dot{\omega}_2 = 0$

$d = 0.02m$



① Place observer. Axis Defined

② Angular Velocity

$\vec{\omega} = \omega_1 \hat{I} - \omega_2 \hat{k}$

$= \omega_1 \hat{\imath} - \omega_2 \hat{k}$

③ Angular Acceleration

$\vec{\alpha} = \dot{\omega}_1 \hat{I} + \omega_1 \dot{\hat{I}} - \dot{\omega}_2 \hat{k} - \omega_2 \dot{\hat{k}}$

$= -\omega_2 [\vec{\omega} \times \hat{k}]$

$= -\omega_2 [(\omega_1 \hat{I} - \omega_2 \hat{k}) \times \hat{k}]$

$= \omega_1 \omega_2 \hat{J}$

Velocity:

④ Find $\vec{v}$ @ A. Use Points A & C

$\vec{v}_A = \vec{v}_C + (\vec{v}_{A/C})_{rel} + \vec{\omega} \times \vec{r}_{A/C}$

⑤ Find $\vec{v}_C$, to plug back in

$\vec{v}_C = \vec{v}_O + \vec{\omega}_{OC} \times \vec{r}_{C/O}$

$= \vec{0} + \omega_1 \hat{I} \times (d \hat{k})$

$= -\omega_1 d \hat{J}$

⑥ w/ $\vec{v}_C$ Solve for $\vec{v}_A$

$\vec{v}_A = -\omega_1 d \hat{J} + \dot{L} \hat{J} + (\omega_1 \hat{\imath} - \omega_2 \hat{k}) \times L \hat{J}$

$= -\omega_1 d \hat{J} + \dot{L} \hat{J} + \omega_1 L \hat{k} + \omega_2 L \hat{\imath}$ (a)

Acceleration:

⑦ Find $\vec{a}$ @ A. Use Points A & C

$\vec{a}_A = \vec{a}_C + (\vec{a}_{A/C})_{rel} + \vec{\alpha} \times \vec{r}_{A/C} + 2\vec{\omega} \times (\vec{v}_{A/C})_{rel} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/C})$

⑧ Find $\vec{a}_C$. to plug back into ⑦

$\vec{a}_C = \vec{a}_O + \vec{\alpha}_{CO} \times \vec{r}_{C/O} + \vec{\omega}_{CO} \times (\vec{\omega}_{CO} \times \vec{r}_{C/O})$ (fixed)

$= \omega_1 \hat{I} \times (\omega_1 \hat{I} \times d \hat{k})$

$= -\omega_1^2 d \hat{k}$

⑨ Solve for $\vec{a}_A$. Knowing $\vec{a}_C$

$\vec{a}_A = -\omega_1^2 d \hat{k} + \vec{0} + \omega_1 \omega_2 \hat{J} \times L \hat{J} + 2(\omega_1 \hat{\imath} - \omega_2 \hat{k}) \times \dot{L} \hat{J} + (\omega_1 \hat{\imath} - \omega_2 \hat{k}) \times [(\omega_1 \hat{\imath} - \omega_2 \hat{k}) \times L \hat{J}]$

$= -\omega_1^2 d \hat{k} + 2\omega_1 \dot{L} \hat{k} + 2\omega_2 \dot{L} \hat{\imath} + (\omega_1 \hat{\imath} - \omega_2 \hat{k}) \times (\omega_1 L \hat{k} + \omega_2 L \hat{\imath})$

$= -\omega_1^2 d \hat{k} + 2\omega_1 \dot{L} \hat{k} + 2\omega_2 \dot{L} \hat{\imath} - \omega_1^2 L \hat{J} - \omega_2^2 L \hat{J}$ (b)

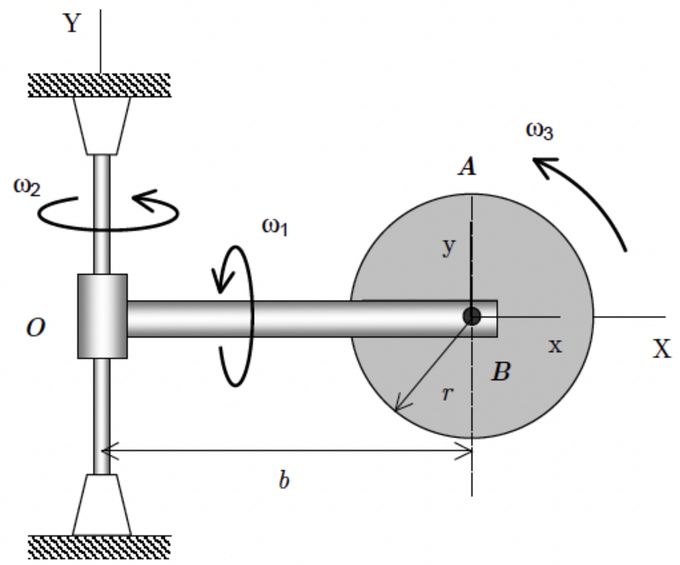
Example 3.B.11

Given: Rotation rates ω_1, ω_2 and ω_3 are all constant. The XYZ axes are fixed, and the xyz axes are attached to the disk. At the instant shown, A is directly above the center B of the disk and the xyz and XYZ axes are aligned.

Find: Determine:

(a) The velocity of point A; and

(b) The acceleration of point A.

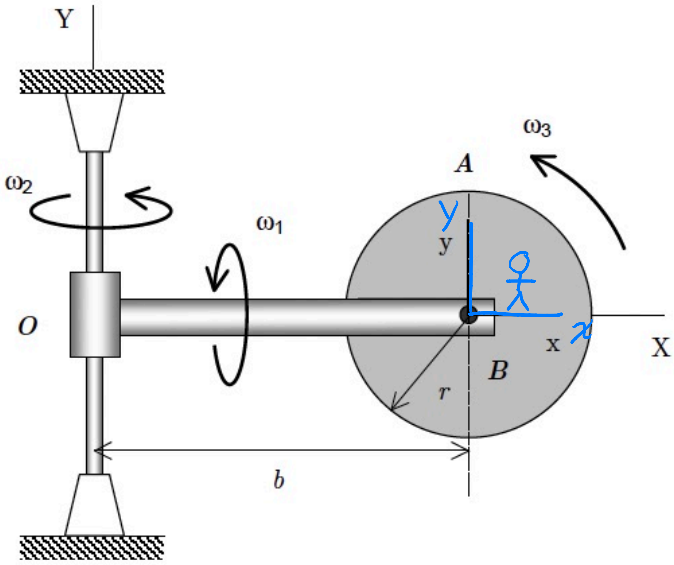


[pg. 173]

Example 3.B.11 p.173

Given: Rotation rates ω_1, ω_2 and ω_3 are all constant. The XYZ axes are fixed, and the xyz axes are attached to the disk. At the instant shown, A is directly above the center B of the disk and the xyz and XYZ axes are aligned.

Find: Determine:
(a) The velocity of point A; and $\vec{v}_A$?
(b) The acceleration of point A. $\vec{a}_A$?



① Change observer in point B on Link OB instead of disk. Makes problem easier to do.

② angular velocity

$\vec{\omega} = \omega_2 \hat{J} - \omega_1 \hat{i}$

$= \omega_2 \hat{J} - \omega_1 \hat{I}$

③ angular acceleration (const spd, fixed, const speed)

$\vec{\alpha} = \dot{\omega}_2 \hat{J} + \omega_2 \dot{\hat{J}} - \dot{\omega}_1 \hat{i} - \omega_1 \dot{\hat{i}}$

$= -\omega_1 (\vec{\omega} \times \hat{i})$

$= -\omega_1 [(\omega_2 \hat{J} - \omega_1 \hat{i}) \times \hat{i}]$

$= -\omega_1 \omega_2 (\hat{J} \times \hat{I})$

$= \omega_1 \omega_2 \hat{K}$

$(v)_{rel} = \omega \cdot r$

helpful for $(v)_{rel}$ & $(a)_{rel}$:

$a = \omega^2 r$

$a = \dfrac{v^2}{r}$

Where:
"a" is acceleration
"ω" is angular velocity
"v" is scalar velocity
"r" is radius

centripetal acceleration eqn

Velocity:

④ Velocity at A. Use point A & B

$\vec{v}_A = \vec{v}_B + (\vec{v}_{A/B})_{rel} + \vec{\omega} \times \vec{r}_{A/B}$

⑤ Find $\vec{v}_B$. to plug back to ④

$\vec{v}_B = \vec{v}_O + \vec{\omega}_{BO} \times \vec{r}_{B/O}$

$= \vec{0} + \omega_2 \hat{J} \times b \hat{I}$

$= -\omega_2 b \hat{K}$

⑥ Solve for $\vec{v}_A$, Knowing $\vec{v}_B$.

$\vec{v}_A = -\omega_2 b \hat{K} + -r\omega_3 \hat{i} + (\omega_2 \hat{J} - \omega_1 \hat{i}) \times r\hat{j}$

$= -\omega_2 b \hat{K} - r\omega_3 \hat{I} - \omega_1 r \hat{K}$

$= (-r\omega_3)\hat{I} + (-\omega_2 b - \omega_1 r)\hat{K}$ (a)

Acceleration:

⑦ Acceleration at A. Use point A & B

$\vec{a}_A = \vec{a}_B + (\vec{a}_{A/B})_{rel} + \vec{\alpha} \times \vec{r}_{A/B} + 2\vec{\omega} \times (\vec{v}_{A/B})_{rel} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/B})$

⑧ Find $\vec{a}_B$. to plug back to ⑦

$\vec{a}_B = \vec{a}_O + \vec{\alpha}_{BO} \times \vec{r}_{B/O} + \vec{\omega}_{BO} \times (\vec{\omega}_{B/O} \times \vec{r}_{B/O})$

$= \vec{0} + \vec{0} + \omega_2 \hat{J} \times (\omega_2 \hat{J} \times b\hat{i})$

$= -\omega_2^2 b \hat{I}$

⑨ Solve for $\vec{a}_A$, Knowing $\vec{a}_B$.

$\vec{a}_A = -\omega_2^2 b \hat{I} + -\omega_3^2 r \hat{j} + \omega_1 \omega_2 \hat{K} \times r\hat{j} + 2(\omega_2 \hat{J} - \omega_1 \hat{i}) \times (-r\omega_3 \hat{i}) + (\omega_2 \hat{J} - \omega_1 \hat{i}) \times [(\omega_2 \hat{J} - \omega_1 \hat{i}) \times r\hat{j}]$

$\vec{a}_A = -\omega_2^2 b \hat{I} - \omega_3^2 r \hat{J} - \omega_1 \omega_2 r \hat{I} + 2\omega_2 \omega_3 r \hat{K} + (\omega_2 \hat{J} - \omega_1 \hat{I}) \times [-\omega_1 r \hat{K}]$

$= -\omega_2^2 b \hat{I} - \omega_3^2 r \hat{J} - \omega_1 \omega_2 r \hat{I} + 2\omega_2 \omega_3 r \hat{K} - \omega_1 \omega_2 r \hat{I} - \omega_1^2 r \hat{J}$

$= (-\omega_2^2 b - 2\omega_1 \omega_2 r)\hat{I} + (-\omega_3^2 r - \omega_1^2 r)\hat{J} + 2\omega_2 \omega_3 r \hat{K}$ (b)

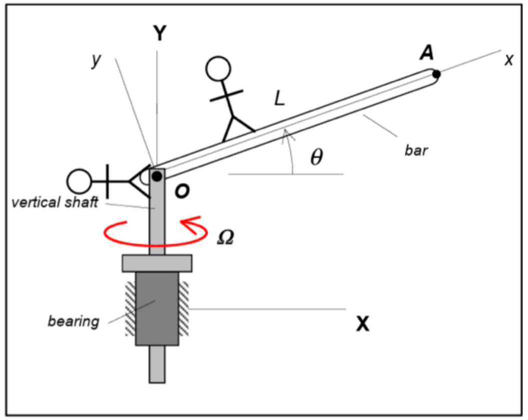
Summary: 3D Moving Reference Frame Kinematics 2

[pg. 158-159]

PROBLEM: A person attached to a moving body (reference frame) is observing the motion of point A.

$$\vec{v}_A = \vec{v}_O + \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\vec{r}_{A/O}$$

$$\vec{a}_A = \vec{a}_O + \left(\vec{a}_{A/O}\right)_{rel} + \vec{\alpha}\times\vec{r}_{A/O} + 2\vec{\omega}\times\left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{A/O}\right)$$



Lec 15 Short Feedback Form:

CHANGING OBSERVERS: For constant rotation rates,

Observer on vertical shaft:

$$\vec{\omega} = \Omega\hat{J}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \vec{0}$$

$$\left(\vec{v}_{A/O}\right)_{rel} = L\dot{\theta}\hat{j}$$

$$\left(\vec{a}_{A/O}\right)_{rel} = -L\dot{\theta}^2\hat{i}$$

Observer on arm OA:

$$\vec{\omega} = \Omega\hat{J} + \dot{\theta}\hat{k}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \dot{\Omega}\hat{J} + \Omega\dot{\hat{J}} + \ddot{\theta}\hat{k} + \dot{\theta}\dot{\hat{k}} = \dot{\theta}\left(\vec{\omega}\times\hat{k}\right)$$

$$\left(\vec{v}_{A/O}\right)_{rel} = \vec{0}$$

$$\left(\vec{a}_{A/O}\right)_{rel} = \vec{0}$$

These give the same result! Try it.