ME 274 Lecture 14

Moving Reference Kinematics: 3D Part 2

Eugenio “Henny” Frias-Miranda

2/16/26

Housekeeping/Announcements

***Reminder for Henny to wear a mic during the lecture.

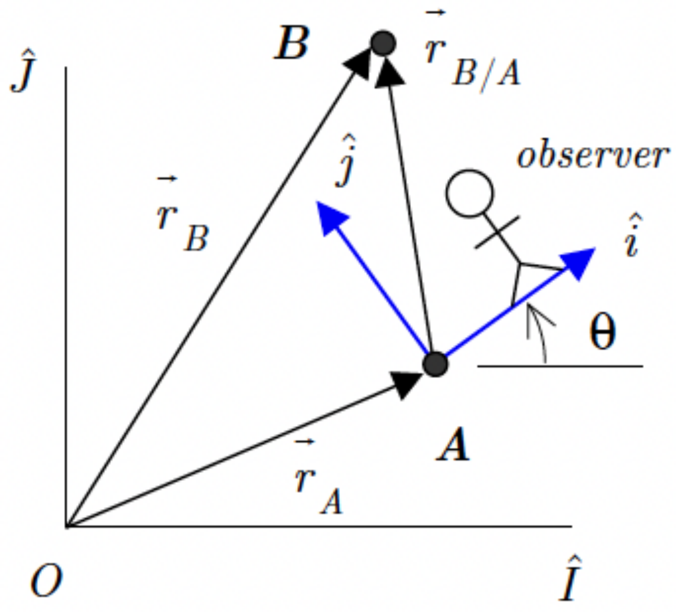
1. **HW due tonight!**
2. **Office hours are changing to ME2008B…**
    - **Second floor of renovated side of ME**

Eugenio Frias Miranda Office Hours 274

Description:	Approved by Beth Hess
Confirmation status:	Confirmed
Room:	ME 2008 Hotel Offices - ME 2008B Private Office
Start time:	01:30:00PM - Monday 16 February 2026
Duration:	1 hours
End time:	02:30:00PM - Monday 16 February 2026
Type:	Internal
Created by:	ewolters
Modified by:	ewolters
Last updated:	03:56:07PM - Friday 13 February 2026
Repeat type:	Weekly
Repeat every:	1 week
Repeat day:	Monday Wednesday Friday
Repeat end date:	Friday 08 May 2026

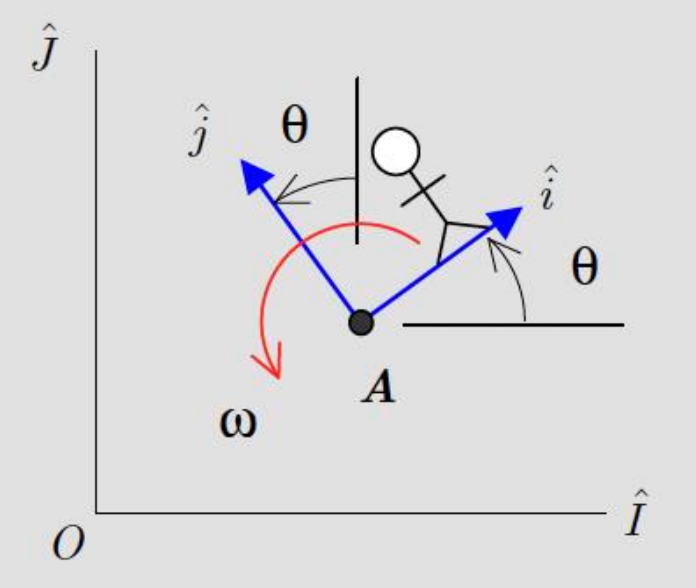
(Reminder/refresher) *Chapter 3 – Moving Reference Frame Kinematics*

- In chapter 2, we saw that if two points A and B are on the **SAME** rigid body.
- It is often the case that we need to use the motion of two points that are **NOT** on the same rigid body.
- Many times, we can use kinematic information about the motion of B obtained from an observer on a "**moving reference frame**" [figure]



[pg. 143 content]

Chapter 3: Moving Reference Frame Kinematics



$$\boxed{\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}}$$

$$\boxed{\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]}$$

- $\vec{v}_A$ and $\vec{v}_B$ are the velocities seen by a **fixed observer** [XYZ]
- $\vec{a}_A$ and $\vec{a}_B$ are the accelerations seen by **fixed observer** [XYZ]
- $\vec{\omega}$ is angular velocity of the **moving observer** [xyz]
- $\vec{\alpha}$ is angular acceleration of the **moving observer** [xyz]

How does observer move?

- $\left(\vec{v}_{B/A}\right)_{rel}$ is the “velocity of point B as seen by the **moving observer** at A”
- $\left(\vec{a}_{B/A}\right)_{rel}$ is the “acceleration of point B as seen by the **moving observer** at A”
- $2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel}$ is known as the **“Coriolis” component** of acceleration.
  - Arises when observer has a non-zero angular velocity

What does the observer see?

[pg. 147 content]

Last Class… 3D Rotating Reference Frames

- For a 3D Rotating Reference Frame system/problem, we will use the same Moving Reference Frame equations as before, except we now have a 'k' term.
  - For derivation, look at pg. 158-159.

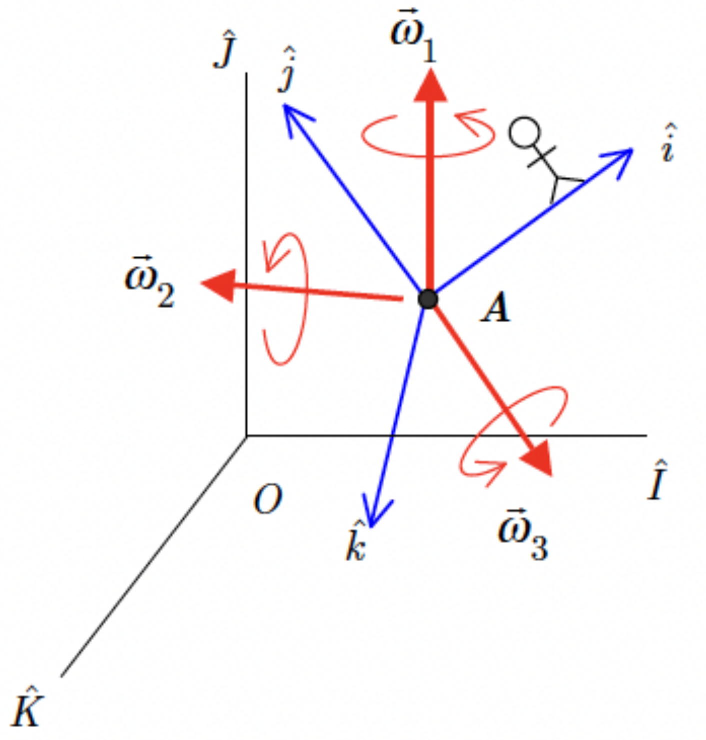
- **Angular Velocity** in 3D motion will usually be made up of several components (omega's). With each component being about a different axis, as shown in the figure.

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3 + \ldots$$

- For **Angular Acceleration**, it is important to note distinction between a **fixed axis** and a **rotating/moving** axis.

- Recalling derivation in pg. 144. **For a rotating axis we see that:**
  - **Straightforward to remember:**

$$\boxed{\frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i}} \qquad \boxed{\frac{d\hat{j}}{dt} = \vec{\omega} \times \hat{j}} \qquad \boxed{\frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}}$$

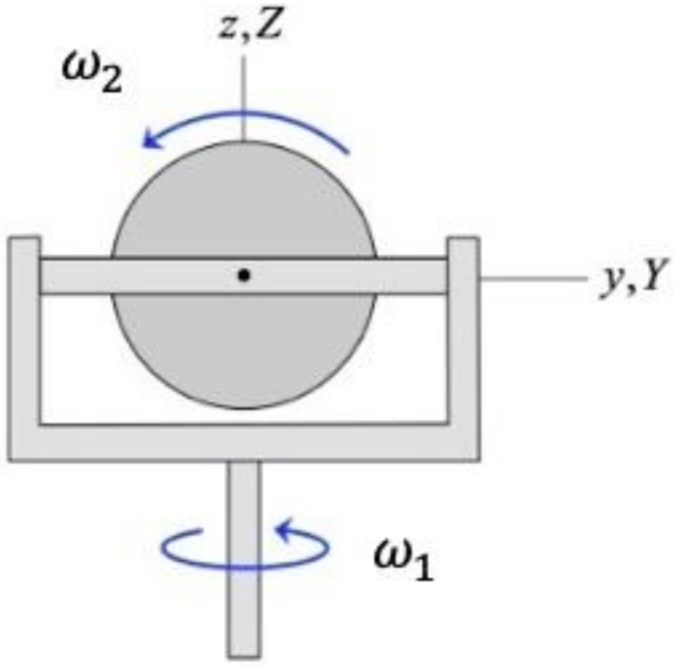


[pg. 157 content]

p. 161 Challenge Question and Animation:
*Does having a **Constant Angular Velocity** mean **Angular Acceleration is equal to zero…?***

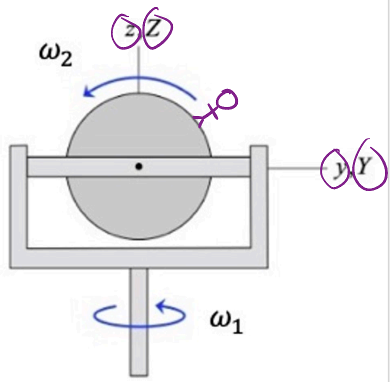
- **TLDR –** In 2D problems yes, **but in 3D problems: NO**

- **Why? Let's to this example problem for example:**
  - A gyro rotor has a constant rotation rate of ω_1 about a fixed vertical axis in addition to a constant rotation rate of ω_2 about the symmetry axis of the rotor. Let the xyz-axes be attached to the rotor, and the XYZ-axes be fixed to ground. **Find the angular acceleration of the rotor.**



p. 161 Challenge Question and Animation:
Does having a **Constant Angular Velocity** mean **Angular Acceleration is equal to zero…?**

- **TLDR –** In 2D problems yes, **but in 3D problems: NO**
- **Why? Let's to this example problem for example:**
  - A gyro rotor has a constant rotation rate of ω_1 about a fixed vertical axis in addition to a constant rotation rate of ω_2 about the symmetry axis of the rotor. Let the xyz-axes be attached to the rotor, and the XYZ-axes be fixed to ground. **Find the angular acceleration of the rotor.**



(1) Attatch observer to rotor. Axes defined.

(2) Find ω, angular vel

$$\vec{\omega} = \omega_1 \hat{K} + \omega_2 \hat{i} \quad ; \quad \hat{K} = \text{fixed} \; ; \; \hat{i} = \text{moving}$$

(3) Find α, angular accel

$$\vec{\alpha} = \dot{\vec{\omega}}$$

$$= \overset{\text{constant}}{\cancel{\dot{\omega}_1 \hat{K}}} + \overset{\text{fixed axis}}{\cancel{\omega_1 \dot{\hat{K}}}} + \overset{\text{constant}}{\cancel{\dot{\omega}_2 \hat{i}}} + \omega_2 \dot{\hat{i}}$$

$$= \omega_2 (\vec{\omega} \times \hat{i})$$

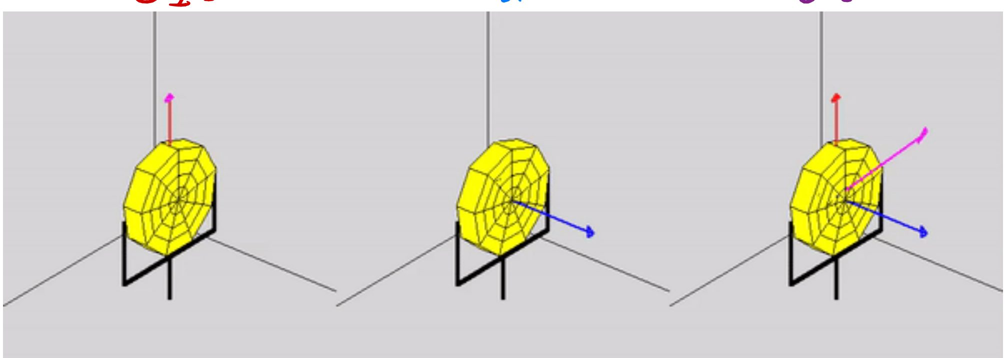
$$= \omega_2 (\omega_1 \hat{K} + \omega_2 \hat{i}) \times \hat{i}$$

$$= \omega_1 \omega_2 \hat{K} \times \hat{i} \quad ; \; \hat{K} = \hat{k}$$

$$= \omega_1 \omega_2$$

(4) Visualization

ω_1: ω_2: $\vec{\omega}$:



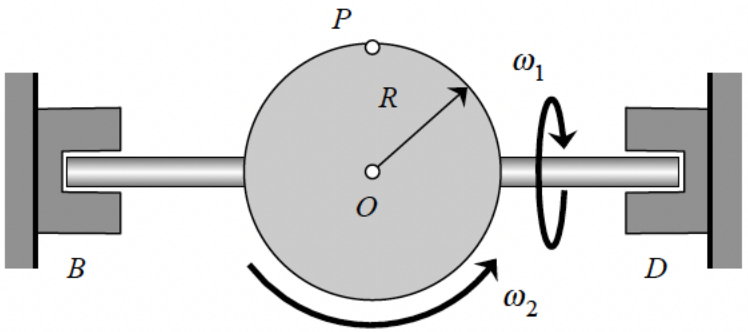
- constant Magnitude
- changes in direction

Example 3.B.6

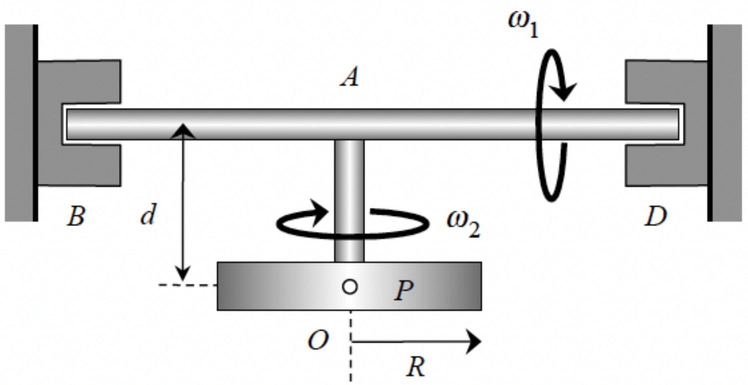
Given: Shaft BD rotates about a fixed axis with a constant rate of ω_1. Shaft OA is rigidly attached to shaft BD with OA being perpendicular to BD. A disk rotates about shaft OA with a constant rate of ω_2 relative to OA.

Find: The acceleration of point P on the edge of the disk for the position shown.

Use the following parameters in your analysis: $\omega_1 = 5$ rad/s, $\omega_2 = 8$ rad/s, $R = 6$ in and $d = 4$ in.



side view



top view

[pg. 168]

Example 3.B.6 p.168

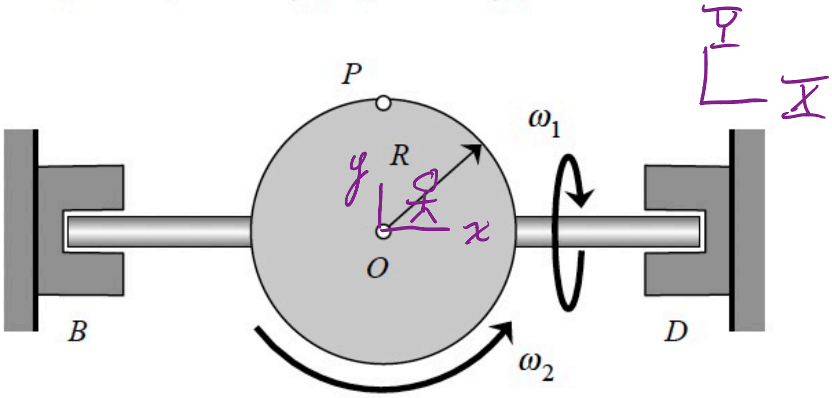
Given: Shaft BD rotates about a fixed axis with a constant rate of ω_1. Shaft OA is rigidly attached to shaft BD with OA being perpendicular to BD. A disk rotates about shaft OA with a constant rate of ω_2 relative to OA.

Find: The acceleration of point P on the edge of the disk for the position shown.

Use the following parameters in your analysis: $\omega_1 = 5$ rad/s, $\omega_2 = 8$ rad/s, $R = 6$ in and $d = 4$ in.

① Define axes & observer

one of few w/o axes drawn



side view

② Find angular velocity, $\vec{\omega}$

$\vec{\omega} = -\omega_1 \hat{I} + \omega_2 \hat{k}$

$= -\omega_1 \hat{\imath} + \omega_2 \hat{k}$

③ Find angular acceleration, $\vec{\alpha}$

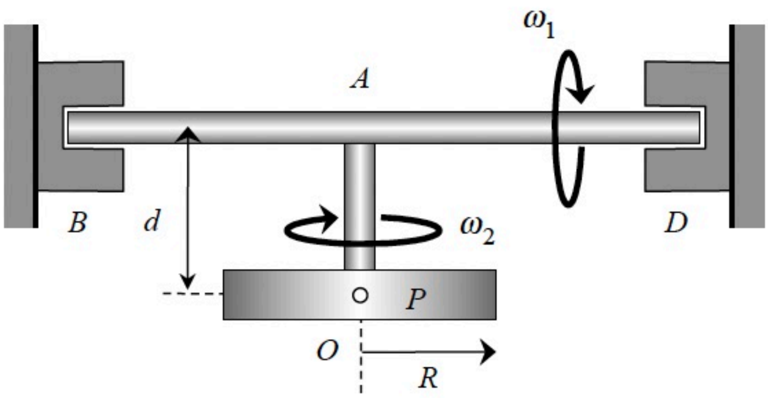
$\vec{\alpha} = \dot{\vec{\omega}}$

$= -\cancel{\dot{\omega}_1} \hat{I} - \cancel{\omega_1 \dot{\hat{I}}} + \cancel{\dot{\omega}_2} \hat{k} + \omega_2 \dot{\hat{k}}$

$= \omega_2 [\vec{\omega} \times \hat{k}]$

$= \omega_2 [(-\omega_1 \hat{\imath} + \omega_2 \hat{k}) \times \hat{k}]$

$= +\omega_2 \omega_1 \hat{j}$



top view

④ Points O & P. Not a Rigid Body

$\vec{a}_P = \vec{a}_O + \cancel{(\vec{a}_{P/O})_{rel}} + \vec{\alpha} \times \vec{r}_{P/O} + 2\vec{\omega} \times \cancel{(\vec{v}_{P/O})_{rel}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O})$

⑤ We Need $\vec{a}_O$. So use Link OA. Rigid body

$\vec{a}_O = \cancel{\vec{a}_A} + \cancel{\vec{\alpha}_{OA}} \times \vec{r}_{O/A} + \vec{\omega}_{OA} \times (\omega_{OA} \times \vec{r}_{O/A})$

$= -\omega_1 \hat{I} \times (-\omega_1 \hat{I} \times d \hat{K})$

$= -\omega_1^2 d \hat{K}$

⑥ w/ $\vec{a}_O$, Solve for $\vec{a}_P$

$\vec{a}_P = -\omega_1^2 d \hat{K} + \omega_2 \omega_1 \hat{j} \times R\hat{j} + (-\omega_1 \hat{\imath} + \omega_2 \hat{k}) \times [(-\omega_1 \hat{\imath} + \omega_2 \hat{k}) \times R \hat{j}]$

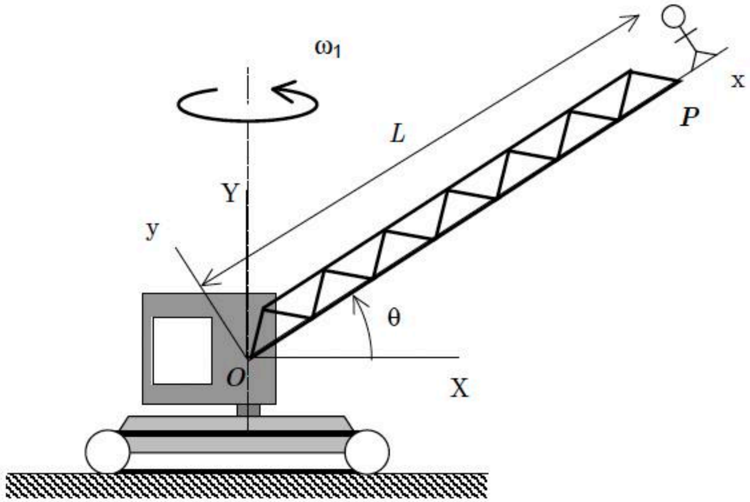
$= -\omega_1^2 d \hat{k} + (-\omega_1 \hat{\imath} + \omega_2 \hat{k}) \times [-\omega_1 R \hat{k} - \omega_2 R \hat{\imath}]$

$= -\omega_1^2 d \hat{k} + -\omega_1^2 R \hat{j} - \omega_2^2 R \hat{j}$

Example 3.B.7

Given: $\omega_1 = 0.30$ rad/s $=$ *constant*, $\dot{\theta} = 0.5$ rad/s $=$ *constant* and $L = 12$ m.

Find: When $\theta = 30°$, determine:
(a) The angular velocity of boom OP;
(b) The angular acceleration of boom OP;
(c) The velocity of end P of the boom; and
(d) The acceleration of end P of the boom.



[pg. 169]

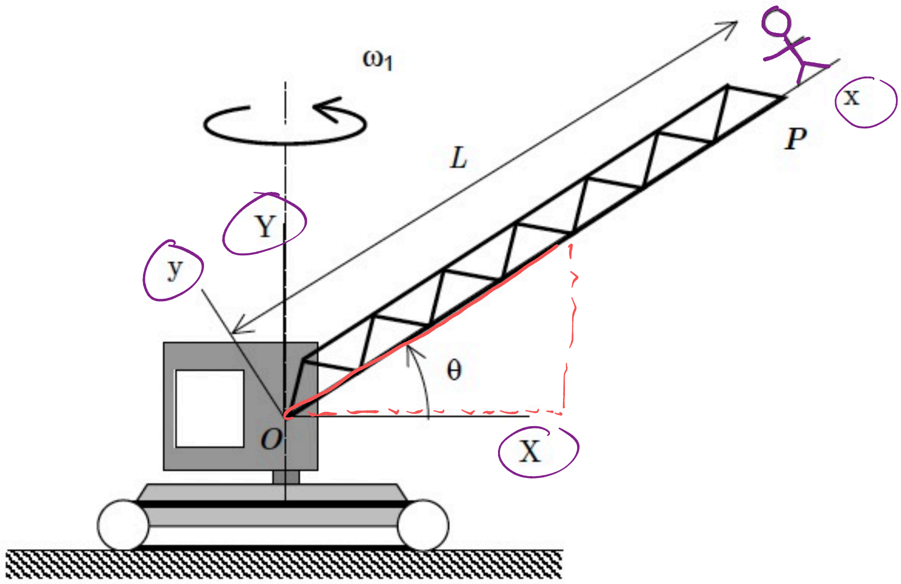
Example 3.B.7

p. 169

Given: $\omega_1 = 0.30$ rad/s $=$ *constant*, $\dot{\theta} = 0.5$ rad/s $=$ *constant* and $L = 12$ m.

Find: When $\theta = 30°$, determine:

(a) The angular velocity of boom OP; ω ?
(b) The angular acceleration of boom OP; α ?
(c) The velocity of end P of the boom; and v_P ?
(d) The acceleration of end P of the boom. a_P ?



(1) Axes are defined

(2) Find angular velocity, $\vec{\omega}$

$\vec{\omega} = \omega_1 \hat{j} + \dot{\theta}\hat{k}$

$= \omega_1 \hat{J} + \dot{\theta}\hat{K}$

(3) Find angular acceleration, $\vec{\alpha}$

$\vec{\alpha} = \dot{\vec{\omega}}$

$= \cancel{\dot{\omega}_1}\hat{J} + \omega_1 \cancel{\dot{\hat{J}}} + \cancel{\ddot{\theta}}\hat{k} + \dot{\theta}\dot{\hat{k}}$

$= \dot{\theta}[\vec{\omega} \times \hat{k}]$ $\hat{k}$ $\hat{K}$

$= \dot{\theta}[(\omega_1 \hat{J} + \cancel{\dot{\theta}\hat{K}}) \times \hat{K}]$

$= \dot{\theta}\omega_1 \hat{I}$

$\vec{r}_{P/O} = L\hat{i}$ $\hat{i} = \cos\theta \hat{I} + \sin\theta \hat{J}$

Good Rule of thumb on observer placement: Place on Rigid body that moves the most.

Velocity:

(4) Points P&O. Not a Rigid Body,

$\vec{v}_P = \vec{v}_O + (\vec{v}_{P/O})_{rel} + \vec{\omega} \times \vec{r}_{P/O}$

$= \vec{0} + \vec{0} + (\omega_1 \hat{J} + \dot{\theta}\hat{K}) \times L(\cos\theta \hat{I} + \sin\theta \hat{J})$

$= -\omega_1 L\cos\theta \hat{K} + \dot{\theta} L\cos\theta \hat{J} - \dot{\theta} L\sin\theta \hat{I}$

Acceleration:

(5) Points P&O. Not a Rigid Body,

$\vec{a}_P = \vec{a}_O + (\vec{a}_{P/O})_{rel} + \vec{\alpha} \times \vec{r}_{P/O} + 2\vec{\omega} \times (\vec{v}_{P/O})_{rel} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O})$

$= \vec{0} + \vec{0} + \dot{\theta}\omega_1 \hat{I} \times L(\cos\theta \hat{I} + \sin\theta \hat{J}) + \vec{0} + (\omega_1 \hat{J} + \dot{\theta}\hat{K}) \times [(\omega_1 \hat{J} + \dot{\theta}\hat{K}) \times L(\cos\theta \hat{I} + \sin\theta \hat{J})]$

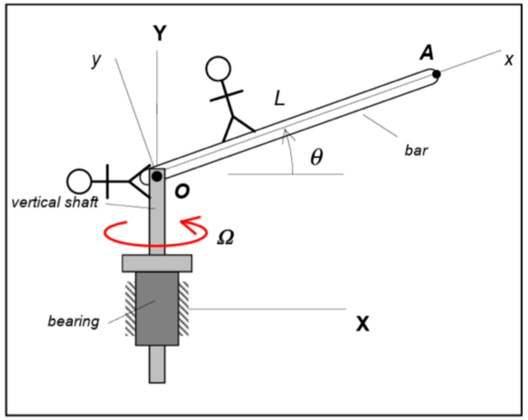
$= \dot{\theta}\omega_1 L\sin\theta \hat{K} + (\omega_1 \hat{J} + \dot{\theta}\hat{K}) \times (-\omega_1 L\cos\theta \hat{K} + \dot{\theta} L\cos\theta \hat{J} - \dot{\theta} L\sin\theta \hat{I})$

$= \dot{\theta}\omega_1 L\sin\theta \hat{K} - \omega_1^2 L\cos\theta \hat{I} + \dot{\theta}\omega_1 L\sin\theta \hat{K} - \dot{\theta}^2 L\cos\theta \hat{I} - \dot{\theta}^2 L\sin\theta \hat{J}$

Summary: 3D Moving Reference Frame Kinematics 2

[pg. 158-159]

PROBLEM: A person attached to a moving body (reference frame) is observing the motion of point A.



$$\vec{v}_A = \vec{v}_O + \left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\vec{r}_{A/O}$$

$$\vec{a}_A = \vec{a}_O + \left(\vec{a}_{A/O}\right)_{rel} + \vec{\alpha}\times\vec{r}_{A/O} + 2\vec{\omega}\times\left(\vec{v}_{A/O}\right)_{rel} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{A/O}\right)$$

CHANGING OBSERVERS: For constant rotation rates,

Observer on vertical shaft:

$$\vec{\omega} = \Omega\hat{J}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \vec{0}$$

$$\left(\vec{v}_{A/O}\right)_{rel} = L\dot{\theta}\hat{j}$$

$$\left(\vec{a}_{A/O}\right)_{rel} = -L\dot{\theta}^2\hat{i}$$

Observer on arm OA:

$$\vec{\omega} = \Omega\hat{J} + \dot{\theta}\hat{k}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \dot{\Omega}\hat{J} + \Omega\dot{\hat{J}} + \ddot{\theta}\hat{k} + \dot{\theta}\dot{\hat{k}} = \dot{\theta}\left(\vec{\omega}\times\hat{k}\right)$$

$$\left(\vec{v}_{A/O}\right)_{rel} = \vec{0}$$

$$\left(\vec{a}_{A/O}\right)_{rel} = \vec{0}$$

These give the same result! Try it.

Lec 14 Short Feedback Form: