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ME 274 Lecture 13

Moving Reference Kinematics: 3D Part 1

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2/11/26

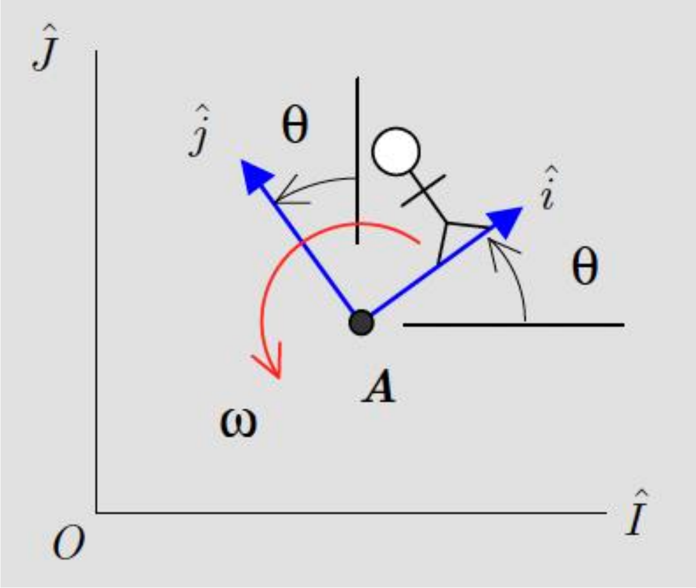
Housekeeping/Announcements

***Reminder for Henny to wear a mic during the lecture.

1. **Exam 1 is tomorrow!**
    - **Details on course website (https://www.purdue.edu/freeform/me274/exams-spring-2026/)**
    - **Recording of review session on course website**
2. No lecture on Friday
    - HW due on Friday

3 Advice on Exam 1

Last Class, Moving Reference Frame Eqns



$$\boxed{\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}}$$

$$\boxed{\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right]}$$

- $\vec{v}_A$ and $\vec{v}_B$ are the velocities seen by a **fixed observer** [XYZ]
- $\vec{a}_A$ and $\vec{a}_B$ are the accelerations seen by **fixed observer** [XYZ]
- $\vec{\omega}$ is angular velocity of the **moving observer** [xyz]
- $\vec{\alpha}$ is angular acceleration of the **moving observer** [xyz]

How does observer move?

- $\left(\vec{v}_{B/A}\right)_{rel}$ is the "velocity of point B as seen by the **moving observer** at A"
- $\left(\vec{a}_{B/A}\right)_{rel}$ is the "acceleration of point B as seen by the **moving observer** at A"
- $2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel}$ is known as the **"Coriolis" component** of acceleration.
  - Arises when observer has a non-zero angular velocity

What does the observer see?

[pg. 147 content]

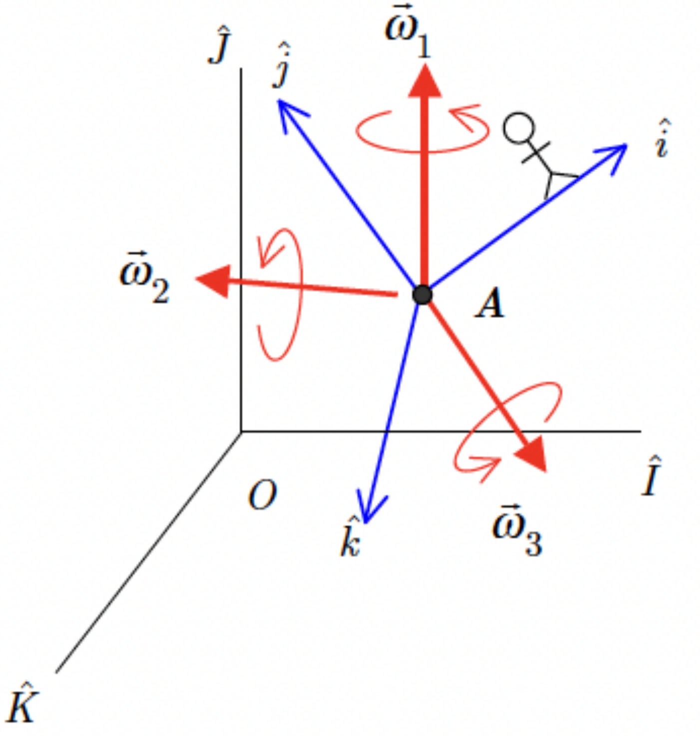
3D Rotating Reference Frames

- For a 3D Rotating Reference Frame system/problem, we will use the same Moving Reference Frame equations as before, except we now have a ‘k’ term.
  - For derivation, look at pg. 158-159.

- Angular Velocity, $\vec{\omega}$ in 3D motion will usually be made up of several components (omega’s). With each component being about a different axis, as shown in the figure.

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3 + \dots$$

- For Angular Acceleration, $\vec{\alpha}$ it is important to note distinction between a Fixed axis and a rotating/moving axis.
  - Why is this important?
    1. Derivative of velocity -> acceleration.
    2. When we derive the angular acceleration, we have to take derivative of i, j, and k unit vectors (product rule).
    3. For fixed axes, the derivative of I, J, and K will be equal to zero.



- Recalling derivation in pg. 144. **For a rotating axis we see that:**
  - **Straightforward to remember:**

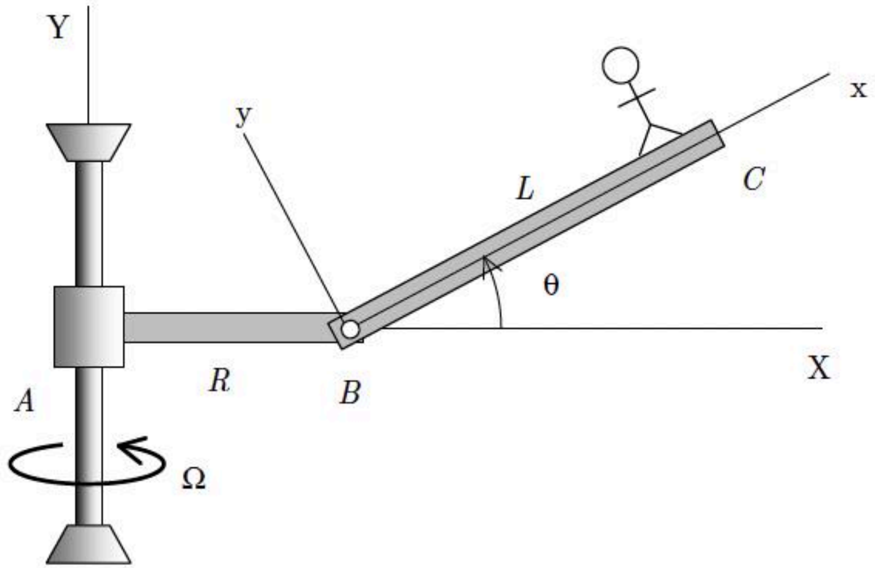
$$\frac{d\hat{i}}{dt} = \omega \times \hat{i} \qquad \frac{d\hat{j}}{dt} = \omega \times \hat{j} \qquad \frac{d\hat{k}}{dt} = \omega \times \hat{k}$$

[pg. 157 content]

Example 3.B.1

Given: Bar BC is pinned at end B to bar AB, which in turn rotates about a fixed vertical axis at a constant rate of $\Omega = 5$ rad/s. The angle θ is increasing at a constant rate of $\dot{\theta} = 4$ rad/s. The observer and the xyz axes are attached to arm BC, and the XYZ axes are fixed.

Find: Determine:
(a) The angular velocity of the observer when $\theta = 90°$; and
(b) The angular acceleration of the observer when $\theta = 90°$.



[pg. 163]

Example 3.B.1

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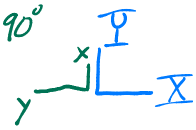
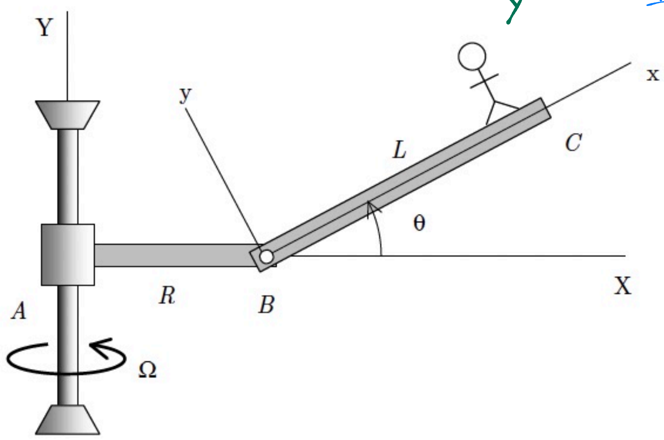
Find: Determine:
(a) The angular velocity of the observer when $\theta = 90°$; and ω ?
(b) The angular acceleration of the observer when $\theta = 90°$. α ?

@ $\theta = 90°$



$\Omega = 5$ rad/s $\dot{\Omega} = 0$

$\theta = 90°$ $\dot{\theta} = 4$ rad/s $\ddot{\theta} = 0$



Angular velocity:

fixed moves

$$\vec{\omega} = \Omega \hat{J} + \dot{\theta} \hat{k}$$

$$= \Omega \hat{J} + \dot{\theta} \hat{K}$$

Angular Acceleration:

prod rule fixed prod. rule

$$\vec{\alpha} = \cancel{\dot{\Omega} \hat{J}} + \cancel{\Omega \dot{\hat{J}}} + \cancel{\ddot{\theta} \hat{k}} + \dot{\theta} \dot{\hat{k}}$$

given

$$= \dot{\theta} \dot{\hat{k}}$$

$$= \dot{\theta}(\vec{\omega} \times \hat{k})$$

$$\dot{\hat{k}} = \vec{\omega} \times \hat{k}$$

$$= \dot{\theta}[(\Omega \hat{J} + \underbrace{\dot{\theta} \hat{k}) \times \hat{k}}_{\vec{0}}]$$

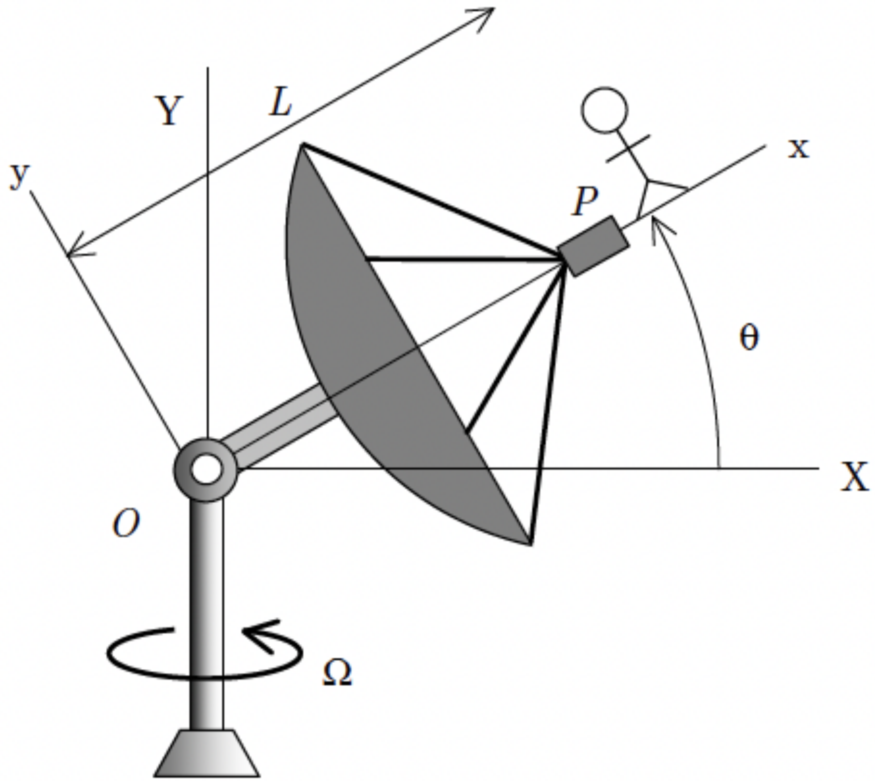
$$= \dot{\theta} \Omega (\hat{J} \times \hat{K})$$

$$= \dot{\theta} \Omega \hat{I}$$

Example 3.B.2

Given: The radar antenna is rotating about a fixed vertical axis at a constant rate of $\Omega = 0.2$ rad/s. The angle θ is increasing at a constant rate of $\dot{\theta} = 0.5$ rad/s. The observer and the xyz axes are attached to the antenna dish, with the XYZ axes being fixed.

Find: Determine:
(a) The angular velocity of the observer when $\theta = 36.87°$; and
(b) The angular acceleration of the observer when $\theta = 36.87°$.



[pg. 164]

Example 3.B.2

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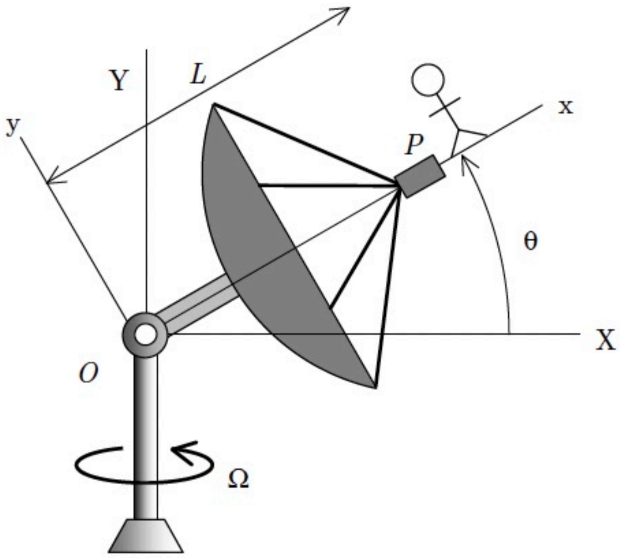
Find: Determine:
(a) The angular velocity of the observer when $\theta = 36.87°$; and
(b) The angular acceleration of the observer when $\theta = 36.87°$.

ω? α? @ 36.87°

$\Omega = 0.2$ rad/s $\quad \dot{\Omega} = 0$

$\theta = 36.87°$ $\quad \dot{\theta} = 0.5$ rad/s $\quad \ddot{\theta} = 0$



① Angular Velocity

$$\vec{\omega} = \Omega \underbrace{\hat{J}}_{\text{fixed}} + \dot{\theta} \underbrace{\hat{k}}_{\text{moves}}$$

$$= \Omega \hat{J} + \dot{\theta} \hat{K}$$

② Angular Acceleration

$$\vec{\alpha} = \underbrace{\cancel{\dot{\Omega} \hat{J}} + \cancel{\Omega \dot{\hat{J}}}}_{\text{prod. Rule}} + \underbrace{\cancel{\ddot{\theta} \hat{k}} + \dot{\theta} \dot{\hat{k}}}_{\text{prod. Rule}}$$

$$\dot{\hat{k}} = \vec{\omega} \times \hat{k}$$

$$= \dot{\theta} (\vec{\omega} \times \hat{k})$$

$$= \dot{\theta} \left[(\Omega \hat{J} + \dot{\theta} \cancel{\hat{k}}) \times \cancel{\hat{k}} \right] \quad \hat{K}$$

$$= \dot{\theta} \Omega (\hat{J} \times \hat{K})$$

$$= \dot{\theta} \Omega \hat{I}$$

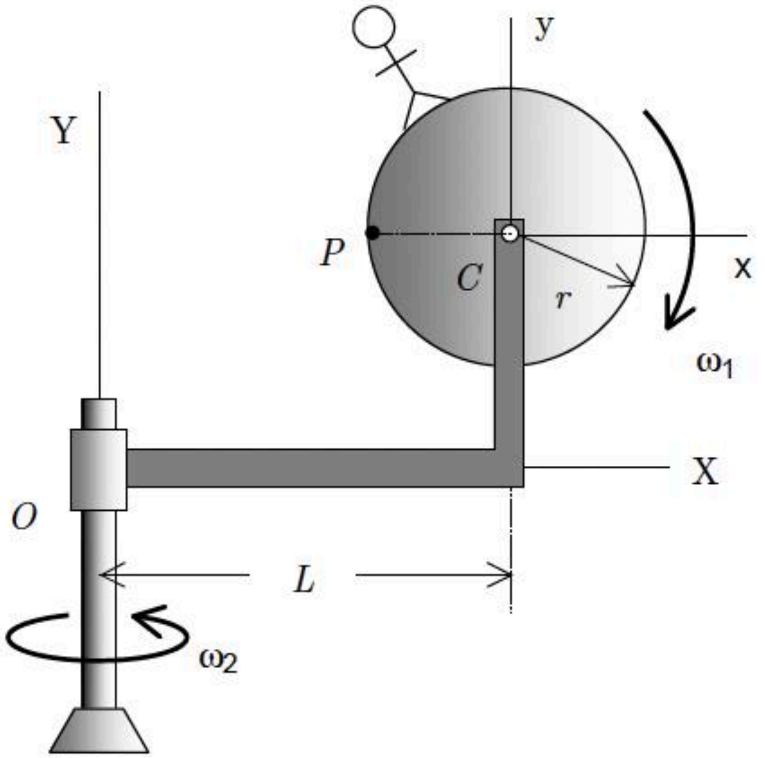
Example 3.B.3

Given: A disk rotates with a constant rate of $\omega_1 = 20$ rad/s with respect to the arm OC as the arm OC rotates about a fixed vertical axis with a constant rate of $\omega_2 = 5$ rad/s. The observer and the xyz axes are attached to the disk, while the XYZ axes are fixed. At this instant, the XYZ and xyz axes are aligned.

Find: Determine:

(a) The angular velocity of the observer at the instant shown; and

(b) The angular acceleration of the observer at the instant shown.



[pg. 165]

Example 3.B.3

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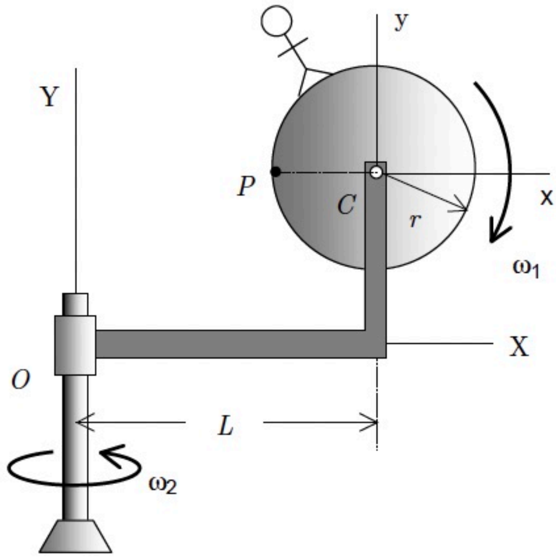
Find: Determine:

(a) The angular velocity of the observer at the instant shown; and ω ?

(b) The angular acceleration of the observer at the instant shown. α ?

$\omega_1 = 20\,\text{rad/s} \quad \dot{\omega}_1 = 0$

$\omega_2 = 5\,\text{rad/s} \quad \dot{\omega}_2 = 5\,\text{rad/s}$



① Angular Velocity:

$$\vec{\omega} = \omega_2 \hat{j} - \omega_1 \hat{k}$$
$$= \omega_2 \hat{J} - \omega_1 \hat{K}$$

② Angular Acceleration:

given, fixed Axis, given

$$\vec{\alpha} = \cancel{\dot{\omega}_2 \hat{j}} + \cancel{\omega_2 \dot{\hat{j}}} - \cancel{\dot{\omega}_1 \hat{k}} - \omega_1 \dot{\hat{k}}$$

$\dot{\hat{k}} = \vec{\omega} \times \hat{k}$

$$= -\omega_1 (\vec{\omega} \times \hat{k})$$
$$= -\omega_1 [(\omega_2 \hat{J} - \omega_1 \hat{K}) \times \hat{K}]$$

consistent vector set IJK...

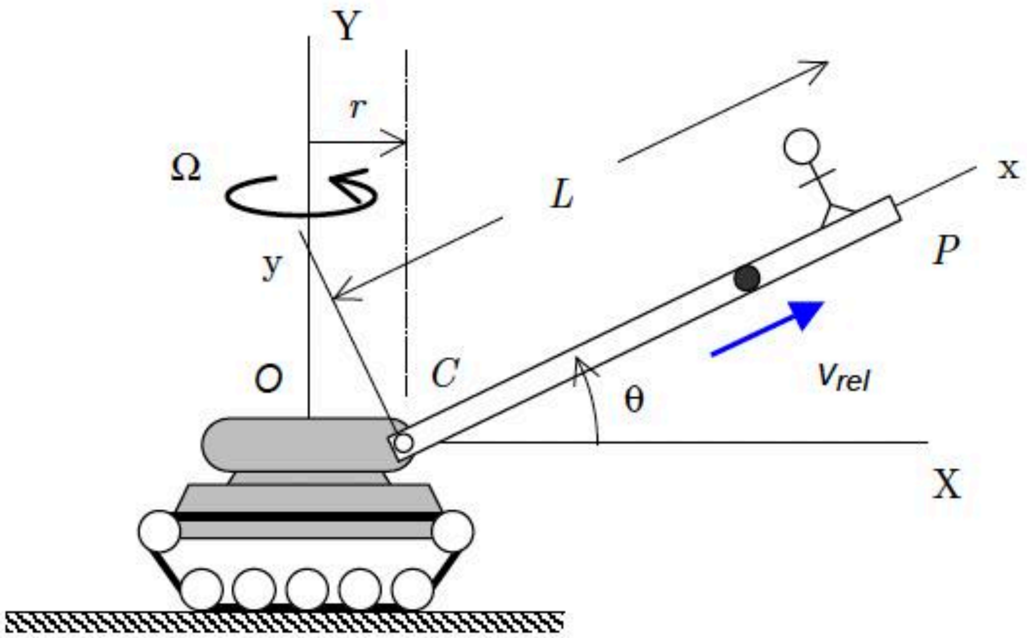
$$= -\omega_1 \omega_2 (\hat{J} \times \hat{K})$$
$$= -\omega_1 \omega_2 \hat{I}$$

Example 3.B.4

Given: The turret on a tank is rotating about a fixed vertical axis at a constant rate of $\Omega = 0.4$ rad/s. The barrel is being raised at a constant rate of $\dot{\theta} = 0.4$ rad/s . A cannon shell is fired with a constant muzzle speed of $v_{rel} = 200$ ft/s relative to the barrel. The observer and the xyz axes are attached to the barrel, while the XYZ axes are fixed. Here, $r = 3$ ft and $L = 15$ ft.

Find:

(a) The angular velocity of the barrel at the instant shown;
(b) The angular acceleration of the barrel at the instant shown; and
(c) The acceleration of the shell as it leaves the barrel at P.



[pg. 166]

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Find:

(a) (a) The angular velocity of the barrel at the instant shown; $\vec{\omega}$?

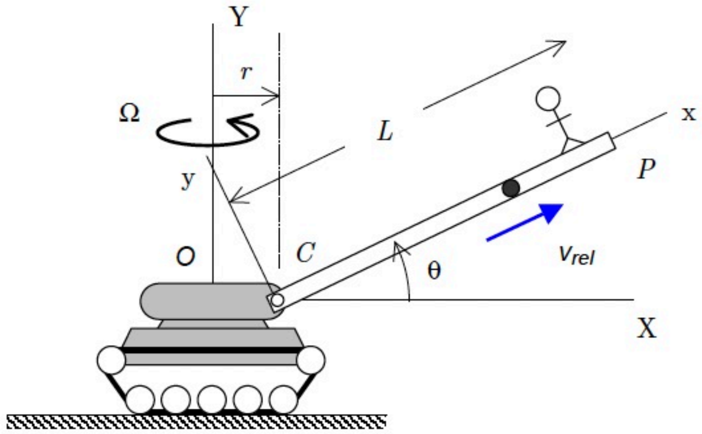
(b) (b) The angular acceleration of the barrel at the instant shown; and $\vec{\alpha}$?

(c) (c) The acceleration of the shell as it leaves the barrel at P. a_P?

$\Omega = 0.4$ rad/s $\quad \dot{\Omega} = 0$

$\dot{\theta} = 0.4$ rad/s $\quad \ddot{\theta} = 0$

$v_{rel} = 200$ ft/s ; $r = 3$ ft ; $L = 15$ ft



① Angular Velocity

$$\vec{\omega} = \Omega \hat{j} + \dot{\theta}\hat{k}$$

(a) $$= \Omega \hat{J} + \dot{\theta}\hat{K}$$

② Angular Acceleration

given fixed axis given

$$\vec{\alpha} = \cancel{\dot{\Omega}\hat{j}} + \cancel{\Omega\dot{\hat{j}}} + \cancel{\ddot{\theta}\hat{k}} + \dot{\theta}\dot{\hat{k}}$$

$\dot{\hat{k}} = \vec{\omega} \times \hat{k}$

$$= \dot{\theta}\dot{\hat{k}} = \dot{\theta}(\vec{\omega} \times \hat{k})$$

$$= \dot{\theta}[(\Omega\hat{j} + \cancel{\dot{\theta}\hat{k}}) \times \hat{k}] \quad \hat{K}$$

$$= \dot{\theta}\Omega(\hat{J} \times \hat{K})$$

consistent vector set IJK...

(b) $$= \dot{\theta}\Omega\hat{I}$$

③ Points CP, Different Rigid Bodies

$$\vec{a}_P = \vec{a}_C + (\vec{a}_{P/C})_{rel} + \vec{\alpha} \times \vec{r}_{P/C} + 2\vec{\omega} \times (\vec{v}_{P/C})_{rel} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/C})$$

④ ASIDE: Get $\vec{a}_C$. Link OC, Same Rigid Body

$$\vec{a}_C = \cancel{\vec{a}_O} + \cancel{\vec{\alpha}_{CO}} \times \vec{r}_{C/O} + \vec{\omega}_{CO} \times (\vec{\omega}_{CO} \times \vec{r}_{C/O}) + \omega \times (\vec{\omega} \times \vec{r}_{P/C})$$

$$\vec{a}_C = \cancel{\vec{a}_O} + \cancel{\vec{\alpha}_{CO}} \times \vec{r}_{C/O} + \vec{\omega}_{CO} \times (\vec{\omega}_{CO} \times \vec{r}_{C/O})$$

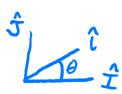
$$= \Omega\hat{J} \times (\Omega\hat{J} \times r\hat{I})$$

$$= -\Omega^2 r\hat{I}$$

⑤ Plug a_c & put in same axes.

$(\vec{a}_{P/C})_{rel} = \vec{0} \qquad \vec{r}_{P/C} = L\hat{i} \qquad (\vec{v}_{P/C})_{rel} = v_{rel}\hat{i}$

$$\vec{a}_P = -\Omega^2 r\hat{I} + \dot{\theta}\Omega\hat{I} \times L\hat{i} + 2(\Omega\hat{J} + \dot{\theta}\hat{K}) \times v_{rel}\hat{i} + (\Omega\hat{J} + \dot{\theta}\hat{k}) \times [(\Omega\hat{J} + \dot{\theta}\hat{K}) \times L\hat{i}]$$



$\hat{i} = \cos\theta\hat{I} + \sin\theta\hat{J}$

$$\vec{a}_P = -\Omega^2 r\hat{I} + \dot{\theta}\Omega\hat{I} \times L(\cos\theta\hat{I} + \sin\theta\hat{J}) + 2(\Omega\hat{J} + \dot{\theta}\hat{K}) \times v_{rel}(\cos\theta\hat{I} + \sin\theta\hat{J}) + (\Omega\hat{J} + \dot{\theta}\hat{K}) \times [(\Omega\hat{J} + \dot{\theta}\hat{K}) \times L(\cos\theta\hat{I} + \sin\theta\hat{J})]$$

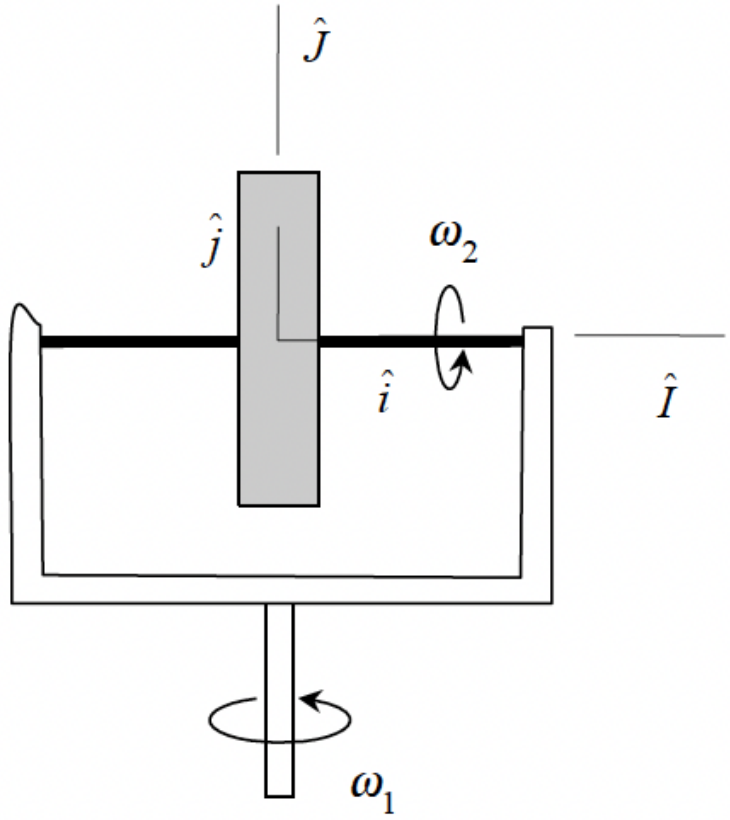
$$= -\Omega^2 r\hat{I} + \dot{\theta}\Omega L\sin\theta\hat{K} - 2\Omega v_{rel}\cos\theta\hat{K} + 2\dot{\theta}v_{rel}\cos\theta\hat{J} - 2\dot{\theta}v_{rel}\sin\theta\hat{I} + (\Omega\hat{J} + \dot{\theta}\hat{K}) \times (-\Omega L\cos\theta\hat{K} + \dot{\theta}L\cos\theta\hat{J} - \dot{\theta}L\sin\theta\hat{I})$$

(c) $$= -\Omega^2 r\hat{I} + \dot{\theta}\Omega L\sin\theta\hat{K} - 2\Omega v_{rel}\cos\theta\hat{K} + 2\dot{\theta}v_{rel}\cos\theta\hat{J} - 2\dot{\theta}v_{rel}\sin\theta\hat{I} - \Omega^2 L\cos\theta\hat{I} + \Omega\dot{\theta}L\sin\theta\hat{K} - \dot{\theta}^2 L\cos\theta\hat{I} - \dot{\theta}^2 L\sin\theta\hat{J}$$

Example 3.B.5

Given: The disk of a gyroscope rotates about its own axis at a constant rate of $\omega_2 = 600$ rev/min. The gimbal support is rotating at a constant rate of $\omega_1 = 10$ rad/s about a fixed vertical axis. The observer and the xyz axes are attached to the disk. The XYZ axes are fixed in space.

Find: Determine:
(a) The angular velocity of the observer at the instant shown; and
(b) The angular acceleration of the observer at the instant shown.



[pg. 167]

Example 3.B.5

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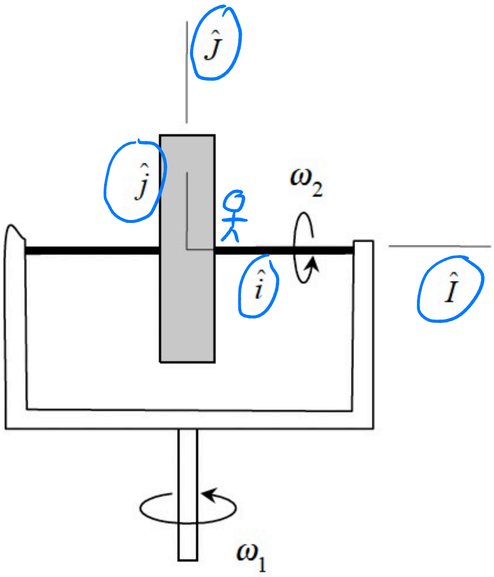
Find: Determine:

(a) The angular velocity of the observer at the instant shown; and $\vec{\omega}$?

(b) The angular acceleration of the observer at the instant shown. $\vec{\alpha}$?

$\omega_1 = 10$ rad/s $\quad \dot{\omega}_1 = 0$

$\omega_2 = 600$ rev/min $\quad \dot{\omega}_2 = 0$



Angular Velocity

$$\vec{\omega} = \omega_1 \hat{J} + \omega_2 \hat{i}$$

$$= \omega_1 \hat{J} + \omega_2 \hat{I}$$

Angular Acceleration:

$$\vec{\alpha} = \cancel{\dot{\omega}_1 \hat{J}} + \cancel{\omega_1 \dot{\hat{J}}} + \cancel{\dot{\omega}_2 \hat{i}} + \omega_2 \dot{\hat{i}}$$

$\dot{\hat{i}} = \vec{\omega} \times \hat{i}$

$$= \omega_2 (\vec{\omega} \times \hat{i})$$

$$= \omega_2 [(\omega_1 \hat{J} + \omega_2 \cancel{\hat{i}}^{\,\hat{I}}) \times \cancel{\hat{i}}^{\,\hat{I}}]$$

consistent vector set

IJK ...

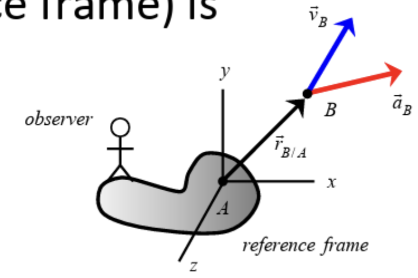
$$= \omega_2 \omega_1 (\hat{J} \times \hat{I})$$

$$= -\omega_1 \omega_2 \hat{K}$$

Summary: 3D Moving Reference Frame Kinematics 1

[pg. 145]

PROBLEM: A person attached to a moving body (reference frame) is observing the motion of point B.



$$\vec{v}_B = \vec{v}_A + \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \left(\vec{a}_{B/A}\right)_{rel} + \vec{\alpha} \times \vec{r}_{B/A} + 2\vec{\omega} \times \left(\vec{v}_{B/A}\right)_{rel} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right)$$

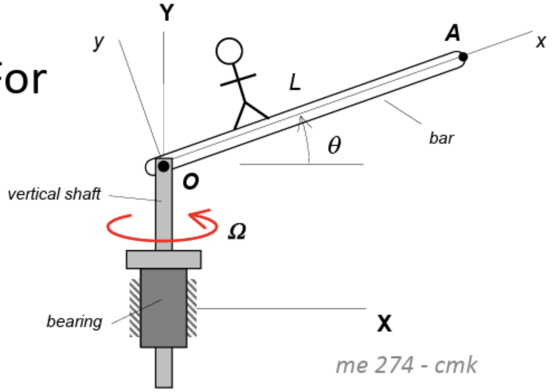
where:

- $\vec{\omega}$ and $\vec{\alpha}$ are the angular velocity/acceleration *of the observer* (*no exceptions*).
- $\left(\vec{v}_{B/A}\right)_{rel}$ and $\left(\vec{a}_{B/A}\right)_{rel}$ are the velocity/acceleration of B *as seen by the observer*.
- A is *ANY* point on the *same* reference frame as the observer.
- Generally, *you* are free to choose your observer.

QUESTION: How is this different from the 2D case? For observer on arm OA:

$$\vec{\omega} = \Omega\hat{J} + \dot{\theta}\hat{k}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \dot{\Omega}\hat{J} + \overset{0}{\cancel{\Omega\dot{\hat{J}}}} + \ddot{\theta}\hat{k} + \dot{\theta}\dot{\hat{k}} = \dot{\Omega}\hat{J} + \ddot{\theta}\hat{k} + \dot{\theta}\left(\vec{\omega} \times \hat{k}\right)$$



Lec 13 Short Feedback Form: