ME 274: Basic Mechanics II

Lecture 5: Planar kinematics: rigid bodies

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Announcements

Homework changes due to snow cancelations:

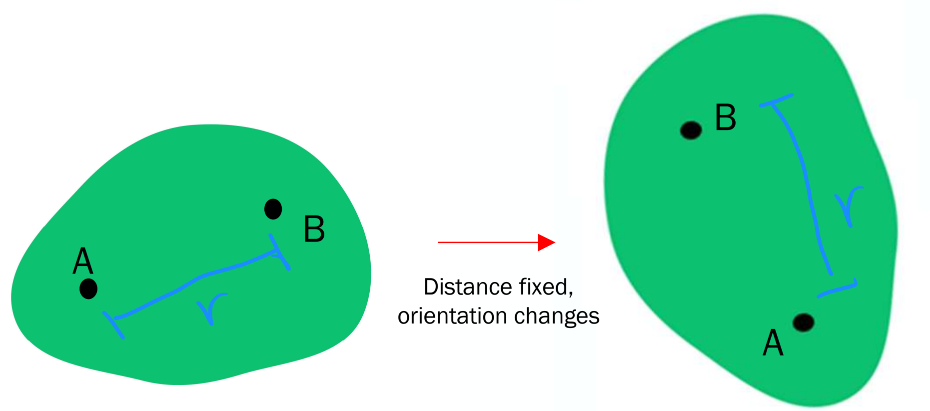
HW 2.B, 2.C, 2.D due on Friday

HW 2.A canceled

HW 1.I and 1.J still due tonight

What are rigid bodies?

Any object for which the distance between any two points on the object remains fixed regardless of the motion of the object



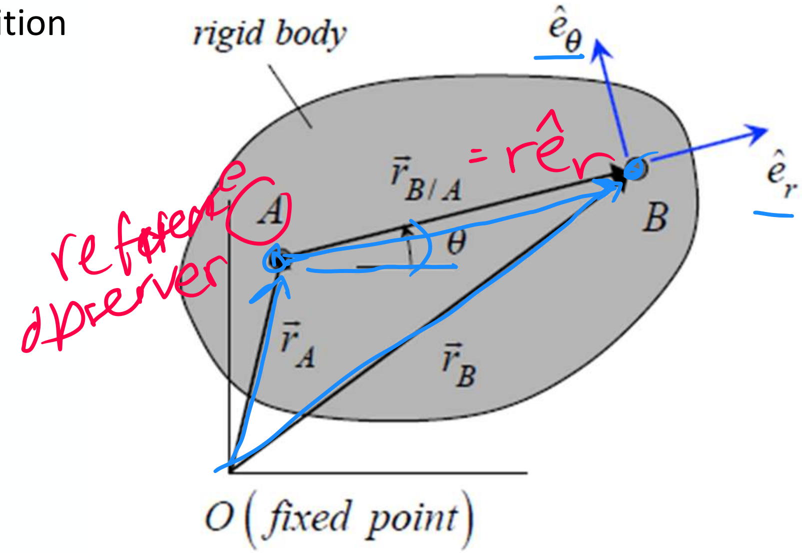
Point kinematics→ we treat the object as a particle $\to \vec{r}, \vec{v}, \vec{a}$
Rigid body kinematics →we care about the **position, geometry, and orientation** of the object

If we know the **motion of one point** and the **rotation of the body**, we can determine the **motion of every point** on the body

Kinematic Equations for Planar Rigid Body Motion

- Position of point B with respect to point A found using relative position expression

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$



Using the polar description of $\vec{r}_{B/A}$:

- Fixed points → $|\vec{r}_{B/A}| = r = \ const.$, $\dot{r}_{B/A} = 0$
- Changing θ → $\vec{r}_{B/A} \neq const.$

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A = r\hat{e}_r$$

Take time derivatives for velocity and acceleration:

$$\vec{V}_{B/A} = \vec{V}_B - \vec{V}_A = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_t$$

Deriving the Rigid Body Velocity Equation

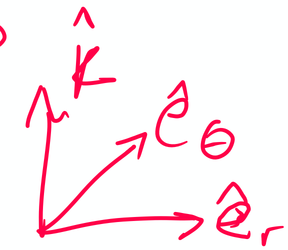
We can derive a **description agnostic expression** by writing in terms of **vector operations**

$\vec{v}_{B/A} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = \vec{v}_B - \vec{v}_A$ ← specific to polar coordinates

we know $|\vec{r}_{B/A}| = r = \text{const}$

$\dot{r} = 0$

$\hat{e}_\theta = \hat{k} \times \hat{e}_r$



$$
\begin{aligned}
\vec{v}_{B/A} &= \vec{v}_B - \vec{v}_A \quad \text{relative} \\
&= \cancel{\dot{r}\hat{e}_r} + r\dot{\theta}\hat{e}_\theta \quad \text{polar} \\
\vec{V}_{B/A} &= r\dot{\theta}\hat{e}_\theta \\
&= r\dot{\theta}\left(\hat{k} \times \hat{e}_r\right) \\
\text{scalar} \quad &= \left(\dot{\theta}\hat{k}\right) \times \left(r\hat{e}_r\right) \\
\vec{V}_{B/A} &= \vec{\omega} \times \vec{r}_{B/A}
\end{aligned}
$$

$r\hat{e}_r \Rightarrow$ relative position vector $\vec{r}_{B/A}$

$\dot{\theta}\hat{k} = \vec{\omega}$

where $\vec{\omega} = \dot{\theta}\hat{k}$ is the "angular velocity" of the rigid body.

Deriving the Rigid Body Acceleration Equation

We can derive a **description agnostic expression** by writing in terms of **vector operations**

$\vec{a}_{B/A} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta = \vec{a}_B - \vec{a}_A$ ← specific to polar coordinates

$|\vec{r}_{B/A}| = r = \text{const} \rightarrow \dot{r} = 0 = \ddot{r}$

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A$$ ← relative

$$= \left(\overset{0}{\cancel{\ddot{r}}} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + \overset{0}{\cancel{2\dot{r}\dot{\theta}}}\right)\hat{e}_\theta$$ ← polar

$$\vec{a}_{B/A} = \left(-r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta}\right)\hat{e}_\theta$$ → in terms of $\hat{k}$

$$= \left(r\dot{\theta}^2\right)\left[\hat{k} \times \left(\hat{k} \times \hat{e}_r\right)\right] + \left(r\ddot{\theta}\right)\left(\hat{k} \times \hat{e}_r\right)$$

$$= \dot{\theta}\hat{k} \times \left[\left(\dot{\theta}\hat{k}\right) \times (r\hat{e}_r)\right] + \left(\ddot{\theta}\hat{k}\right) \times (r\hat{e}_r)$$

$$= \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right] + \vec{\alpha} \times \vec{r}_{B/A}$$

$$\left.\begin{aligned} \hat{e}_\theta &= \hat{k} \times \hat{e}_r \\ \hat{e}_r &= -\hat{k} \times (\hat{k} \times \hat{e}_r) \end{aligned}\right\}$$

where $\vec{\alpha} = \theta k$ is the "angular acceleration" of the rigid body.

To find the velocity/acceleration of point B

→

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right] + \vec{\alpha} \times \vec{r}_{B/A}$$

translation rotation

$$\vec{a}_{B/A} = \cancel{\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})} + \vec{\alpha} \times \vec{r}_{B/A} \leftarrow 3D$$

$$-\omega^2 \vec{r}_{B/A}$$

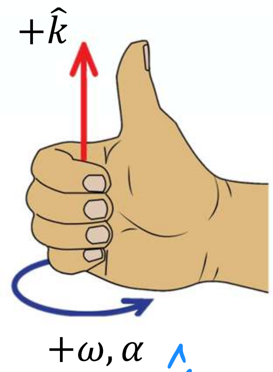
$$2D \quad \vec{a}_{B/A} = -\omega^2 \vec{r}_{B/A} + \vec{\alpha} \times \vec{r}_{B/A}$$

Angular Velocity and Acceleration

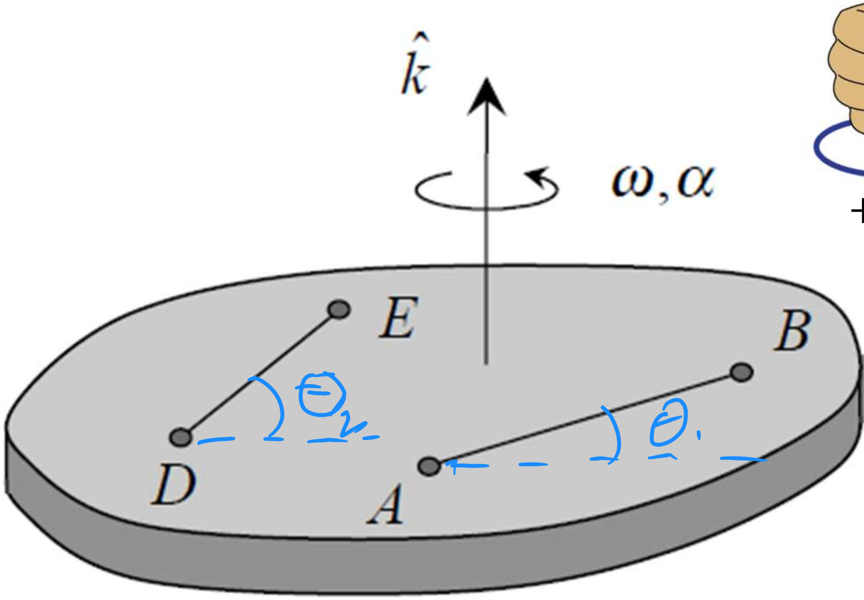
Angular velocity: $\vec{\omega} = \omega\hat{k} = \dot{\theta}\hat{k}$

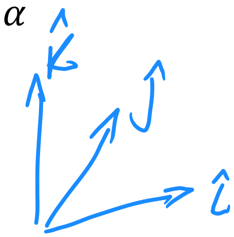
Angular acceleration: $\vec{\alpha} = \alpha\hat{k} = \ddot{\theta}\hat{k}$

Describe the motion of the **BODY** and is the same for any set of points – i.e. a rigid body will only have one angular velocity and acceleration



The direction $\hat{k}$ denotes the axis the body rotates around. The sign denotes the direction (ex. Clockwise, ccw)





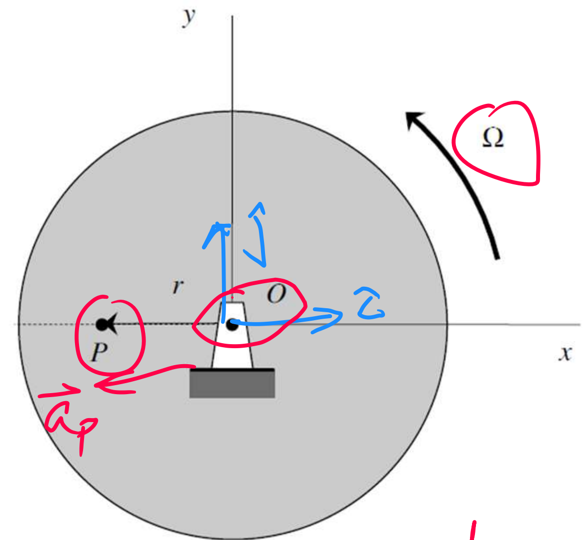
$$\theta_2 = \theta_1 + \text{const}$$

$$\dot{\theta}_2 = \dot{\theta}_1 + 0 = \omega$$

$$\ddot{\theta}_2 = \ddot{\theta}_1 = \alpha$$

Example 2.A.1

Given: The disk shown is rotating at a non-constant rate of Ω about a fixed axis passing through its center O. At a particular instant, the acceleration vector of point P on the disk is $\vec{a}_P$.



Find: Determine:

(a) The angular velocity of the disk at this instant; and

(b) The angular acceleration of the disk at this instant.

Use the following parameters in your analysis: $\vec{a}_P = 3\hat{\imath} + 4\hat{\jmath}$ m/s^2 and $r = 0.4$ m. Also, be sure to write your answers as vectors.

$$\vec{a}_P = \vec{a}_O + \vec{\alpha} \times \vec{r}_{P/O} - \omega^2 \vec{r}_{P/O}$$

O → pinned

$\vec{v}_O = 0 \quad \vec{a}_O = 0$

$$a_{Px}\hat{\imath} + a_{Py}\hat{\jmath} = 0 + \alpha\hat{k} \times -r\hat{\imath} - \omega^2(-r\hat{\imath})$$

$$= -\alpha r\hat{\jmath} + \omega^2 r\hat{\imath}$$

$$\hat{\imath}: \quad a_{Px} = \omega^2 r \rightarrow \omega = \pm\sqrt{\frac{a_{Px}}{r}} = \pm\sqrt{\frac{3}{0.4}} = \pm 2.74 \frac{rad}{s}$$

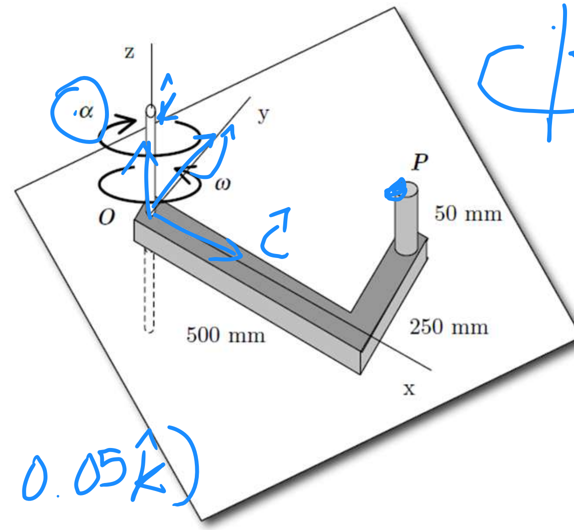
$$\vec{\omega} = \pm 2.74 \frac{rad}{s}\hat{k}$$

$$\hat{\jmath}: \quad a_{Py} = -\alpha r \rightarrow \alpha = -\frac{a_{Py}}{r} = -\frac{4}{0.4} = -10 \text{ rad/s}^2$$

$$\vec{\alpha} = -10\hat{k} \text{ rad/s}^2$$

Example 2.A.3

Given: The system shown below rotates about a vertical shaft at point O, such that $\omega = 2$ rad/s and $\alpha = 3$ rad/s^2.



Find: Determine:

(a) The velocity of point P; and
(b) The acceleration of point P.

O pinned $\rightarrow$ $V_O = 0$, $a_O = 0$

$$\vec{V}_P = \cancelto{0}{\vec{V}_O} + \vec{\omega} \times \vec{r}_{P/O}$$

$$= \vec{\omega} \times \vec{r}_{P/O} = \omega \hat{k} \times (0.5\hat{i} + .25\hat{j} + 0.05\hat{k})$$

$$\vec{V}_P = 0.5\omega\hat{j} - .25\omega\hat{i}$$

$$0.5(2)\hat{j} - 0.25(2)\hat{i} = \boxed{1\hat{j} - 0.5\hat{i} \text{ m/s}}$$

$$\vec{a}_P = \cancelto{0}{\vec{a}_O} + \alpha \times \vec{r}_{P/O} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O})$$

$$= \alpha\hat{k} \times (0.5\hat{i} + .25\hat{j} + 0.05\hat{k}) + \omega\hat{k} \times [\omega\hat{k} \times (0.5\hat{i} + 0.25\hat{j} + 0.05\hat{k})]$$

$$= 0.5\alpha\hat{j} - .25\alpha\hat{i} + \omega\hat{k} \times (0.5\omega\hat{j} - 0.25\omega\hat{i})$$

$\rightarrow$ $0.5\omega^2\hat{i} - 0.25\omega^2\hat{j}$

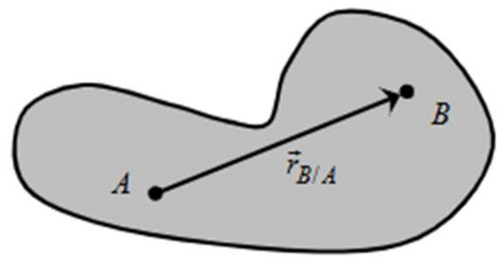
$$\vec{a}_P = (-.25\alpha - .5\omega^2)\hat{i} + (.5\alpha - .25\omega^2)\hat{j}$$

$\rightarrow$ plug in + solve

Summary: Rigid Body Kinematics 1

PROBLEM: Two points A and B on the same rigid body undergoing planar motion.

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$
$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right)$$



COMMENTS:

- $\vec{\omega}$ and $\vec{\alpha}$ are the *__angular velocity__* and *__angular acceleration__* vectors of the body. These are the same for ANY two points A and B.
- $\vec{r}_{B/A}$ points *__FROM__* point A *__TO__* point B.
- If A and B lie in the same plane, then: $\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}$
- From where did these equations come? From the general motion of two points (Chapter 1) with the constraint that $\left|\vec{r}_{B/A}\right|$.

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Lecture 7: Rigid Body Motion

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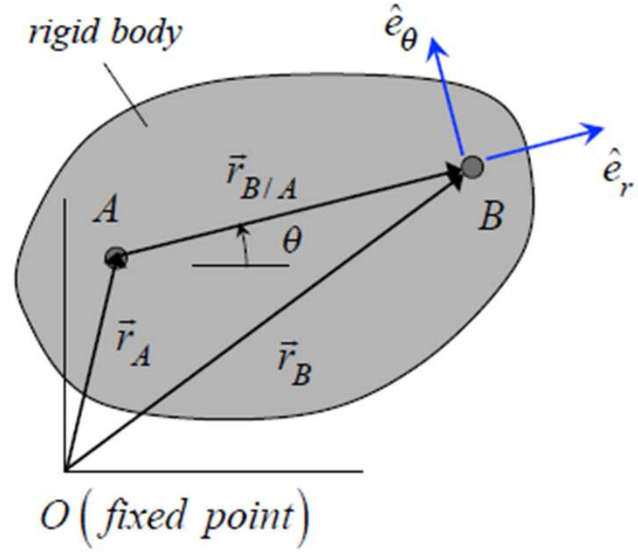
Review: Rigid Body Motion

If we know the motion of one point and the rotation of the body, we can determine the motion of any point on the body



Rigid body:

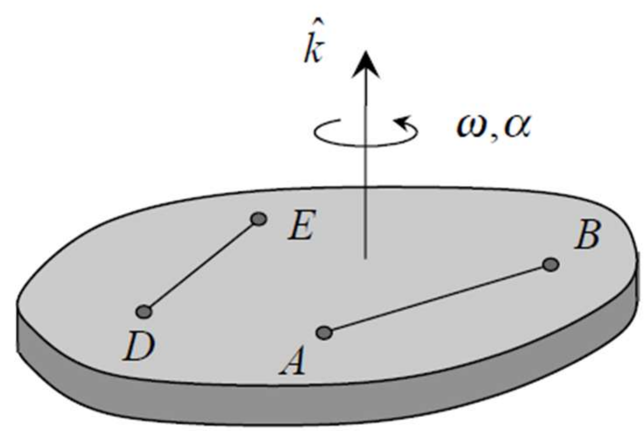
$$r = \left| \vec{r}_{B/A} \right| = \text{CONSTANT and } \dot{r} = 0$$

Velocity and acceleration of point B with respect to point A:

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right] + \vec{\alpha} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}.$$



Angular velocity ω, and angular acceleration α are properties of the body

- $\vec{\omega} = \omega\hat{k} = \dot{\theta}\hat{k}$, $\vec{\alpha} = \alpha\hat{k} = \ddot{\theta}\hat{k}$
- Sign of ω, α determines direction of rotation (using right hand rule)

Example 2.A.7

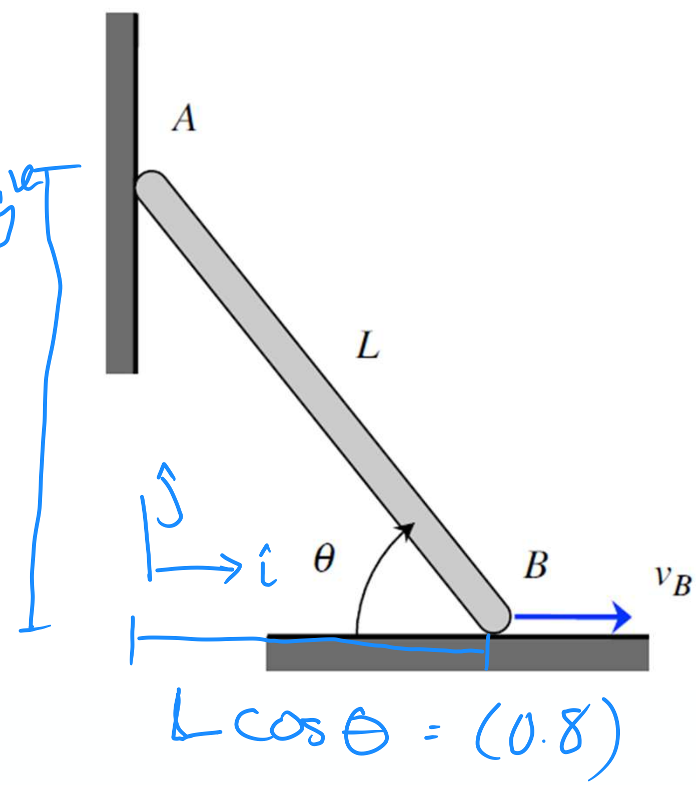
Given: End B of the link moves to the right with a constant speed v_B.

Find: Determine:
(a) The angular velocity of link AB; and
(b) The angular acceleration of link AB.

Use the following parameters in your analysis: $v_B = 3$ m/s, $L = 0.5$ m and $\theta = 36.87°$. Also, be sure to express your answers as vectors.

3,4,5 triangle

$L\sin\theta$
(0.6)



1) draw unit vectors
2) what do we know?

$$\vec{V}_B = V_B\,\hat{i}, \quad \vec{V}_A = V_A\,\hat{j}$$

$$\vec{r}_{A/B} = (-L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j})$$

3) solve for V_A in terms of V_B

$$\vec{V}_A = \vec{V}_B + \vec{\omega} \times \vec{r}_{A/B}$$

$$V_A\,\hat{j} = V_B\,\hat{i} + \omega\hat{k} \times (-L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j})$$

$$= V_B\,\hat{i} - \omega L\cos\theta\,\hat{j} - \omega L\sin\theta\,\hat{i}$$

4) split components

$\hat{i}$: $\quad 0 = V_B - \omega L \sin\theta$

$\hat{j}$: $\quad V_A = -\omega L \cos\theta$

solve for ω & V_A

$$\vec{\omega} = \frac{V_B}{L\sin\theta} = \frac{3}{0.5(0.6)} = 10\hat{k}$$

$$V_A = -10(0.5)(0.8) = -4 \text{ m/s} \qquad \vec{V}_A = -4\hat{j} \text{ m/s}$$

5) solve for $\vec{a}_A$

$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}_{A/B} - \omega^2 \vec{r}_{A/B}$$

$$a_A\hat{j} = a_B\hat{i} + \alpha\hat{k} \times (-L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j}) - \omega^2(-L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j})$$

$$= a_B\hat{i} - \alpha L\cos\theta\,\hat{j} - \alpha L\sin\theta\,\hat{i} + \omega^2 L\cos\theta\,\hat{i} - \omega^2 L\sin\theta\,\hat{j}$$

6) split components

$\hat{i}$: $0 = a_B - \alpha L \sin\theta + \omega^2 L \cos\theta$

$\hat{j}$: $a_A = -\alpha L \cos\theta - \omega^2 L \sin\theta$

sub in known/found values & solve

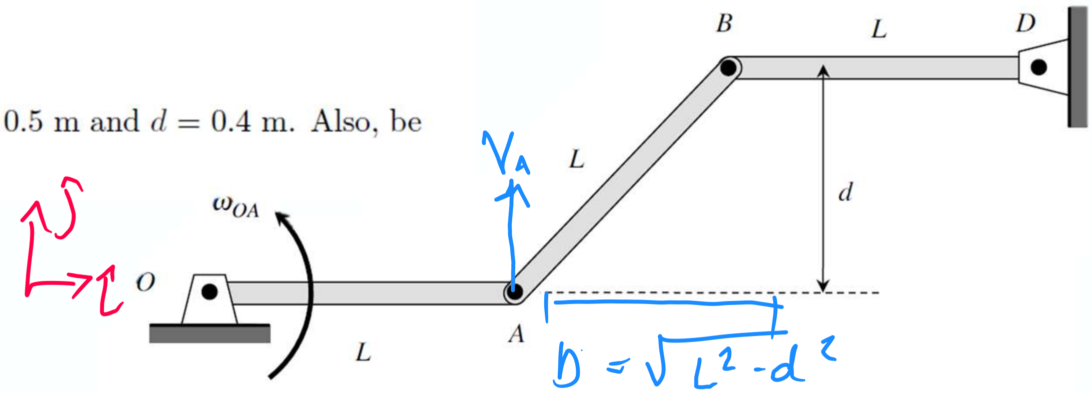
Example 2.A.10

Given: At the instant shown, link OA rotates counterclockwise about pin O with a constant angular speed of ω_{OA}. At the instant shown, links OA and BD are horizontal.

Find: Determine at this instant:

(a) The angular acceleration of link AB; and
(b) The angular acceleration of link BD.

Use the following parameters in your analysis: $\omega_{OA} = 3$ rad/s, $L = 0.5$ m and $d = 0.4$ m. Also, be sure to express your answers as vectors.



1) draw unit vectors

2) define geometry

$$b = \sqrt{L^2 - d^2}$$

3) find V_A from V_O

$$\vec V_A = \underline{\vec V_O} + \vec\omega \times r_{A/O}$$

$$0 + \omega_{OA}\hat k \times L\hat\imath$$

$$\vec V_A = \omega_{OA} L \hat\jmath$$

4) find V_B using V_A

$$\vec V_B = \vec V_A + \vec\omega_{AB} \times r_{B/A}$$

$$= V_A\hat\jmath + \omega_{AB}\hat k \times (b\hat\imath - d\hat\jmath)$$

$$= V_A\hat\jmath + \omega_{AB} b\hat\jmath - \omega_{AB} d\hat\imath$$

5) find V_B using V_D

$$\vec V_B = \vec V_D + \vec\omega_{BD} \times r_{B/D}$$

$$= 0 + \omega_{BD}\hat k \times (-L\hat\imath)$$

$$\vec V_B = -\omega_{BD} L \hat\jmath$$

6) set eqns for V_B equal

$$-\omega_{AB} d\,\hat{i} + (V_A + \omega_{AB} b)\hat{j} = -\omega_{BD} L\,\hat{j}$$

7) split components

$$\hat{i}: \quad -\omega_{AB} d = 0 \implies \omega_{AB} = 0 \qquad \vec{\omega}_{AB} = \vec{0}$$

$$\hat{j}: \quad V_A + \omega_{AB} b = -\omega_{BD} L$$

$$V_A = -\omega_{BD} L \implies \omega_{BD} = \frac{-V_A}{L} = \frac{\omega_{OA} L}{L}$$

$$\vec{\omega}_{BD} = -\omega_{OA}\,\hat{k}$$

now for acceleration:

find a_A from a_O

$$\vec{a}_A = \cancel{\vec{a}_O}^{\,0} + \cancel{\alpha_{OA}}^{\,0} \times \vec{r}_{A/O} - \omega_{OA}^2 r_{A/O}$$

fixed pt. ← given

$$\vec{a}_A = -\omega_{OA}^2 L \hat{i}$$

a_B from a_A

$$\vec{a}_B = \vec{a}_A + \alpha_{AB} \times \vec{r}_{B/A} - \omega_{AB}^2 \vec{r}_{B/A}$$

$$= \vec{a}_A \hat{i} + \alpha_{AB} \hat{k} \times (b\hat{i} - d\hat{j}) - \cancel{\omega_{AB}^2}^{\,0} (b\hat{i} - d\hat{j})$$

$$= \vec{a}_A \hat{i} + \alpha_{AB} b \hat{j} - \alpha_{AB} d \hat{i}$$

a_B from a_D ← fixed pt.

$$\vec{a}_B = \cancel{\vec{a}_D}^{\,0} + \alpha_{BD} \times r_{B/D} - \omega_{BD}^2 \vec{r}_{B/D}$$

$$+ \alpha_{BD} \hat{k} \times -L\hat{i} - \omega_{BD}^2 (-L)\hat{i}$$

$$\vec{a}_B = -\alpha_{BD} L \hat{j} + \omega_{BD}^2 L \hat{i}$$

Set eqns for $\vec{a}_B$ =

$$\vec{a}_A \hat{i} + \alpha_{AB} b \hat{j} - \alpha_{AB} d \hat{i} = -\alpha_{BD} L \hat{j} + \omega_{BD}^2 L \hat{i}$$

Separate terms:

$\hat{i}$: $\vec{a}_A - \alpha_{AB} d = \omega_{BD}^2 L$

$\hat{j}$: $\alpha_{AB} b = -\alpha_{BD} L$

2 eqn, 2 unknown

Sub in values and solve

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Lecture 7: Rigid Body Motion: Rolling without Slipping

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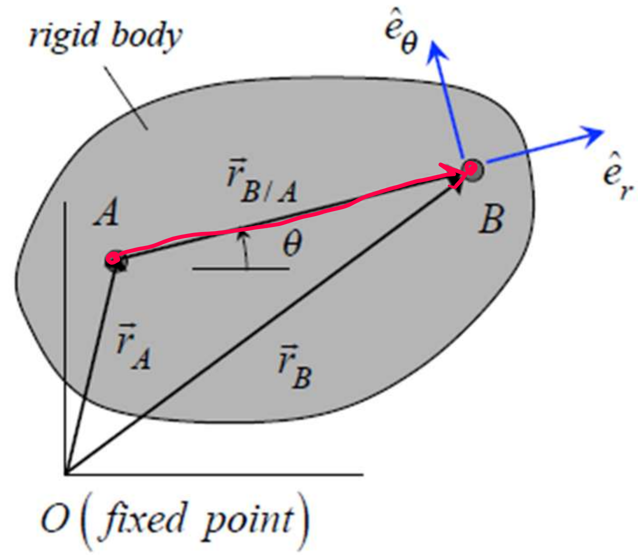
Review: Rigid Body Motion

If we know the motion of one point and the rotation of the body, we can determine the motion of any point on the body



Rigid body:

$$r = |\vec{r}_{B/A}| = \text{CONSTANT and } \dot{r} = 0$$

Velocity and acceleration of point B with respect to point A:

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

ang. velocity → 2D & 3D

$$\vec{a}_B = \vec{a}_A + \vec{\omega} \times [\vec{\omega} \times \vec{r}_{B/A}] + \vec{\alpha} \times \vec{r}_{B/A}$$

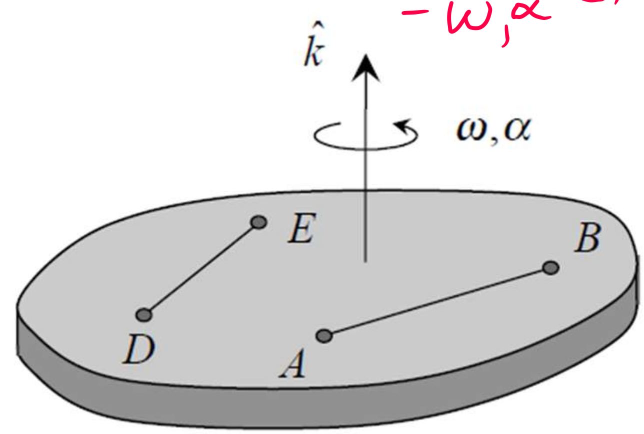
→ 3D

ang accel.

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}.$$

→ 2D

$+\omega, \alpha \rightarrow$ ccw
$-\omega, \alpha \rightarrow$ cw

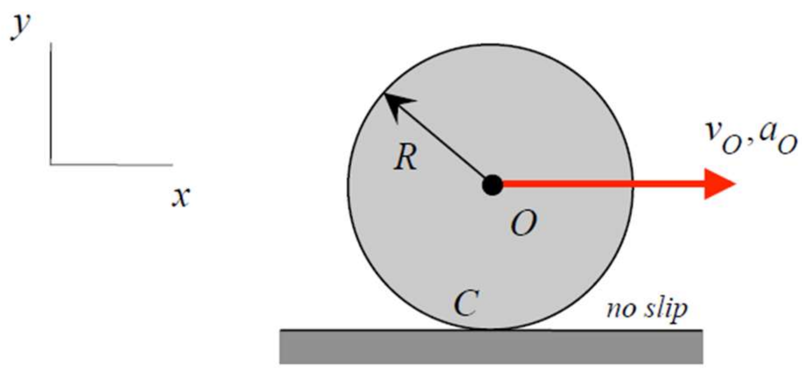


Angular velocity ω, and angular acceleration α are properties of the body

- $\vec{\omega} = \omega\hat{k} = \dot{\theta}\hat{k}$, $\vec{\alpha} = \alpha\hat{k} = \ddot{\theta}\hat{k}$
- Sign of ω, α determines direction of rotation (using right hand rule)

Special case: Rolling without Slipping

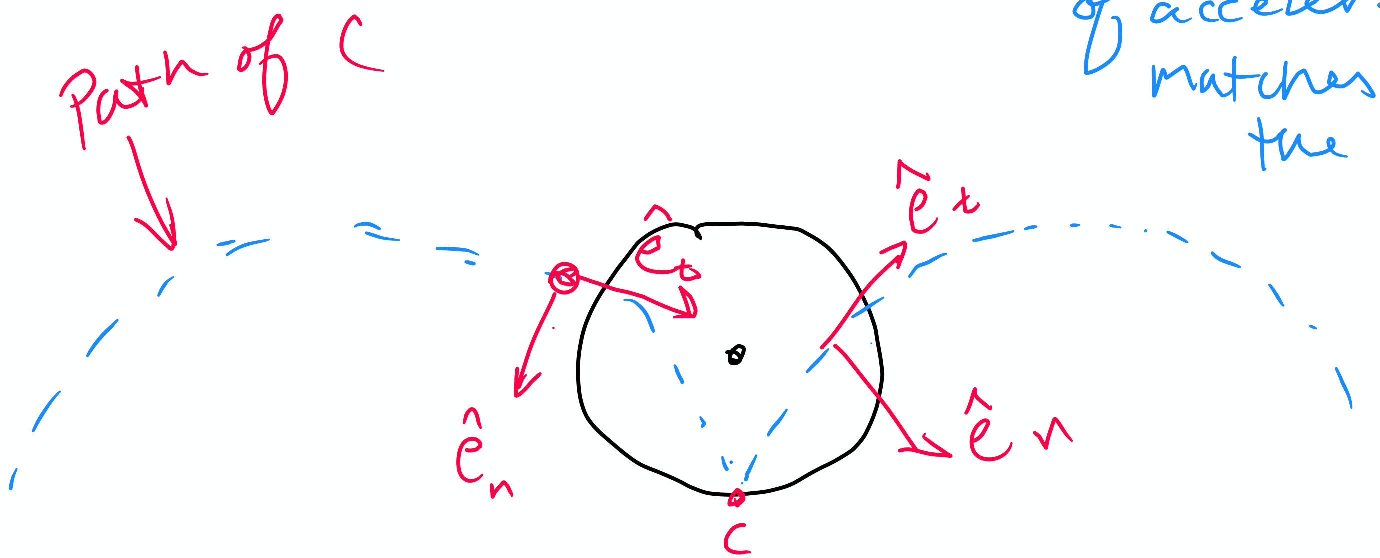
The wheel rolls on a rough, stationary surface. The center of the wheel O has a velocity and acceleration v_O, a_O . It is assumed that sufficient friction acts between the wheel and surface such that the contact point, C, does not slip



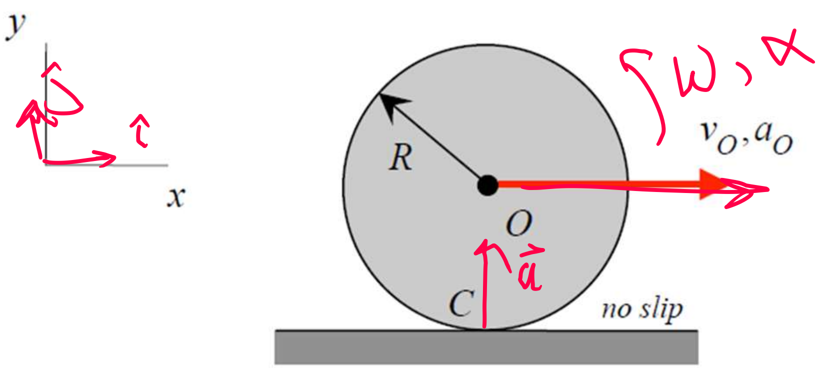
"No slip" condition:

$$v_{cx} = 0, \qquad a_{cx} = 0$$

tangential component of acceleration & velocity matches that of the contact surface



Question: If C is a no-slip point, what are the y-components for the velocity and acceleration of C?



Known information:

- O moves on a straight, horizontal path

$\vec{V}_O = V_o \hat{i}$, $\vec{a}_O = a_o \hat{i}$

- "No slip" condition

$V_{cx} = 0\hat{i}$, $\vec{a}_{cx} = 0\hat{i}$

Using our rigid body equation for velocity: $\vec{v}_C = \vec{v}_O + \vec{\omega} \times \vec{r}_{C/O}$

$\cancel{\vec{V}_{cx}} + \vec{V}_{cy} = V_o\hat{i} + \omega\hat{k} \times (-R\hat{j})$

$V_{cy}\hat{j} = V_o\hat{i} + \omega R\hat{i}$

$\hat{j}: V_{cy} = 0$

$\hat{i}: 0 = V_o + \omega R \rightarrow V_o = -\omega R$

rotating cw

$\omega = -\frac{V_o}{R}$

Using our rigid body equation for acceleration: $\vec{a}_c = \vec{a}_O + \vec{\alpha} \times \vec{r}_{C/O} - \omega^2 \vec{r}_{C/O}$

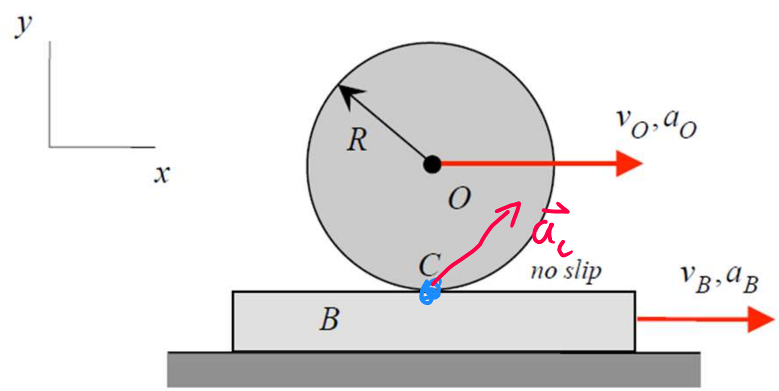
$\cancel{a_{cx}\hat{i}}^{\,0} + a_{cy}\hat{j} = a_o\hat{i} + \alpha\hat{k} \times (-R\hat{j}) - \omega^2(-R\hat{j})$

$a_{cy}\hat{j} = a_o\hat{i} + \alpha R\hat{i} + \omega^2 R\hat{j}$

$i: 0 = a_o + \alpha R \rightarrow \alpha = -\frac{a_o}{R}$

$j: a_{cy} = \omega^2 R\hat{j}$

Question: What if C is a no-slip point, but the surface it is on is moving?



Known information:

- O moves on a straight, horizontal path $\vec{V}_o = V_o\hat{i}$, $\vec{a}_o = a_o\hat{i}$
- B is translating in the x-direction $\vec{V}_B = V_B\hat{i}$ $\vec{a}_B = a_B\hat{i}$
- "No slip" condition $V_{cx} = V_B$ $a_{cx} = a_B$

Using our rigid body equation for velocity: $\vec{v}_C = \vec{v}_O + \vec{\omega} \times \vec{r}_{C/O}$

$V_{cx}\hat{i} + V_{cy}\hat{j} = V_o\hat{i} + \omega\hat{k} \times (-R\hat{j})$

$V_B\hat{i} + V_{cy}\hat{j} = V_o\hat{i} + \omega R\hat{i}$

$\hat{i}: \; V_B = V_o + \omega R \rightarrow \omega = \dfrac{V_B - V_o}{R}$

$\hat{j}: \; V_{cy} = 0$

Using our rigid body equation for acceleration: $\vec{a}_c = \vec{a}_O + \vec{\alpha} \times \vec{r}_{C/O} - \omega^2\vec{r}_{C/O}$

$a_{cx}\hat{i} + a_{cy}\hat{j} = a_o\hat{i} + \alpha\hat{k} \times (-R\hat{j}) - \omega^2(-R\hat{j})$

$a_B\hat{i} + a_{cy}\hat{j} = a_o\hat{i} + \alpha R\hat{i} + \omega^2 R\hat{j}$

$\hat{i}: \; a_B = a_o + \alpha R \longrightarrow \alpha = \dfrac{a_B - a_o}{R}$

$\hat{j}: \; a_{cy} = \omega^2 R$

Example 2.A.5

Given: A wheel rolls without slipping on a rough horizontal surface. At one instant, when $\theta = 90°$, the center of the wheel O is moving to the right with a speed of $v_O = 5$ ft/s with this speed decreasing at a rate of 3 ft/s^2. $a_o = -3\ ft/s^2$

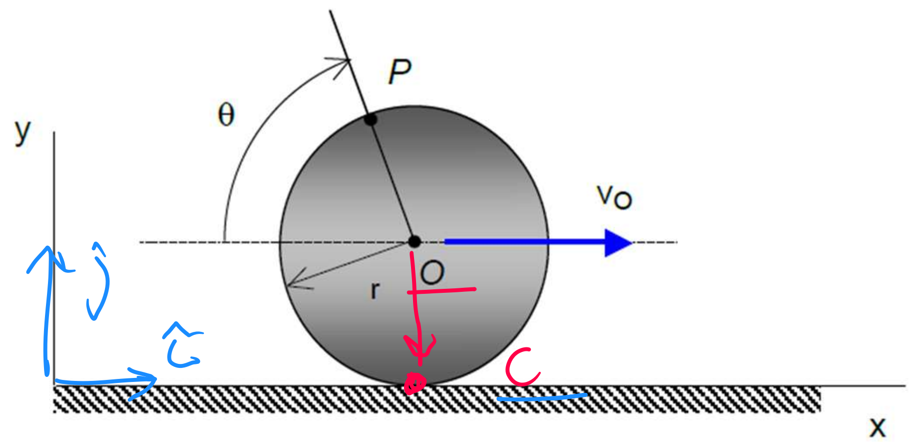
Find: Determine the acceleration of point P on the circumference of the wheel at this instant, if $r = 2$ ft. Make a sketch of this acceleration vector at P.

1) draw unit vectors
2) relate pts. with the most info

pt O: $\vec{V}_o = V_o\hat{\imath}$

pt C: $\vec{V}_{cx} = 0$, $\vec{a}_{cx} = 0$

$\vec{V}_{cy} = 0$



3) rigid body motion → velocity

$\vec{V}_c = \vec{V}_o + \vec{\omega} \times r_{c/o}$

$V_{cx}\hat{\imath} + V_{cy}\hat{\jmath} = \vec{V}_o\hat{\imath} + \omega\hat{k} \times (-r\hat{\jmath})$

$= V_o\hat{\imath} + \omega r\hat{\imath}$

$\hat{\imath}$: $V_{cx} = V_o + \omega r \longrightarrow \omega = \frac{\cancel{V_{cx}}^{0} - V_o}{r}$ $\omega = \frac{-V_o}{r} = -\frac{5}{2} = -2.5 \frac{rad}{s}\hat{k}$

$\hat{\jmath}$: $\underline{V_{cy}} = 0$

4) rigid body motion → acceleration

$$\vec{a}_c = \vec{a}_o + \vec{\alpha} \times r_{c/o} - \omega^2 r_{c/o}$$

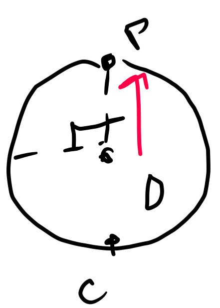
$$a_{cx}\hat{i} + a_{cy}\hat{j} = a_o\hat{i} + \alpha\hat{k} \times (-r\hat{j}) + \omega^2 r\hat{j}$$

$$a_{cx}\hat{i} + a_{y}\hat{j} = a_o\hat{i} + \alpha r\hat{i} + \omega^2 r\hat{j}$$

$$\hat{i}: \cancel{a_{cx}}^{0} = a_o + \alpha r \rightarrow \alpha = -\frac{a_o}{r} = -\frac{3}{2} = -1.5 \text{ rad/s}^2\hat{k}$$

$$\hat{j}\quad a_{cy} = \omega^2 r = \left(\frac{-5}{2}\right)^2 2 = 12.5 \text{ ft/s}^2$$

5) motion of pt. P



$$\vec{a}_p = \vec{a}_o + \vec{\alpha} \times r_{p/o} - \omega^2 r_{p/o}$$

$$\vec{a}_p = a_o\hat{i} + \alpha\hat{k} \times (r\hat{j}) - \omega^2 r\hat{j}$$

$$= a_o\hat{i} - \alpha r\hat{i} - \omega^2 r\hat{j}$$

$$\vec{a}_p = [3 - (-1.5 \cdot 2)]\hat{i} - \left(\left(-\frac{5}{2}\right)(2)\right)^{2} =$$

$$\vec{a}_p = 0\hat{i} - 12.5\hat{j} \text{ ft/s}^2$$

Example 2.A.4

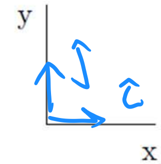
Given: Rack A moves to the right with a speed of v_A and an acceleration of a_A. Rack B moves to the left with a constant speed of v_B. Assume $v_A = 0.8$ m/s, $a_A = 2$ m/s^2, $v_B = 0.6$ m/s, $r = 0.1$ m and $R = 0.16$ m.

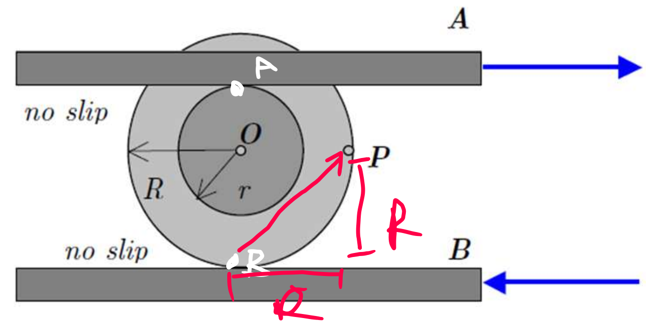
$a_{Bx} = 0$

Find: Determine:

(a) The velocity of point P on the outer rim of the gear; and

(b) The acceleration of point P on the outer rim of the gear.





1) unit vectors

2) known info

pt. A: $\vec{V}_A = v_A \hat{i}$, $\vec{a}_A = a_{Ax}\hat{i} + a_{Ay}\hat{j}$

pt. B: $V_B = -V_B\hat{i}$, $\vec{a}_B = 0\hat{i} + a_{By}\hat{j}$

3) find ω using pt A & B

$$\vec{V}_A = \vec{V}_B + \vec{\omega} \times \vec{r}_{A/B}$$

$$V_A\hat{i} = -V_B\hat{i} - \omega\hat{k} \times (R+r)\hat{j}$$

$$V_A\hat{i} = -V_B\hat{i} - \omega(R+r)\hat{i}$$

$$-\frac{V_A+V_B}{(R+r)}\vec{k} = \vec{\omega}$$

4) find α using pt A & B

$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}_{A/B} - \omega^2 \vec{r}_{A/B}$$

$$a_{Ax}\hat{i} + a_{Ay}\hat{j} = \cancel{a_{Bx}}\hat{i} + a_{By}\hat{j} + \alpha\hat{k} \times (R+r)\hat{j} - \omega^2 (R+r)\hat{j}$$

$$= a_{By}\hat{j} - \alpha(R+r)\hat{i} - \omega^2(R+r)\hat{j}$$

$$\hat{i} : a_{Ax} = -\alpha(R+r) \rightarrow \alpha = -\frac{a_{Ax}}{R+r}$$

5) use ω, α to find $\vec{V}_P$

$$\vec{V}_P = \vec{V}_B + \vec{\omega} \times \vec{r}_{P/B}$$

$$= V_B\hat{i} + \omega\hat{k} \times (R\hat{i} + R\hat{j})$$

$$= V_B\hat{i} + \omega R\hat{j} - \omega R\hat{i}$$

$$\vec{V}_P = (V_B - \omega R)\hat{i} + \omega R\hat{j} = \left(V_B - \frac{(V_A + V_B)}{(R+r)}R\right)\hat{i} + \omega R\hat{j}$$

(sub in & solve)

6) use ω, α to find $\vec{a}_p$

$$a_p = \vec{a}_B + \vec{\alpha} \times \vec{r}_{P/B} - \omega^2 r_{P/B}$$

$$= a_{By}\hat{j} + \alpha\vec{k} \times (R\hat{i} + R\hat{j}) - \omega^2(R\hat{i} + R\hat{j})$$

$$= \underbrace{a_{By}\hat{j}} + \alpha R\hat{j} - \alpha R\hat{i} - \omega^2 R\hat{i} - \underline{\omega^2 R}\hat{j}$$

$\underline{\omega^2 R}\,\hat{j}$ $\Leftarrow$ from earlier

$$\boxed{\vec{a}_p = -R(\alpha + \omega^2)\hat{i} + \alpha R\hat{j}}$$

ME 274: Basic Mechanics II

Lecture 9: Instant centers

PURDUE UNIVERSITY | School of Mechanical Engineering

Announcements:

Updated office hrs (ME 2008A):
T: 9:30- 11 am, W: 3:30-4:20, Th: 3-4:30

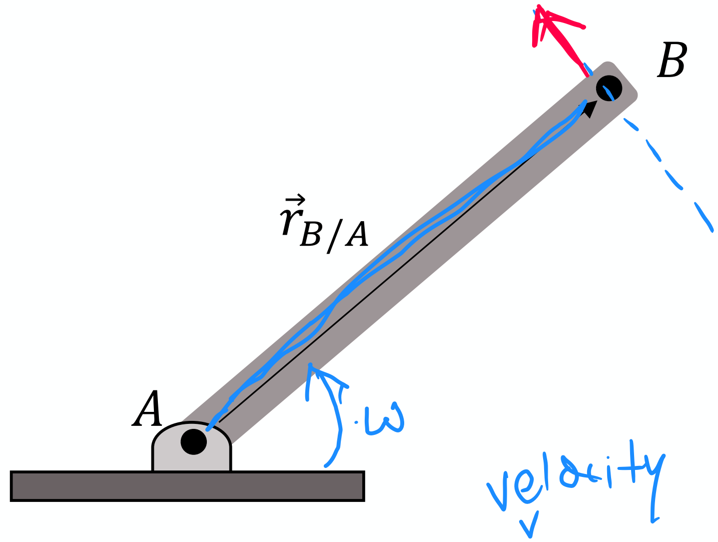
Exam 1:
- 8-9:30 pm Thursday, Feb 12
- Room: MA 175
- Students with DRC accommodations: HIKS G980

Review Sessions:
- Pi Tau Sigma: Thursday, February 5, 6:30-7:30 PM, WTHR 172
- ME 274 Instructor, Krousgrill: Tuesday, February 10, 7:00 PM
- Samples of past exams can be found on WeeklyJoys – note division of course material has changed over the years

Exam format:
- Topics covered: Ch.1 & Ch.2
- Problems 1 & 2: Long form problems similar to HWs
- Problem 3: Several true/false, multiple choice, or fill in the blank conceptual questions

Rigid body motion of an object rotating around a pin

Consider a body pinned at point A such that $\vec{v}_A = 0$. The body is therefore rotating about point A, or equivalently A can be called the center of rotation.



What do we know about the direction and magnitude of point B?

$$\vec{v}_B = \cancelto{0}{\vec{v}_A} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\vec{V}_B = \vec{\omega} \times r_{B/A}$$

direction $\vec{V}_B \perp \vec{r}_{B/A}$

magnitude?: $|\vec{V}_B| = \omega |\vec{r}_{B/A}|$

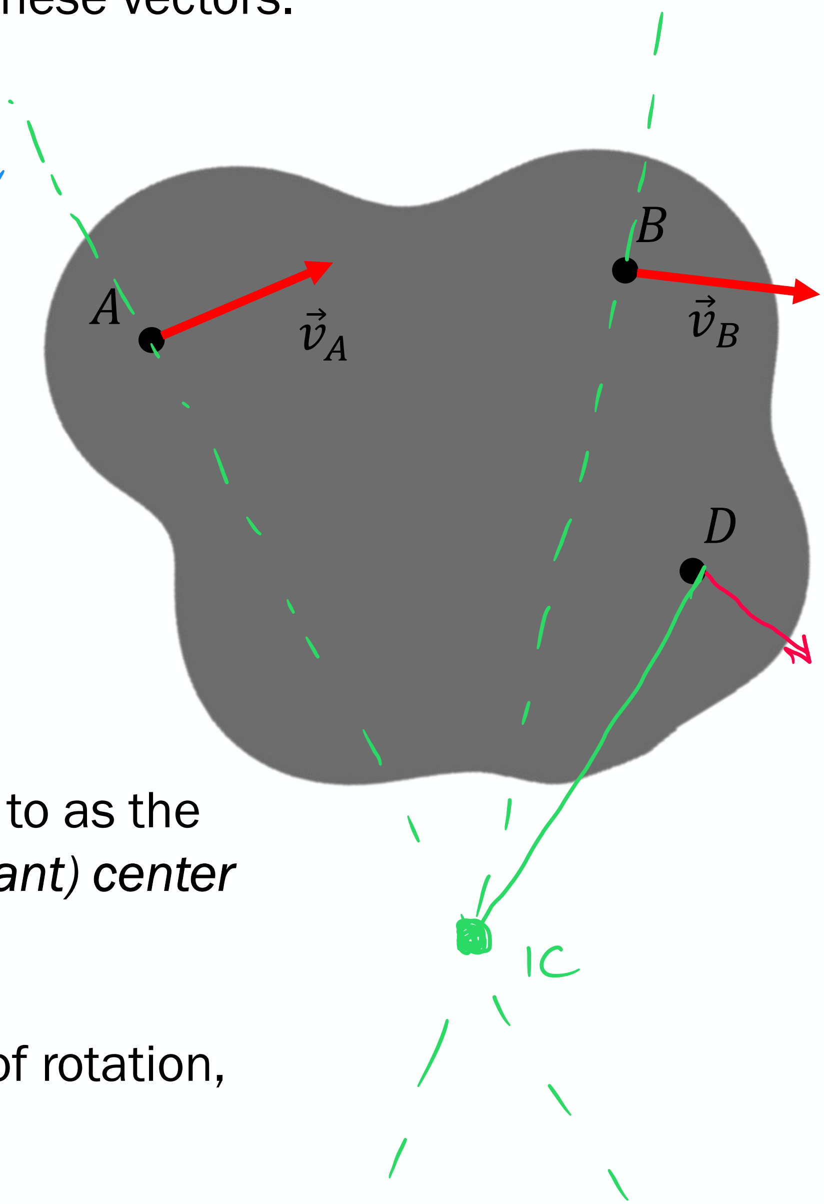
Reminder: the cross product of two vectors produces a vector **perpendicular** to both input vectors

Given velocity, how do we determine find the point the body rotates about?

Consider a rigid body with known velocities at points A and B.
We can find the **center of the body's rotation** by finding the **point of intersection** of lines drawn **perpendicular** to these vectors.

$\vec{V}_A = \vec{V}_B + \vec{\omega} \times \vec{r}_{A/}$

solve ω



Reversing this process can give us information about the velocity of point D

$V_A = \omega |\vec{r}_{A/C}|$

$V_B = \omega |r_{B/C}|$

This point is referred to as the *instantaneous* (or *instant*) *center of rotation*.

Since C is the center of rotation,

$$\vec{v}_C = \vec{0}$$

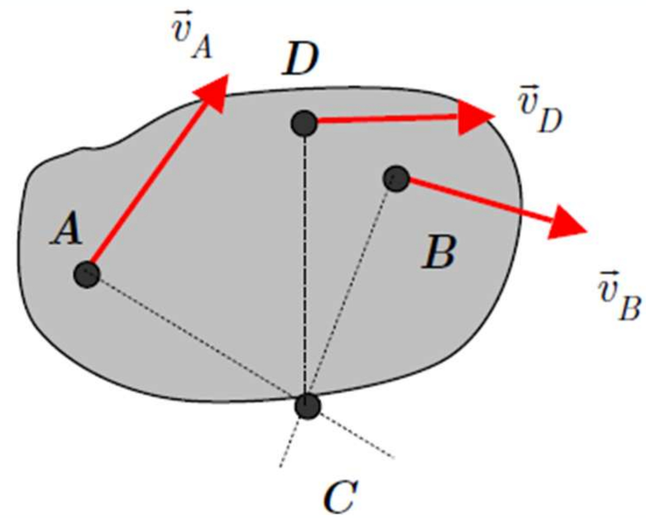
$\vec{V}_A = \cancel{\vec{V}_C}^{\,0} + \vec{\omega} \times r_{A/C} = \vec{\omega} \times \vec{r}_{A/C}$

Instant centers (ICs): more key concepts

- The **speed** of a point on the rigid body is **proportional to its distance** from the IC.

  - Remember, ω is a property of the body.

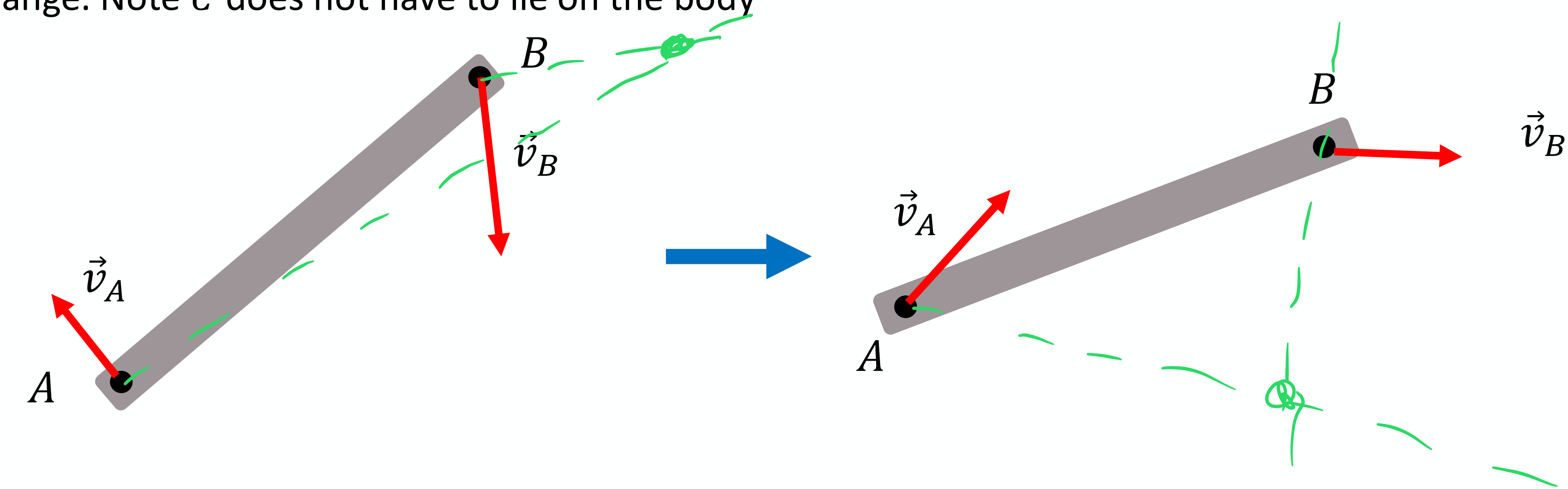
$$\omega = \frac{|v_A|}{|r_{A/C}|} = \frac{|v_B|}{|r_{B/C}|}$$



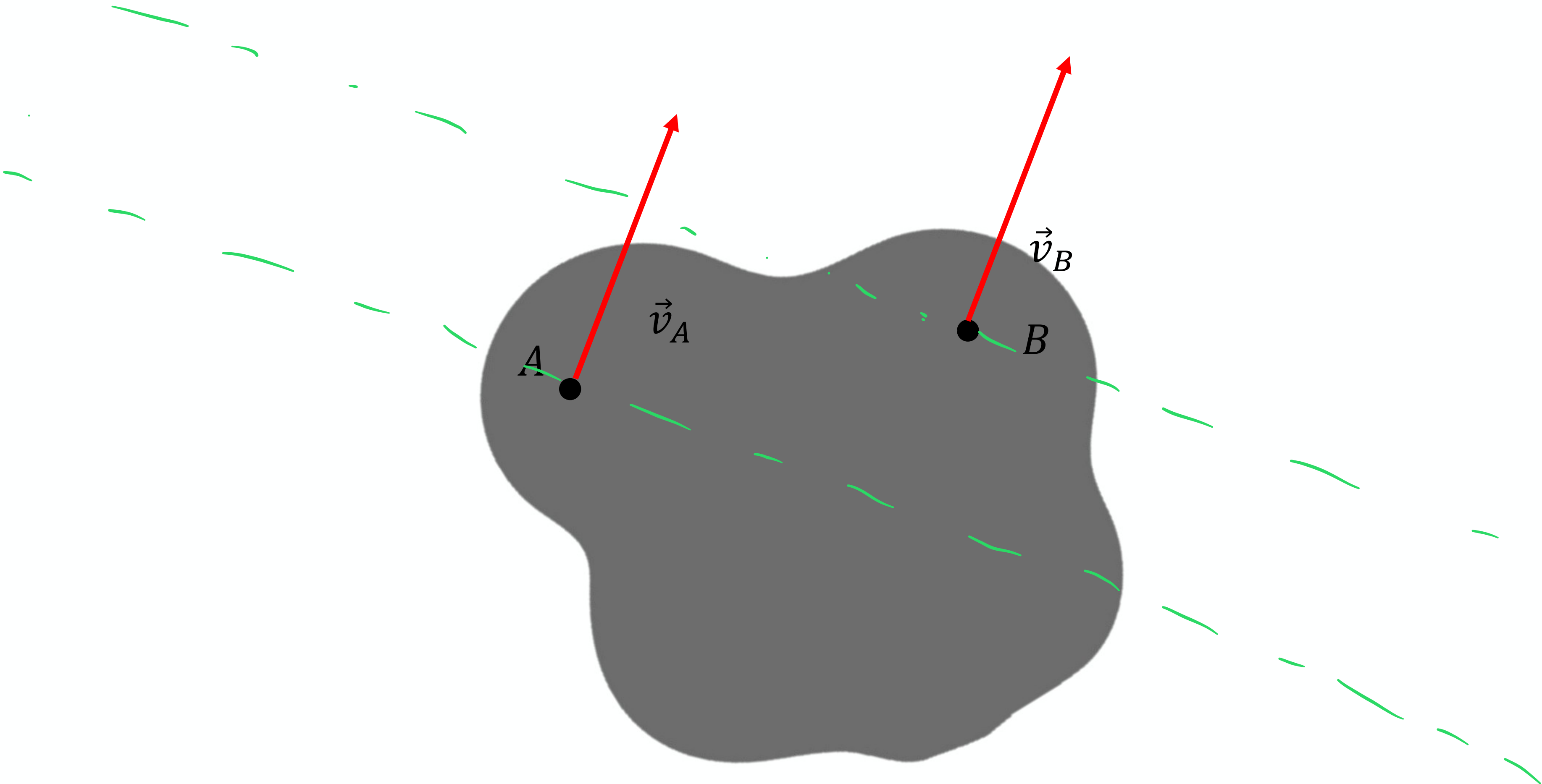
- The **sense of rotation** (i.e. the sign of ω or CW/CCW) is determined by visualizing the rotation of the body around C.

CW $-\hat{k}$

- The velocity vectors of each point changes with time, meaning the location of point C will also change. Note C does not have to lie on the body



Question: What if $|\vec{r}_{A/C}|$, $|\vec{r}_{B/C}| = \infty$?



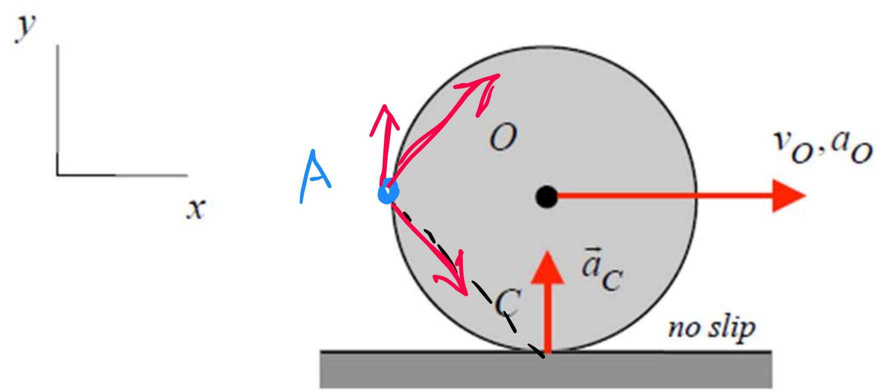
$$\omega = \frac{\vec{V}_A}{|r_{A/C}|} = \frac{\vec{V}_B}{|r_{B/C}|} = 0$$

∞ ∞

no rotation,
only translation

Question: The instant center of rotation for a body has zero velocity. Does the instant center also have zero acceleration?

Consider our rolling without slipping example:



$V_c = 0$

$C = IC$

$\vec{a}_c = \omega^2 r \hat{j}$

$$\vec{a}_A = \underline{\vec{a}_c} + \vec{\alpha} \times \underline{\vec{r}_{A/c}} - \omega^2 \underline{r_{A/c}}$$

3 components $\rightarrow$ not $\perp$ to $\vec{r}_{A/c}$

As a general rule, you cannot use instant centers when determining accelerations.

Method: Instant Centers

Recommended steps for instant center analysis for planar motion of a rigid body:

1. Locate two points A and B on the rigid body for which you know some information of their motion.
   - Let A be a point for which you know BOTH the magnitude and direction of its velocity, $\vec{v}_A$.
   - Let B be a point for which you know the direction of its velocity, $\vec{v}_B$.
2. On a sketch of the body, draw the directions of the velocity vectors $\vec{v}_A$ and $\vec{v}_B$.
3. Draw lines that are perpendicular to the respective directions of $\vec{v}_A$ and $\vec{v}_B$.
4. The intersection of the two lines drawn above locates the instant center C of the body. That is, for the instant corresponding to the position of the body in your sketch, we know that $\vec{v}_C = \vec{0}$.
5. From this, we can find:
   - magnitude of the angular velocity of the body, ω, from:

     $$\omega = \frac{|\vec{v}_A|}{|\vec{r}_{A/C}|}$$

     where $|\vec{r}_{A/C}|$ is the distance from C to A. Recall from above that we have assumed that we know the speed of point A.
   - the direction of the angular velocity from examining the figure. Since C is the instant center of the body, the body is instantaneously rotating about C. Knowing the direction of $\vec{v}_A$, we can determine if the body is rotating clockwise or counterclockwise about C.
6. The velocity of ANY point D on the body is perpendicular to the line connecting C and D. The speed of D is found from:

   $$|\vec{v}_D| = |\vec{\omega}|\ |\vec{r}_{D/C}|$$

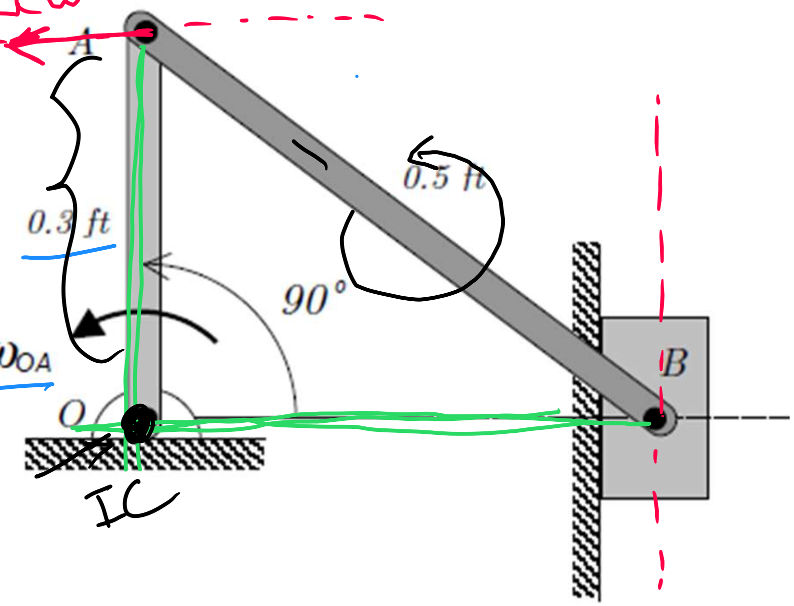
   The direction of is found from recalling again that the body is instantaneously rotating about point C.

Refer to lecture book for step-by-step process for IC analysis

What is the sense of rotation for link AB for the following two mechanisms at the positions shown?

$\pm \omega$? cw ccw

$V_A = 0.3\,\omega_{OA}$



1. Locate two points A and B on the rigid body for which you know some information of their motion.
   - Let A be a point for which you know BOTH the magnitude and direction of its velocity, $\vec{v}_A$.
   - Let B be a point for which you know the direction of its velocity, $\vec{v}_B$.
2. On a sketch of the body, draw the directions of the velocity vectors $\vec{v}_A$ and $\vec{v}_B$.
3. Draw lines that are perpendicular to the respective directions of $\vec{v}_A$ and $\vec{v}_B$.
4. The intersection of the two lines drawn above locates the instant center C of the body. That is, for the instant corresponding to the position of the body in your sketch, we know that $\vec{v}_C = \vec{0}$.
5. From this, we can find:
   - magnitude of the angular velocity of the body, ω, from:

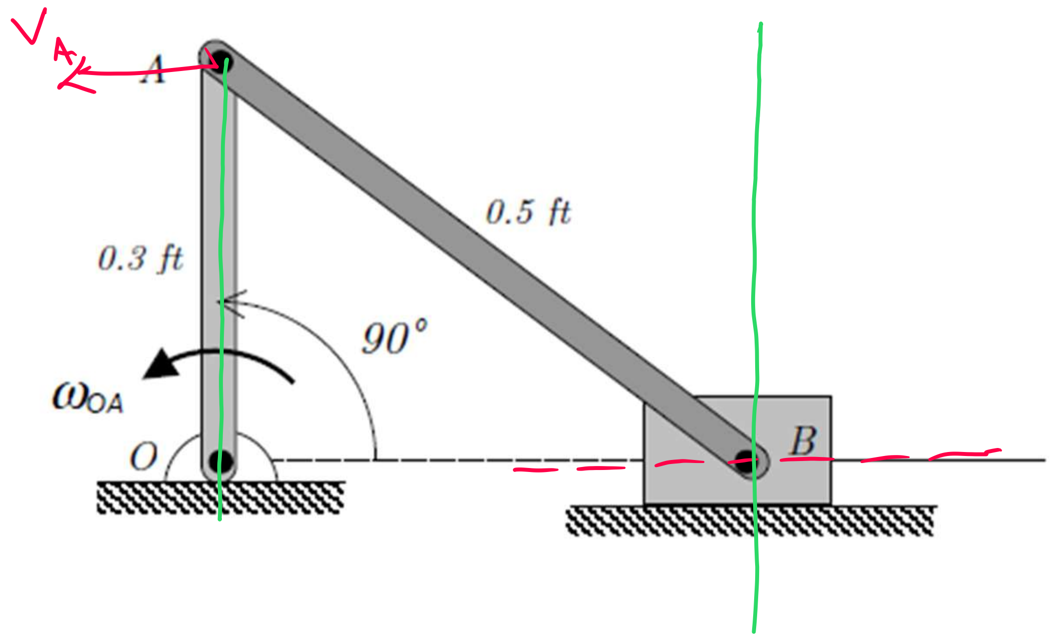
     — link AB

$$\omega = \frac{|\vec{v}_A|}{|\vec{r}_{A/C}|} = \frac{0.3\,\omega_{OA}}{0.3}$$

$$\omega_{AB} = +\omega_{OA}\,\hat{k}$$

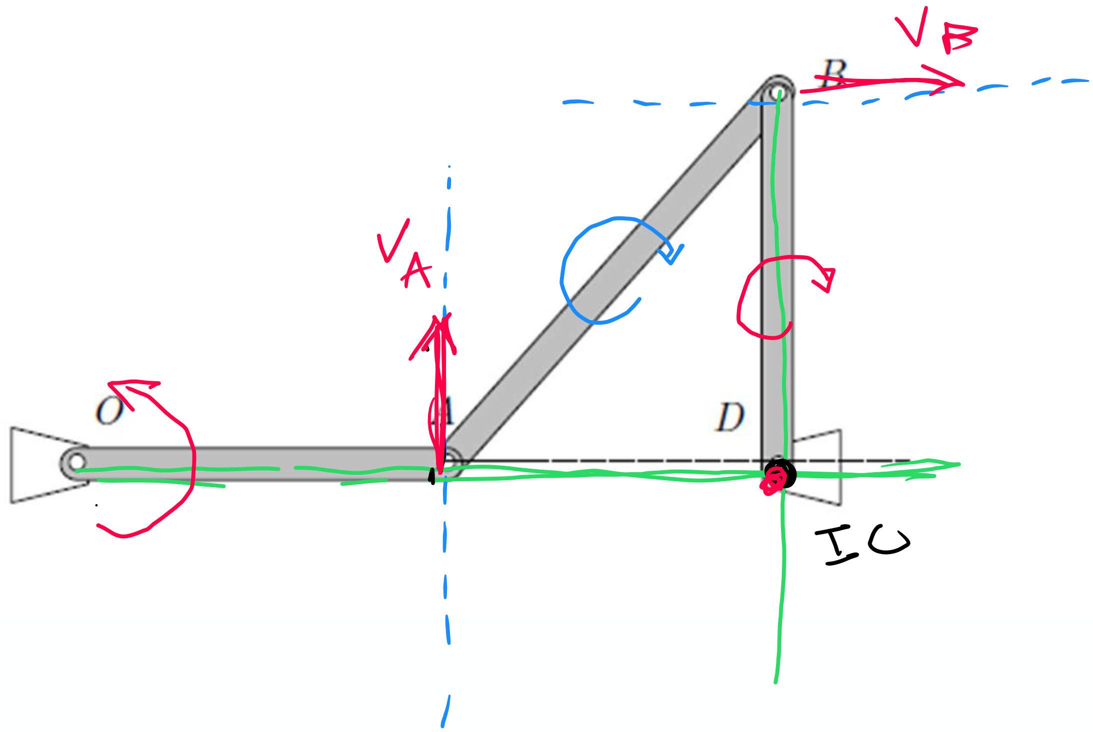
     where $|\vec{r}_{A/C}|$ is the distance from C to A. Recall from above that we have assumed that we know the speed of point A.
   - the direction of the angular velocity from examining the figure. Since C is the instant center of the body, the body is instantaneously rotating about C. Knowing the direction of $\vec{v}_A$, we can determine if the body is rotating clockwise or counterclockwise about C.

What is the sense of rotation for link AB for the following two mechanisms at the positions shown?

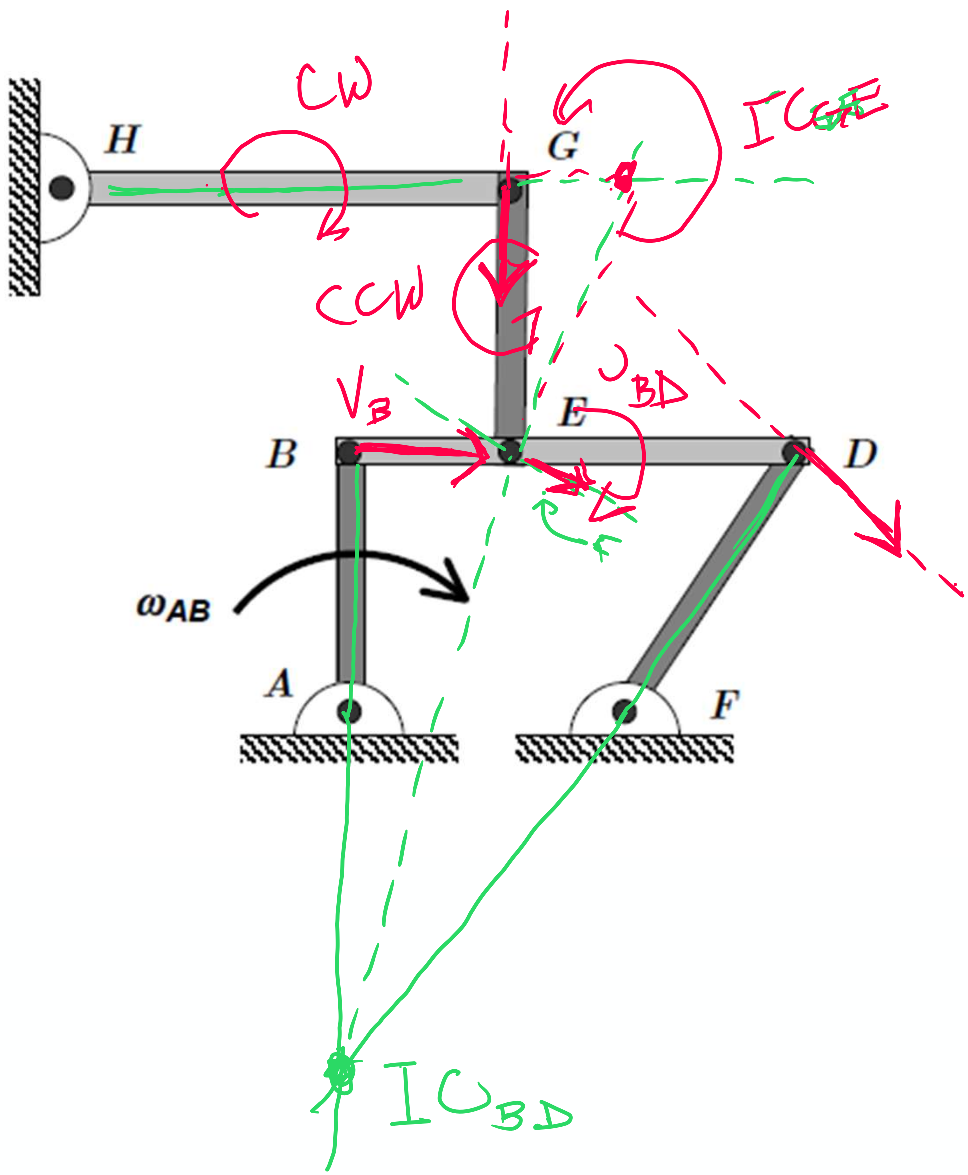


$\vec{V}_A \parallel \vec{V}_B \rightarrow \omega_{AB} = 0$

What is the sense of rotation of links OA and BD, if link AB is rotating CW?



Link AB is rotating in the clockwise direction. What is the sense of rotation for the other links of this mechanism?



Example 2.B.1

Given: Link OA rotates with an angular speed of $\omega_{OA} = 3$ rad/s with a counterclockwise sense about pin O. At the instant shown, link OA is horizontal, AB is vertical and $\theta = 36.87°$.

Find:

(a) Locate the instant center IC_{AB} for link AB.

(b) Using the location of IC_{AB}, determine the angular velocities of links AB and DB.

$V_A = \omega_{OA} L$

Link AB

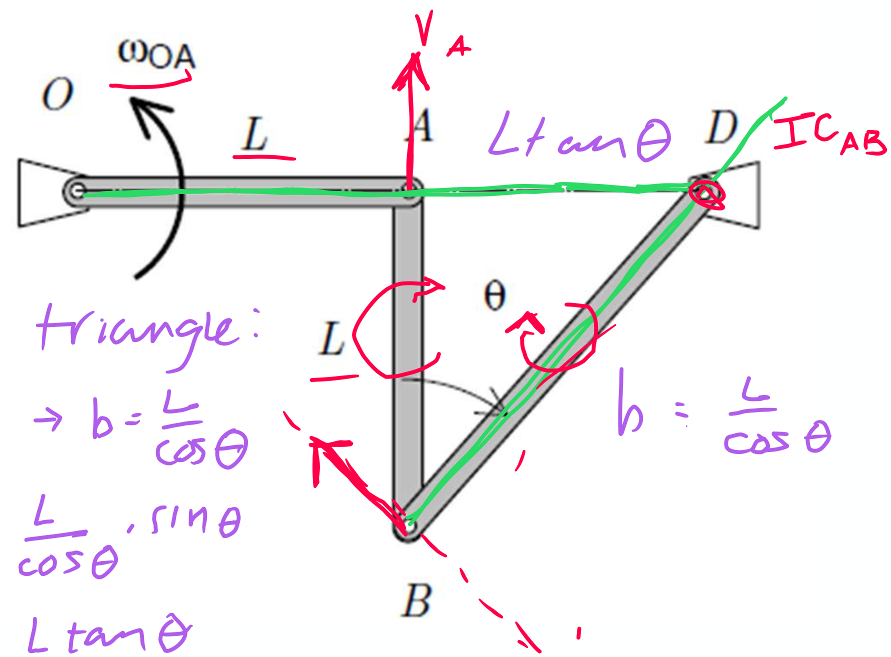
$\underline{V_A} = \omega_{AB} \underline{|r_{A/IC_{AB}}|}$

$\omega_{AB} = \frac{V_A}{L\tan\theta}$

Link BD

$V_B = \omega_{AB} L\cos\theta$

$V_B = \omega_{BD} L\cos\theta$



trig for triangle:

$L = b\cos\theta \rightarrow b = \frac{L}{\cos\theta}$

$\overline{AD} = b\sin\theta = \frac{L}{\cos\theta} \cdot \sin\theta$

$= L\tan\theta$

$\omega_{AB} \cancel{L\cos\theta} = \omega_{BD} \cancel{L\cos\theta}$

$\omega_{AB} = \omega_{BD}$

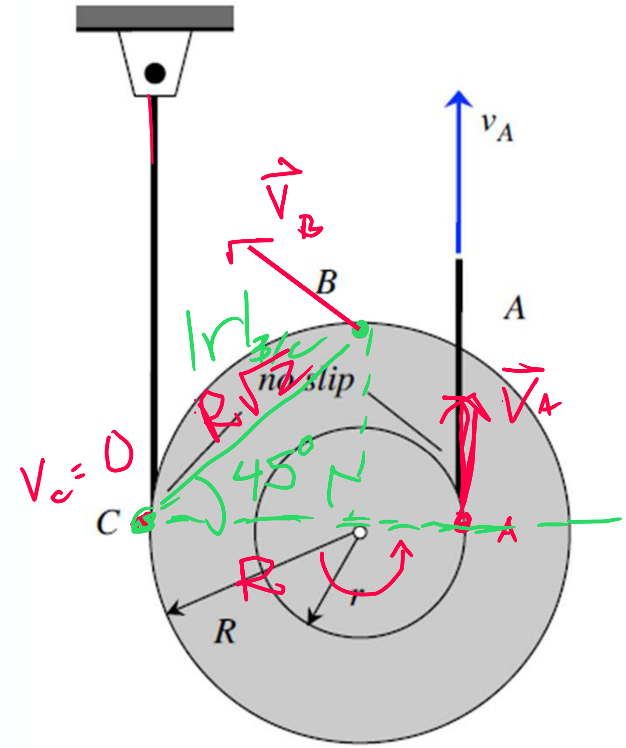
Example 2.B.2

Given: A cable, wrapped around the inner radius of the pulley shown, is being raised at a rate of v_A. A second cable is wrapped around the outer radius of the same pulley with the upper end of this cable attached to ground. Assume that the pulley does not slip on either cable.

Find: Determine:

(a) The location of the instant center for the pulley; and

(b) The velocity of point B on the outer radius of the pulley when B is directly above the center O of the pulley. Sketch this velocity vector.



$$\omega = \frac{V_A}{|r_{A/C}|} = \frac{V_A}{(r+R)}$$

$$V_B = \omega |\vec{r}_{B/C}|$$

$$V_B = \omega\sqrt{2R^2}$$

$$\vec{V}_B = V_B(-\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$= \omega R\sqrt{2}\left(-\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}\right)$$

$$\vec{V}_B = -\omega R\,\hat{i} + \omega R\,\hat{j}$$

ME 274: Basic Mechanics II

Lecture 10: Summary of Kinematic Analysis of Planar Mechanisms

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Announcements:

Updated office hrs (ME 2008A):
T: 9:30- 11 am, W: 3:30-4:20, Th: 3-4:30

Exam 1: See course website for info on Exam 1, review sessions, equations sheet etc.

A reminder on required HW formatting:

DUE DATE *(month, day, year)*	**PROBLEM NO.** *(H#.#)*	**PAGE NO.** *(# of #)*	**NAME** *(last, first)*

Given: A concise statement (in your own words) of the information given.

Find: A concise statement (in your own words) of the information sought.

Solution: **Sketch the system** to be studied. USE A STRAIGHT EDGE for drawing lines. Always draw in the UNIT VECTORS for the coordinate systems that you use in your solution.

For kinetics problems, follow the four-step plan:

1. Draw FBD's
2. Write down the fundamental kinetics equations (Newton/Euler, work/energy, linear impulse/momentum, angular impulse momentum equations)
3. Kinematics
4. Solve

Work the problem symbolically.
At the end convert all quantities to a consistent set of units and substitute into the equations to obtain the answers.

Check your answers for correctness and feasibility.

Check your vector notation and units. In particular, check that you are not equating vector quantities to scalar quantities. It is important that you demonstrate that you know the difference between scalars and vectors. So pay attention to your notation.

Label the answers.____________________________________ANSWER

Review: Rigid Body Motion

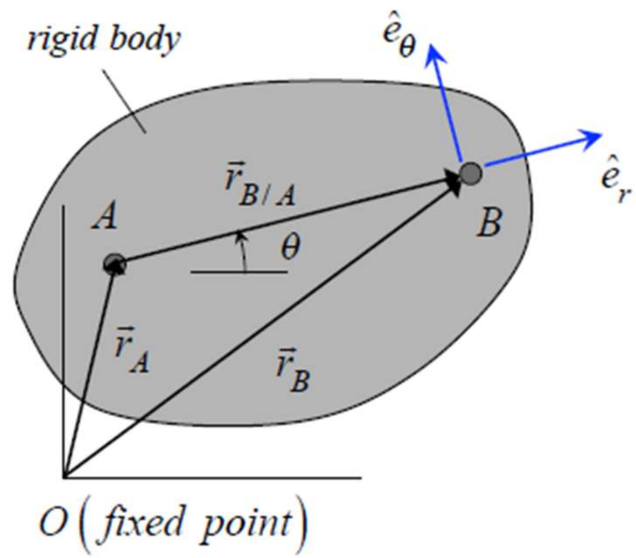
If we know the motion of one point and the rotation of the body, we can determine the motion of any point on the body

Velocity and acceleration of point B with respect to point A :

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

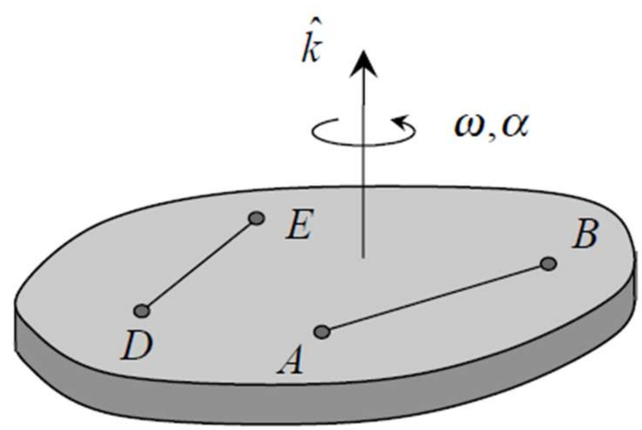
$$\vec{a}_B = \vec{a}_A + \vec{\omega} \times \left[\vec{\omega} \times \vec{r}_{B/A}\right] + \vec{\alpha} \times \vec{r}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}.$$



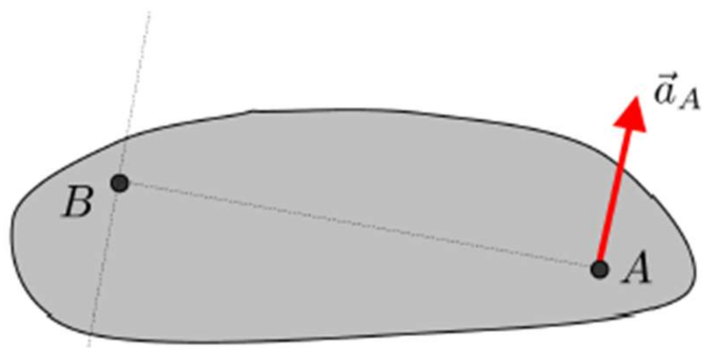
Angular velocity ω, and angular acceleration α are properties of the body

- $\vec{\omega} = \omega \hat{k} = \dot{\theta} \hat{k}, \;\; \vec{\alpha} = \alpha \hat{k} = \ddot{\theta} \hat{k}$
- Sign of ω, α determines direction of rotation (using right hand rule)



Question C2.1

The rigid body shown below has a counterclockwise angular velocity ω and a clockwise angular acceleration α. The direction of the acceleration of point A, $\vec{a}_A$, is shown in the figure with $\vec{a}_A$ being perpendicular to line AB. Make a sketch showing the direction of the acceleration vector for point B. This sketch does not need to be accurate; simply show its direction relative to the lines in the figure, where these two lines are perpendicular and parallel to $\vec{a}_A$.



Steps for solving velocity and acceleration problems

1. Define your coordinate axis

2. Write down your velocity/acceleration equation for each body
   - If not shown graphically or explicitly stated, assume ω, α are positive, a negative end result tells you to flip this assumption
   - Remember that $\vec{r}_{B/A}$ points from A to B
   - A and B must be on the same linkage
   - Give each ω, α a unique subscript

3. Combine your equations from step 2
   - Split vector equations into scalar equations (break into $\hat{\imath}, \hat{\jmath}$ components)
   - Count your equations and unknowns, if they match, solve for your unknowns
   - If they don't match, you've missed an equation. Go back to step 2.
   - Write you answers as vectors, noting ω, α will be in the $\hat{k}$ direction

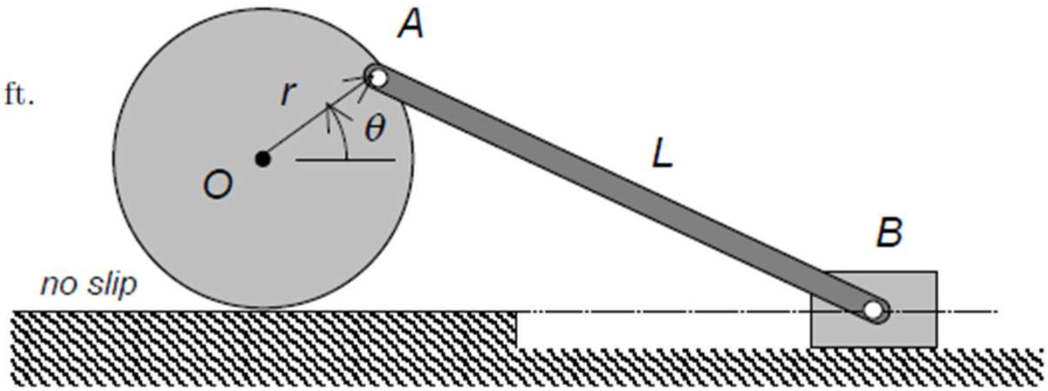
4. Use the concept of instant centers to check the direction of your velocity and the sense of rotation of your bodies

Example 2.C.1

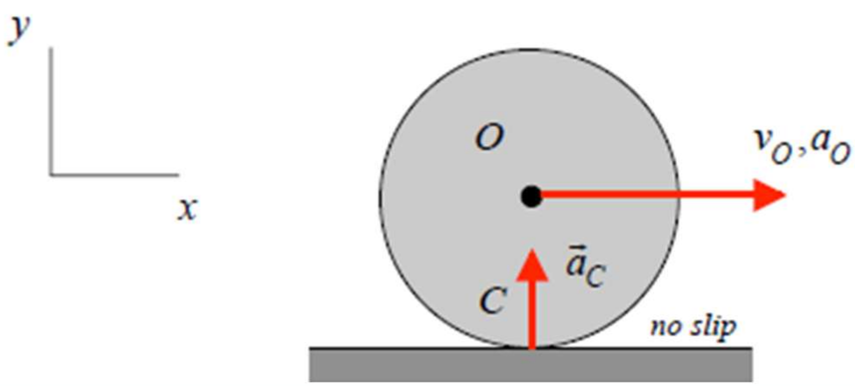
Given: The wheel rolls without slipping in such a way that slider B moves to the left with a constant speed of $v_B = 5$ ft/s.

Find: Determine:
(a) The angular velocity of the wheel when $\theta = 0$; and
(b) The angular acceleration of the wheel when $\theta = 0$.

Use the following parameters in your analysis: $L = 2$ ft and $r = 0.5$ ft.



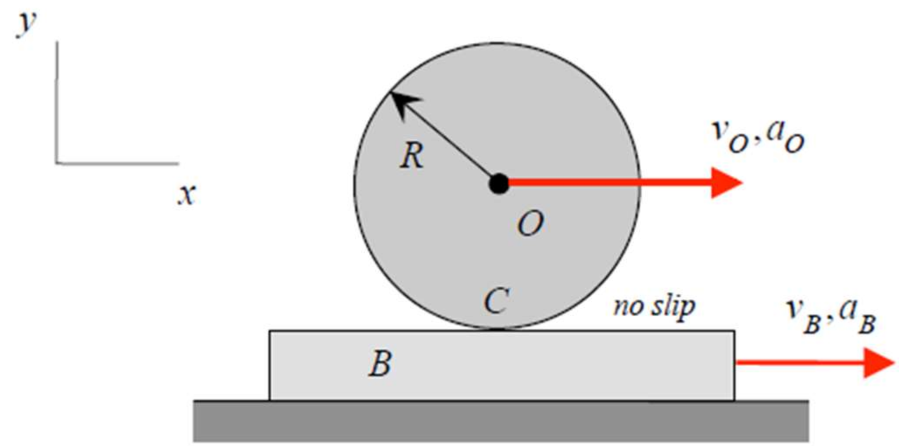
Review: Rolling without slip



$$v_{cx} = 0$$

$$a_{cx} = 0$$

$$v_{Cy} = 0 \text{ and } a_{Cy} \neq 0.$$



$$v_{cx} = v_B$$

$$a_{cx} = a_B$$

$$v_{Cy} = 0 \text{ and } a_{Cy} \neq 0.$$

Stationary Surface:

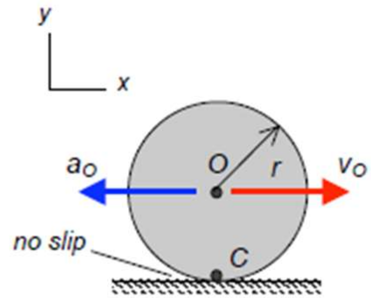
- Contact point is **instant center**
- Only the **tangential component** of acceleration will be **zero**

Moving Surface:

- Contact point is **not** instant center
- Components of velocity and acceleration **tangent** to the surface **will be the same** for the body and the surface.
- Acceleration components **normal** to the surface **will be different.**

Question C2.2

A sphere of radius r rolls without slipping to the right on a rough, horizontal surface. The center of the sphere, O, has a speed of v_O, with this speed decreasing at a rate of a_O.



Circle the figure below that most accurately represents the direction of the acceleration of the contact point C.

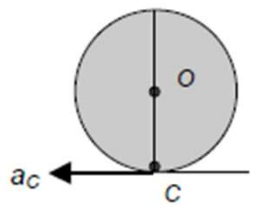


Figure A

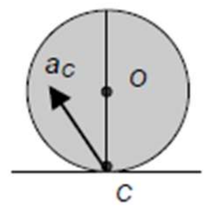


Figure B

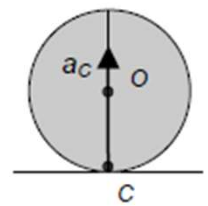


Figure C

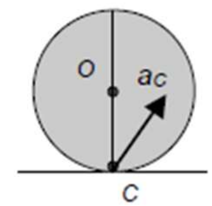


Figure D

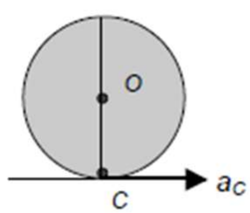


Figure E

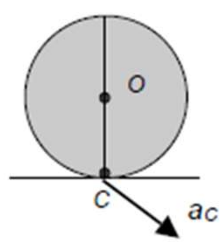


Figure F

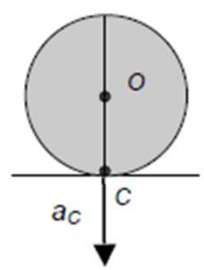


Figure G

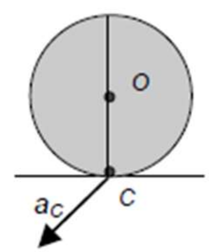


Figure H

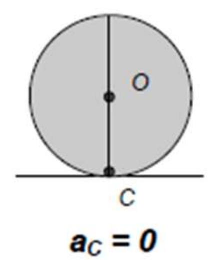


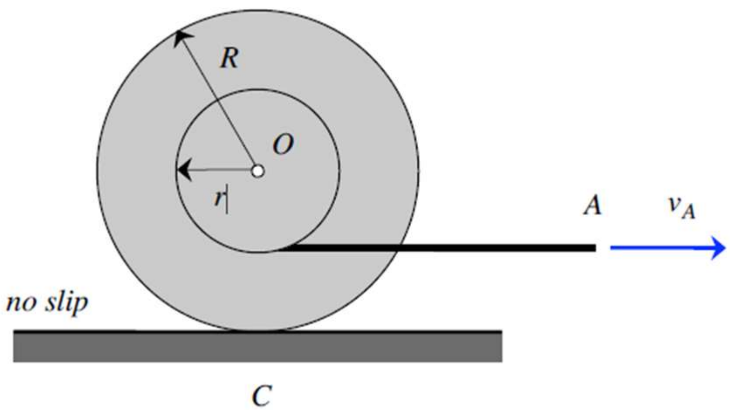
Figure I

Example 2.B.3

Given: A cable is wrapped around the inner radius of a spool. End A of the cable is moving to the right with a speed of v_A. The spool is able to roll without slipping on a rough horizontal surface.

Find: Using the method of instant centers, determine:
- (a) The velocity of the center O of the spool; and
- (b) The angular velocity of the spool.

Use the following parameters in your analysis: $v_A = 0.8$ m/s, $r = 0.3$ m and $R = 0.9$ m.



Review: Instant Centers of Rotation

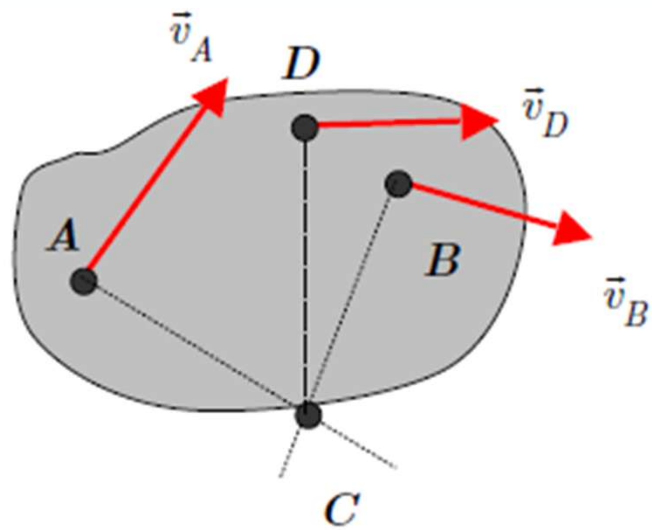
The "center of rotation" (instant center) for a body is located at the intersection of the perpendiculars to the velocities of two points on the body.

The IC is a point in space where $\vec{v}_{IC} = \vec{0}$

The sense of rotation (i.e. sign of ω) can be determined visually by the direction of the velocity and position of IC

The speed of a point on a body is proportional to its distance from the IC

$$|\vec{v}_D| = |\vec{\omega}| \; \left|\vec{r}_{D/C}\right| \qquad \omega = \frac{|\vec{v}_A|}{\left|\vec{r}_{A/C}\right|}$$



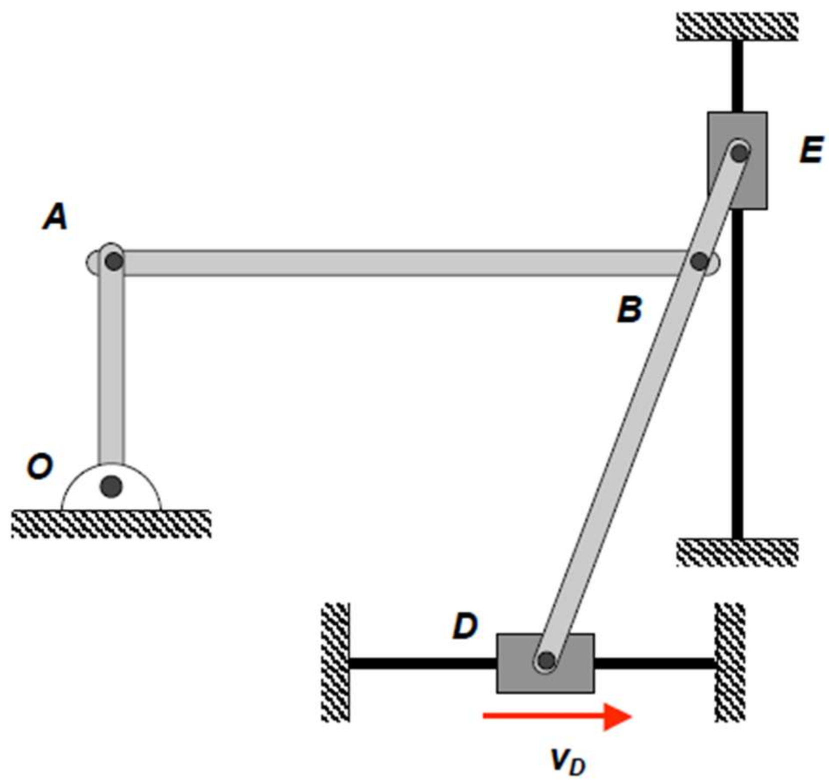
Question C2.5

A mechanism is made up of links OA, AB and DE. Pins D and E on link DE are constrained to move along straight guides. Link OA is pinned to ground at O and pinned to link AB at A. Link AB is also pinned to link DE at point B. Pin D moves to the right with a speed of v_D. For the position shown:

(a) Accurately locate the instant center for link DE.

(b) Accurately locate the instant center for link AB.

(c) Determine if link AB is rotating counterclockwise, rotating clockwise or is instantaneously at rest. Provide an argument for your answer.



Question C2.6

The mechanism shown below is made up of links AB, BD, DF and EH. Links AB and DF are pinned to ground at pins A and F, respectively. Link EH is pinned to link DF at E. Pin H is constrained to move along a straight, horizontal path. Link AB is rotating counterclockwise, as shown.

(a) Accurately locate the instant center for link BD.

(b) Accurately locate the instant center for link EH.

(c) What is the direction of motion for pin H (left, right or instantaneously stationary)? Provide an argument for your answer.