ME 274: Basic Mechanics II

Lecture 1: Course Intro & Point Kinematics, Cartesian Description

PURDUE UNIVERSITY | School of Mechanical Engineering

ME 274 Course Structure

- Textbook: Dynamics – A Lecturebook
  - https://www.purdue.edu/freeform/me274/daily-schedule-sp-2026/
  - Will be **used daily during lecture**
  - Can be purchased for $90 from the University Book Store

- Course website: **https://www.purdue.edu/freeform/me274**
  - Homework assignments, solution videos, and discussion posts
  - Additional example problems
  - Section specific announcements on instructor page

Assignments and Grading

Grade scale: 97-100% A+, 93-97% A, 90-93% A-, 87-90% B+, etc.….

Homework and Quizzes: 25%

Homework
- Two homework problems per lecture, submitted to Gradescope by **11:59 PM the day of the next lecture**.
- One problem graded for correctness, one problem graded for completion.
- Three dropped assignments.

Quizzes
- In-class quizzes will be given throughout the semester and will be **unannounced**.
- Quizzes serve as practice for you and an assessment of class understanding for me.
- Grading will be based off completion/demonstrated effort.
- If you need to miss a lecture email me **before the start of class** with the subject line **ME 274 Absence** (no body text needed).

Exams: 75%

- Two 1.5 hour evening midterms, Th 2/12 & Th 4/2
- 2 hour final exam, date TBD during finals week
- Weighting of midterm average and final: **Higher score → 50%** of course grade, **Lower score → 25%** of course grade

Teaching Team and Help Resources

- Dr. Andress office hours: T 9:30-10:30 & W 3:30-4:30 in ME 2008A, (or by appointment)

- See syllabus for additional faculty office hours

- ME tutorial rooms – staffed by TAs during normal business hours

- Discussion thread on course website
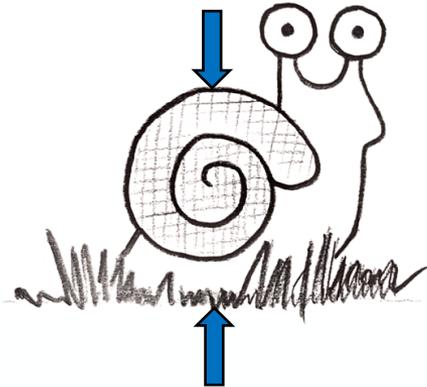
- 800+ of your peers!

Moving from statics to dynamics

Statics (ME 270):

- Systems in equilibrium
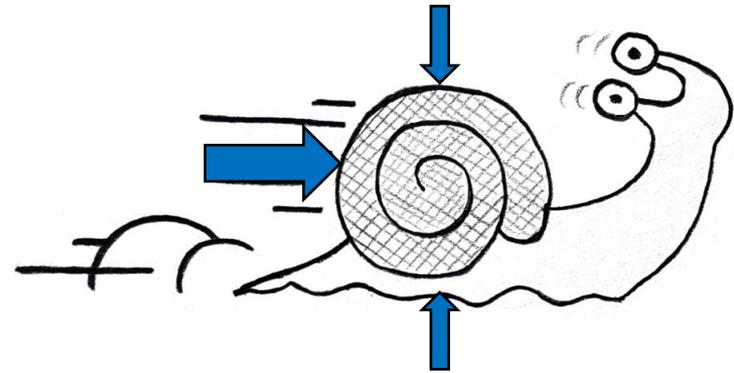
$$\sum F = 0 \,, \sum M = 0$$

- No acceleration

Dynamics (ME 274):

- Net forces result in acceleration

$$\sum F = m\, a$$

Course Objectives:
1. Describe motion precisely
2. Relate motion to forces
3. Predict system behavior over time

Image: https://enterfea.com/difference-between-static-and-dynamic-analysis/

Describing motion: particle kinematics

Kinematics describe particle motion without reference to forces – **how the system moves** with respect to a chosen coordinate system.

Describes:
- Position, $\vec{r}(t)$
- Velocity, $\vec{v}(t)$
- Acceleration, $\vec{a}(t)$

Path of P

P

$\vec{r}$
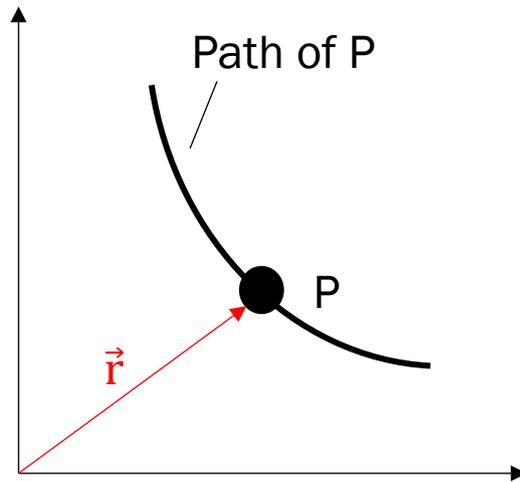
For all coordinate systems:
$$\vec{v} = \frac{d\vec{r}}{dt} \, ,$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}$$

Cartesian Kinematics

- The path of particle P is expressed in terms of **x** and **y** components

$$\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

- All vectors are represented as linear combinations of the **unit vectors $\hat{\imath}$ and $\hat{\jmath}$**
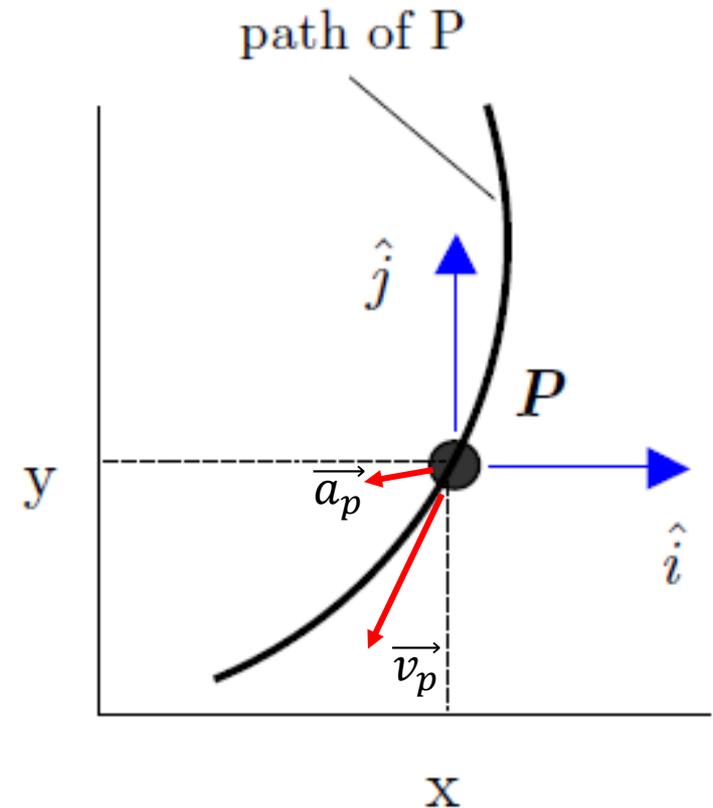
$$\frac{d\hat{\imath}}{dt} = 0, \frac{d\hat{\jmath}}{dt} = 0$$

Velocity components come from **time derivatives of position components**

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$

Acceleration components come from **second time derivatives of position components**

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$

path of P

In Cartesian kinematics, position can be defined in two ways:

Explicit time dependence

- Position components given directly as a function of time:

$$x = x(t), \qquad y = y(t)$$

- Velocity and acceleration found by direct differentiation

$$\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}$$
$$\ddot{x} = \frac{d^2x}{dt^2}, \ddot{y} = \frac{d^2y}{dt^2}$$

Implicit time dependence

- Geometry defines the path and one coordinate depends on time **indirectly**

$$x = x(t), \qquad y = f(x) \quad \textbf{or}$$
$$y = y(t), \qquad x = f(y)$$

- Differentiation requires the **chain rule:**

if $x = x(t)$, $y = f(x)$,

$$\dot{x} = \frac{dx}{dt},$$
$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \dot{x}\frac{dy}{dx}$$

Example: Implicit time dependence

Problem Statement: Suppose that $y = \sin(x)$ and $\dot{x} = 3$ m/s $= constant$, and we want to know the velocity and acceleration when $x = \pi/2$.

Find: Cartesian components of velocity and acceleration

Solution:

x components of velocity and acceleration are known:

$$x = \frac{\pi}{2}, \ \dot{x} = 3 \text{ m/s = constant} \ \rightarrow \ \ddot{x} = 0$$

To find the y component of velocity, differentiate using the chain rule:

$$y = \sin(x) \ \rightarrow \ \dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) = \cos x, \ \frac{dx}{dt} = \dot{x} \ \Rightarrow \dot{y} = \dot{x}\cos x = 3\cos\frac{\pi}{2} = 0 \text{ m/s}$$

To find the y component of acceleration, differentiate velocity with respect to time:

$$\ddot{y} = \frac{d}{dt}(\dot{y}) = \frac{d}{dt}(\dot{x}\cos x) \quad \leftarrow \text{ Use chain \textbf{and} product rule}$$

$$\ddot{y} = \ddot{x}\cos x - \dot{x}^2 \sin x \ \Rightarrow \ddot{y} = (0)\cos\frac{\pi}{2} - 3^2 \sin\frac{\pi}{2} = -9 \text{ m/s}$$

Answer: $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = 3\hat{\imath} + 0\hat{\jmath} \text{ m/s}, \quad \vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = 0\hat{\imath} - 9\hat{\jmath} \text{ m/s}^2$

Given: A particle P moves on a path whose Cartesian components are given by the following functions of time (where both components are given in inches and time t is given in seconds):

$$x(t) = t^3 + 10$$
$$y(t) = 2\cos 4t$$

Find: Determine at the time $t = 2$ s:
 (a) The velocity vector of P;
 (b) The acceleration of P; and
 (c) The angle between the velocity and acceleration vectors of P.

Solution:

a) Velocity components can be found directly by differentiating wrt t: $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$

$$\dot{x} = 3t^2, \qquad \dot{y} = -8\sin 4t$$

$$\vec{v} = 3t^2\hat{\imath} - 8\sin 4t\,\hat{\jmath} \Rightarrow \text{plug in t = 2 and solve}$$

b) Acceleration components can be found by taking the second derivative wrt t: $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

$$\ddot{x} = 6t, \qquad \ddot{y} = -32\cos 4t$$

$$\vec{a} = 6t\hat{\imath} - 32\cos 4t\,\hat{\jmath} \Rightarrow \text{plug in t = 2 and solve}$$

c) From linear algebra remember: $\vec{v} \cdot \vec{a} = |\vec{v}||\vec{a}|\cos(\theta)$

$$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{a}}{|\vec{v}||\vec{a}|}\right) = \cos^{-1}\left(\frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}\,\sqrt{\ddot{x}^2+\ddot{y}^2}}\right) \Rightarrow \text{plug in t = 2 and solve}$$

Given: Pin P is constrained to move along a elliptical ring whose shape is given by $x^2/a^2+y^2/b^2 = 1$ (where x and y are given in mm). The pin is also constrained to move within a horizontal slot that is moving upward at a constant speed of v.

Find: Determine:
 (a) The velocity of pin P at the position corresponding to $y = 6$ mm; and
 (b) The acceleration of pin P at the position corresponding to $y = 6$ mm.

Use the following parameters in your analysis: $a = 5$ mm, $b = 10$ mm, $v = 30$ mm/s.
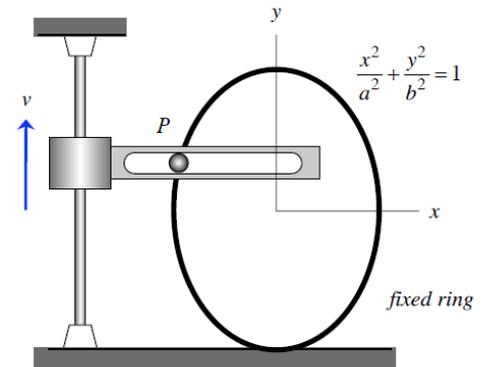
Solution:

a) Velocity components are defined as: $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$

Known: vertical motion of P constrained by slot $\rightarrow \dot{y} = v$

Known: the slot moves upward at a constant speed $\rightarrow \ddot{y} = 0$

Known: x position constrained with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \rightarrow$ implicitly differentiate and solve for $\dot{x}$

$$\frac{2x\dot{x}}{a^2} + \frac{2\dot{y}\dot{y}}{b^2} = 0 \Rightarrow b^2 x\dot{x} + a^2 y\dot{y} = 0$$

Solve for x from original constraint eqn: $x = \pm a\sqrt{\left(1 - \dfrac{y^2}{b^2}\right)}$

Solve for $\dot{x}$: $\dot{x} = -\dfrac{a^2 y\dot{y}}{b^2 x} = -\dfrac{ay\dot{y}}{b^2\sqrt{\left(1-\frac{y^2}{b^2}\right)}}$ $\leftarrow$Plug in known numerical values and solve

Given: Pin P is constrained to move along a elliptical ring whose shape is given by $x^2/a^2 + y^2/b^2 = 1$ (where x and y are given in mm). The pin is also constrained to move within a horizontal slot that is moving upward at a constant speed of v.

Find: Determine:

(a) The velocity of pin P at the position corresponding to $y = 6$ mm; and
(b) The acceleration of pin P at the position corresponding to $y = 6$ mm.

Use the following parameters in your analysis: $a = 5$ mm, $b = 10$ mm, $v = 30$ mm/s.

fixed ring
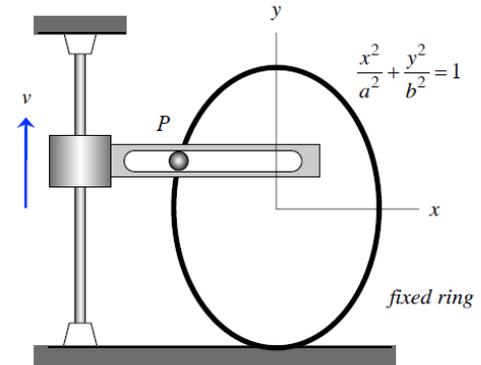
Solution:

a) Acceleration components defined as: $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

Known from part (a): $b^2 x\dot{x} + a^2 y\dot{y} = 0 \rightarrow$ implicitly differentiate and solve for $\ddot{x}$

$$\dot{x}^2 b^2 + x\ddot{x}b^2 + \dot{y}^2 a^2 + y\ddot{y}a^2 = 0$$

Known from part (a): $\ddot{y} = 0 \rightarrow \dot{x}^2 b^2 + x\ddot{x}b^2 + \dot{y}^2 a^2 = 0$

Plug in known numerical values and solve for $\ddot{x}$

Note: once you solve for $\dot{x}, \ddot{x}, \dot{y}, \ddot{y}$, remember to plug back into the vector equations for $\vec{a}, \vec{v}$

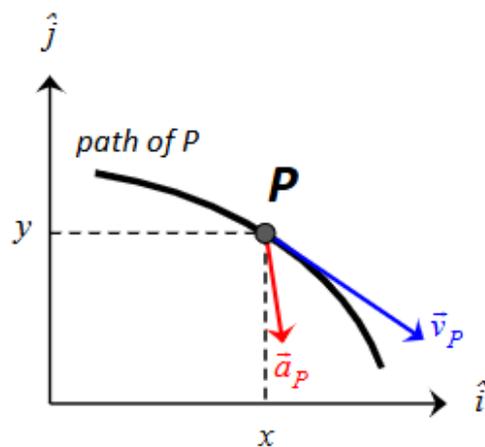# *Summary*: Particle Kinematics – Cartesian Description

1. *PROBLEM*: Motion of a point is described in Cartesian *xy*-coordinates.

2. *FUNDAMENTAL EQUATIONS*:

$$\vec{v}_P = \dot{x}\hat{i} + \dot{y}\hat{j} = velocity\ of\ P$$

$$\vec{a}_P = \ddot{x}\hat{i} + \ddot{y}\hat{j} = acceleration\ of\ P$$
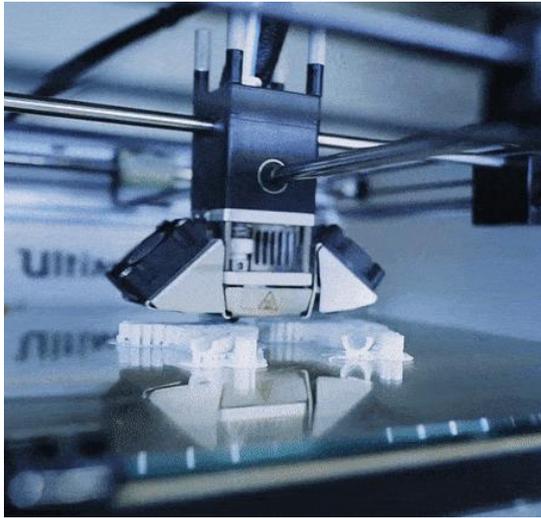
with $\quad \dot{x} = \dfrac{dx}{dt}$, etc.

3. *CHAIN RULE OF DIFFERENTIATION*: Suppose that *y* is given in terms of *x* (instead of time *t*) – how do you find $\dot{y} = dy/dt$??

The chain rule!! $\dot{y} = \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dot{x}\dfrac{dy}{dx}$ (⬅ remember this!)

4. *COMMENT*: The Cartesian description is easy to use, but not as useful as other descriptions. More later…

FDM 3D Printing

CNC milling

Gantry vending machine

ME 274: Basic Mechanics II

Lecture 3: Point Kinematics, Polar Description
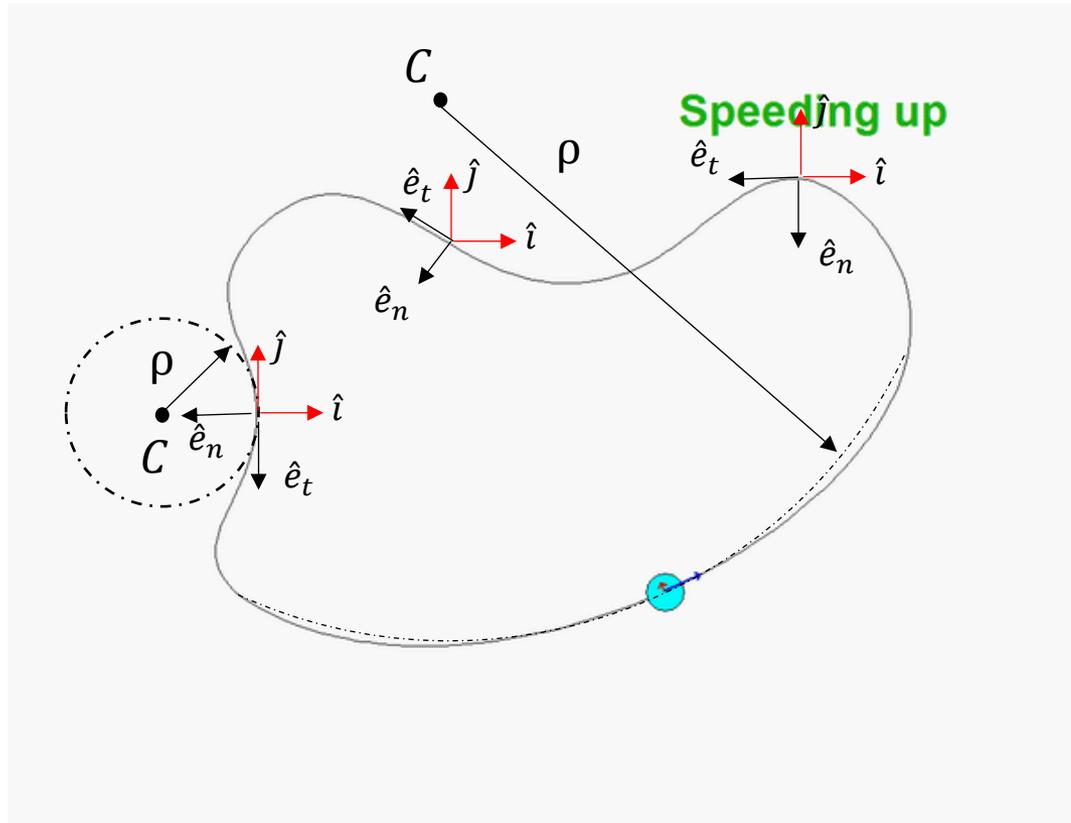
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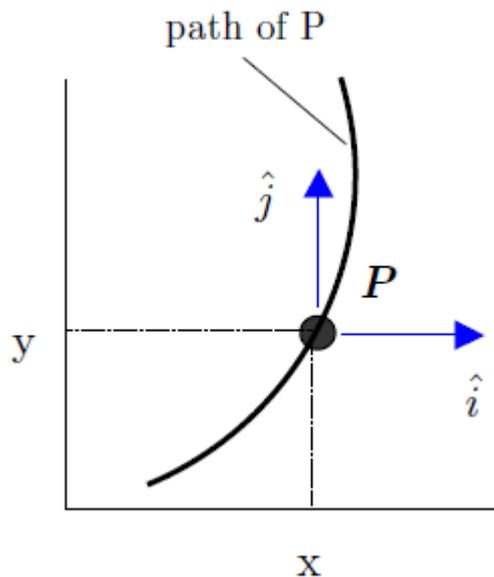
$$\vec{v} = v\hat{e}_t \qquad\qquad \vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

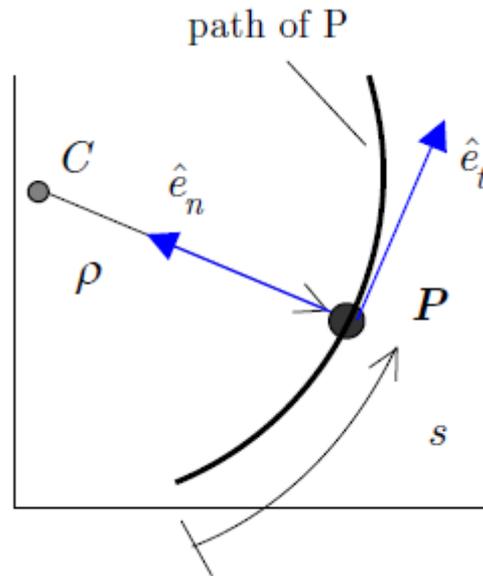# *Kinematic Descriptions*

Cartesian description:
- Position: $\vec{r} = x\hat{i} + y\hat{j}$
- Fixed direction basis vectors $\hat{i}, \hat{j}$
- $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$
- $\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$
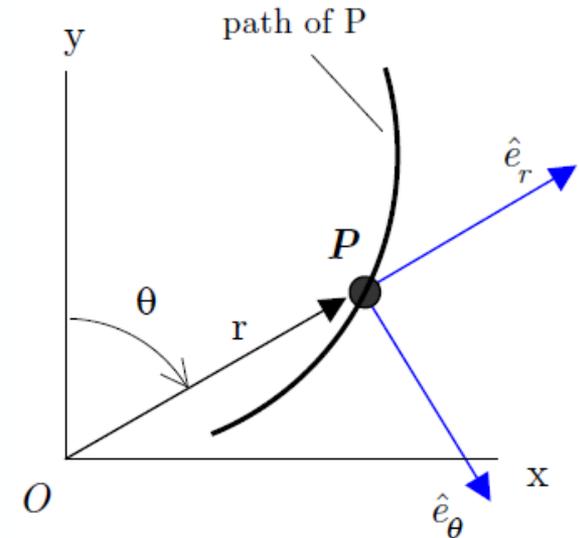

Path description:
- Distance along path: $s(t)$
- $\hat{e}_t, \hat{e}_n$ depend on path geometry
- $\vec{v} = v\hat{e}_t$
- $\vec{a} = \dot{v}\hat{e}_t + \dfrac{v^2}{\rho}\hat{e}_n$

Today

Polar description:
- Position: $\vec{r} = r\hat{e}_r$
- $\hat{e}_r, \hat{e}_\theta$ change with particle motion
- $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$
- $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$
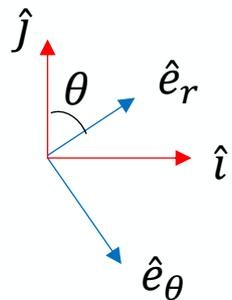
Polar Kinematics

Position: $\vec{r} = r\hat{e}_r$

Velocity: $\vec{v} = \frac{d\vec{r}}{dt}$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

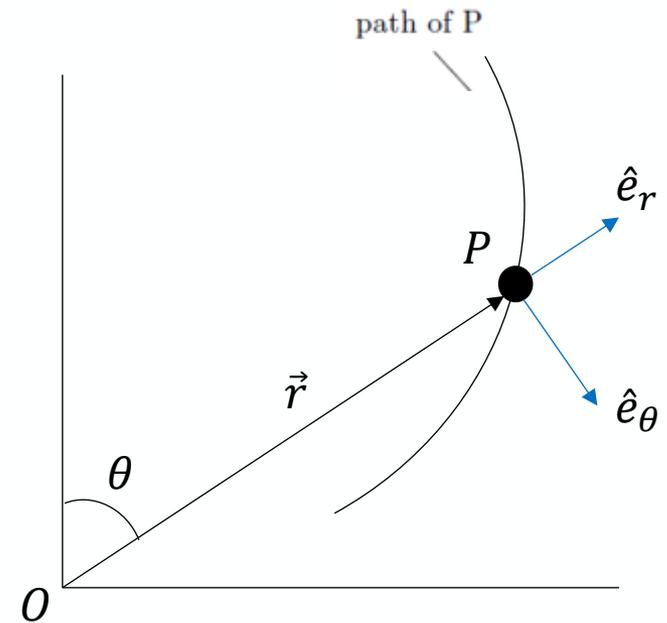$$\frac{d\hat{e}_r}{d\theta}\frac{d\theta}{dt}$$

Writing $\hat{e}_r, \hat{e}_\theta$ in terms of $\hat{i}, \hat{j}$:

$$\hat{e}_r = \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\hat{e}_\theta = \cos\theta\,\hat{i} - \sin\theta\,\hat{j}$$

$$\frac{d\hat{e}_r}{d\theta} = cos\theta\hat{i} - sin\theta\hat{j} = \hat{e}_\theta$$

Velocity: $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$
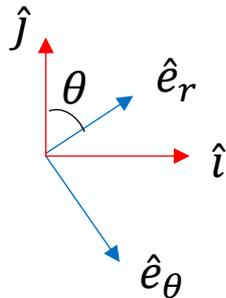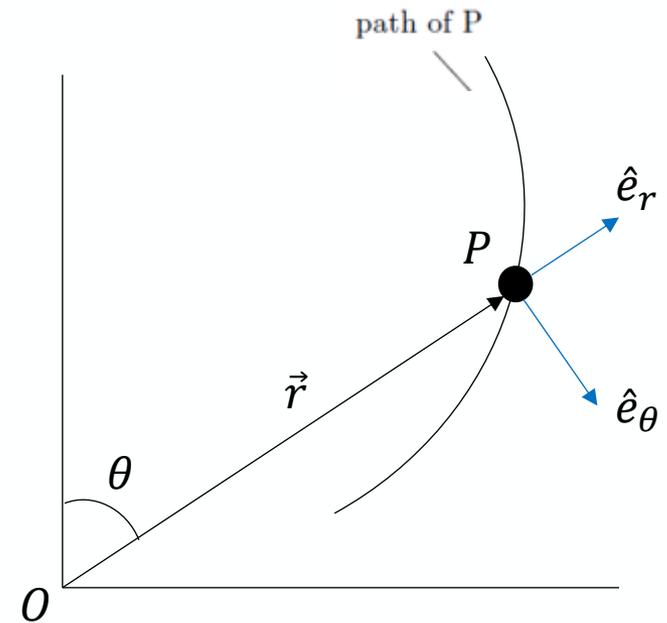
Polar Kinematics

Position: $\vec{r} = r\hat{e}_r$

Velocity: $\vec{v} = \dfrac{d\vec{r}}{dt} = \vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

Acceleration: $\vec{a} = \dfrac{d\vec{v}}{dt}$

$$\boxed{\dot{\theta}\hat{e}_\theta} \qquad = \frac{d}{dt}(\dot{r}\hat{e}_r) + \frac{d}{dt}(r\dot{\theta}\hat{e}_\theta)$$

$$\boxed{\frac{d\hat{e}_\theta}{d\theta}\frac{d\theta}{dt}}$$

$$= \ddot{r}\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$

$$\hat{e}_r = \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$
$$\hat{e}_\theta = \cos\theta\,\hat{i} - \sin\theta\,\hat{j}$$
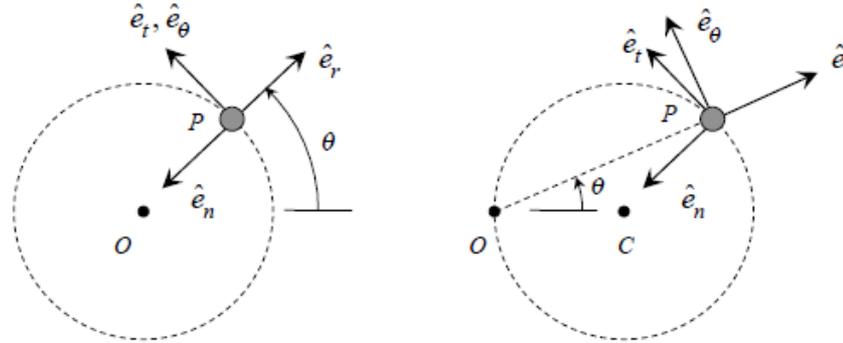
$$\Rightarrow$$

$$\frac{d\hat{e}_r}{d\theta} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j} = \hat{e}_\theta$$
$$\frac{d\hat{e}_\theta}{d\theta} = -\sin\theta\,\hat{i} - \cos\theta\,\hat{j} = -\hat{e}_r$$

$$\boxed{\begin{aligned}\vec{a} &= \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r) \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\end{aligned}}$$
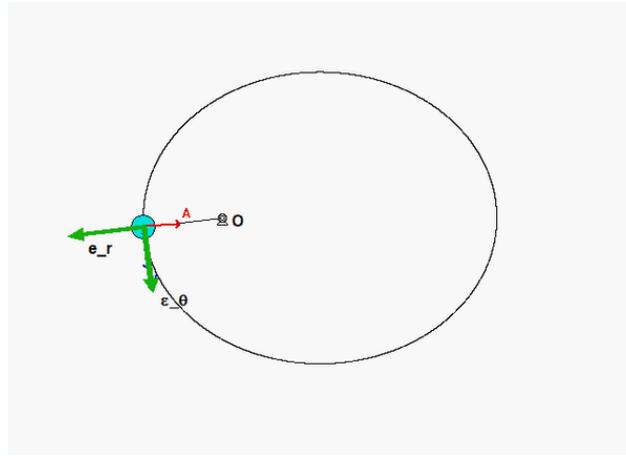
- The values of the **vector components depend** on the location of your **reference point** and your **angle convention**

- When the path of P is given as $r = r(\theta)$, you will need the **chain rule** of differentiation to find time **derivatives in terms of $\dot{\theta}, \ddot{\theta}$**

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta}, \qquad \ddot{r} = \frac{d^2r}{d\theta^2}\dot{\theta} + \frac{dr}{d\theta}\ddot{\theta}$$

- Polar description of motion is useful in cases where there is an **observer of motion**

Particle P travels on a path described by the polar coordinates $r = 2\cos\theta$, where r is given in feet and θ is given in radians. When $\theta = \pi/3$ radians, it is known that $\dot{\theta} = -3$ rad/s and $\ddot{\theta} = 0$.

At this instant

(a) $\dot{r} < 0$

(b) $\dot{r} = 0$

(c) $\dot{r} > 0$

Solution:

$$\dot{r} = \frac{dr}{dt}$$

$$= \frac{dr}{d\theta}\frac{d\theta}{dt}$$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(2cos\theta) = -2sin\theta$$

$$\frac{d\theta}{dt} = \dot{\theta}$$

$$\dot{r} = -2\dot{\theta}sin\theta = -2(-3)\sin\left(\frac{\pi}{3}\right) = 3\sqrt{3}\ ft/s$$
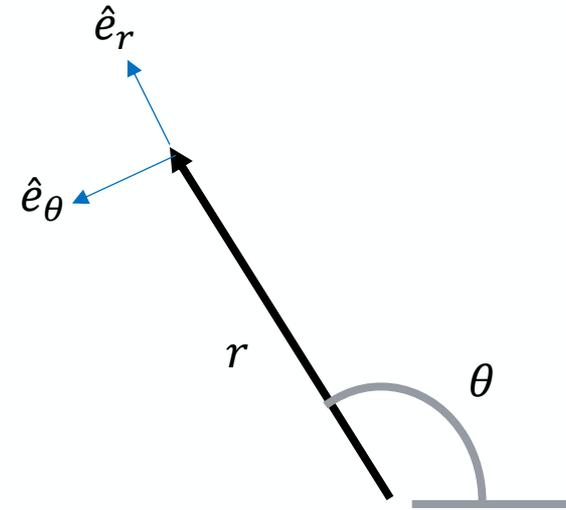
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At this instant

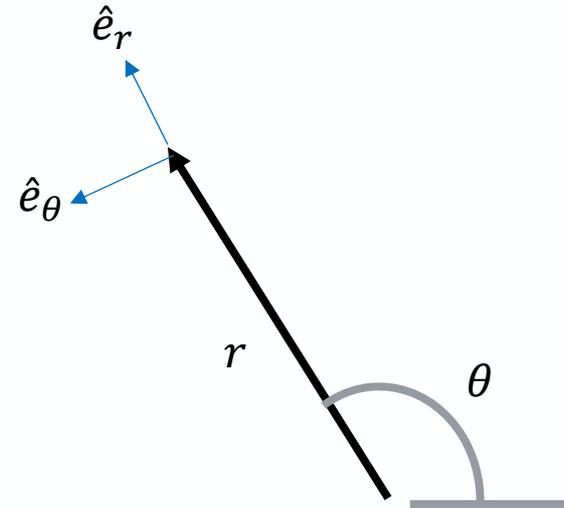(a) $\ddot{r} < 0$

(b) $\ddot{r} = 0$

(c) $\ddot{r} > 0$

$\hat{e}_r$

$\hat{e}_\theta$

r

θ

Solution:

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d}{dt}(-2\dot{\theta}\sin\theta)$$

$$= -2\left(\ddot{\theta}\sin\theta + \dot{\theta}\frac{d}{dt}(\sin\theta)\right)$$

$$\frac{d}{dt}(\sin\theta) = \cos\theta\,\dot{\theta}$$

$$\ddot{r} = -2\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)$$

$$\Rightarrow -2\left((0)\sin\left(\frac{\pi}{3}\right) + (-3)^2\cos\left(\frac{\pi}{3}\right)\right) = -9 \text{ ft/s}^2$$

Question C1.8

A particle P travels on a path given in terms of polar coordinates as: $r = \cos(3\theta)$, where r is given in feet, θ is given in radians and $\dot{\theta} = 2$ rad/s $= constant$. Determine the magnitude of the acceleration vector of P when $\theta = \pi/3$.

Solution:

Remember $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\overset{0}{\ddot{\theta}} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

Calculate derivatives of r:

$$r = \cos(3\theta) = \cos\left(\frac{3\pi}{3}\right) = -1\,ft$$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \dot{\theta}(-3sin3\theta) = 0\,ft/s$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = -3(\ddot{\theta}sin3\theta + 3\dot{\theta}^2cos3\theta) = -3((0)\sin(\pi) + 3(2)^2\cos(\pi)) = 36ft/s^2$$

Plug known values into expression for $\vec{a}$

$$\vec{a} = (36 - (-1)(2)^2)\hat{e}_r + ((-1)(0) + 2(0)(2))\hat{e}_\theta$$

$$\vec{a} = 40\hat{e}_r\,ft/s^2$$

Given: Particle P travels along an elliptical path shown with $\dot{\theta} = constant$

Find: for the position of P corresponding to $\theta = \frac{\pi}{2}$

a) Determine $\dot{R}$ and $\ddot{R}$. Use b = 2m, $\dot{\theta} = 3\frac{rad}{s}$

Solution:

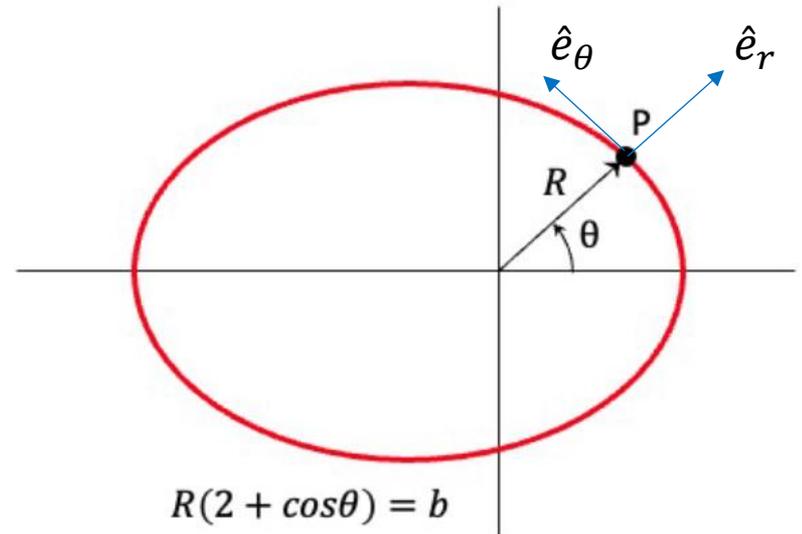Path defined by: $R(2 + cos\theta) = b$

At $\theta = \frac{\pi}{2}$,

$$R = \frac{b}{(2 + cos\theta)} = \frac{2}{2 + cos\left(\frac{\pi}{2}\right)} = 1m$$

Solve for $\dot{R}$:

$$\frac{d}{dt}: \dot{R}(2 + cos\theta) + R\left(-\dot{\theta}sin\theta\right) = 0$$

Rearranging: $\dot{R} = \frac{R\dot{\theta}sin\theta}{2+cos\theta}$

Plug in: $\dot{R} = \frac{(1)(3)\sin\left(\frac{\pi}{2}\right)}{2+\cos\left(\frac{\pi}{2}\right)} = \frac{3}{2}m/s$

$\hat{e}_\theta$ $\hat{e}_r$

P

R

θ

$R(2 + cos\theta) = b$

Additional lecture Example 1.3

Given: Particle P travels along an elliptical path shown with $\dot{\theta} = constant$

Find: for the position of P corresponding to $\theta = \frac{\pi}{2}$

a) Determine $\dot{R}$ and $\ddot{R}$. Use b = 2m, $\dot{\theta} = 3\frac{rad}{s}$

Solution:
Path defined by: $R(2 + cos\theta) = b$

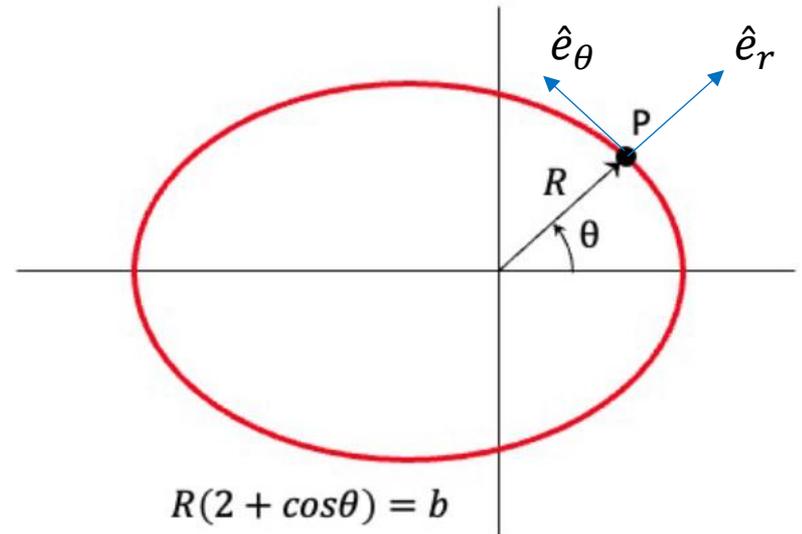At $\theta = \frac{\pi}{2}$, $R = 1m, \dot{R} = \frac{3}{2}m/s$

Path eqn. derivatives:

$$\frac{d}{dt}: \dot{R}(2 + cos\theta) + R(-\dot{\theta}sin\theta) = 0$$

$$\frac{d^2}{dt^2}: \ddot{R}(2 + cos\theta) + \dot{R}(-\dot{\theta}sin\theta) + \dot{R}(-\dot{\theta}sin\theta)$$
$$+ R(-\ddot{\theta}sin\theta) + R(-\dot{\theta}^2 cos\theta) = 0$$

Simplify: $2\ddot{R} - 2\dot{R}\dot{\theta} = 0$

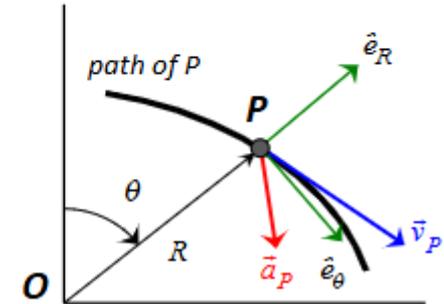Rearrange and solve: $\ddot{R} = \dot{R}\dot{\theta} = \frac{9}{2}m/s^2$

$R(2 + cos\theta) = b$

Summary: Particle Kinematics – Polar Description

1. *PROBLEM*: Motion of a point described in polar coordinates, R and θ.

2. *FUNDAMENTAL EQUATIONS*:

$$\vec{v}_P = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta = \textit{velocity of } P$$

$$\vec{a}_P = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{e}_\theta = \textit{acceleration of } P$$

where $\hat{e}_R$ and $\hat{e}_\theta$ are the radial and transverse unit vectors.

3. *OBSERVATIONS: In regard to the polar description kinematics, we see*

- <u>You</u> are free to choose the observation point O.

- $\hat{e}_R$ <u>*always*</u> points OUTWARD from O to P. $\hat{e}_\theta$ is perpendicular to $\hat{e}_R$ and in direction of increasing θ.

- Polar description is useful for problems with observers or rotations about fixed axes.

- Do not confuse the unit radial vector $\hat{e}_R$ with the unit normal vector $\hat{e}_n$.

ME 274: Basic Mechanics II

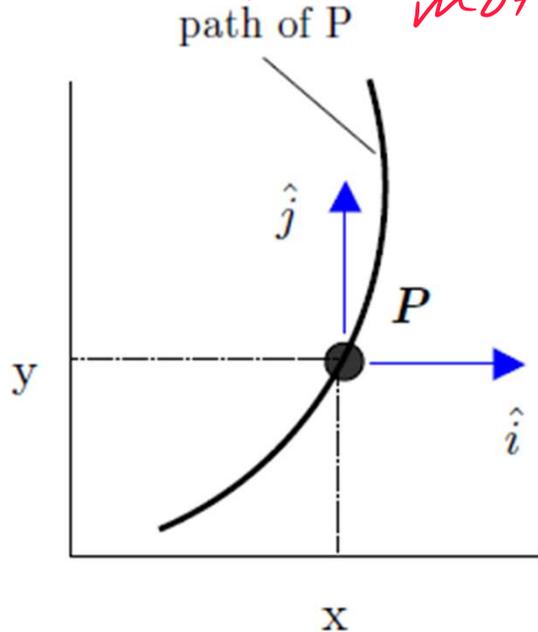Lecture 4: Point Kinematics, Joint Description

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Kinematic Descriptions

Cartesian description:
- Position: $\vec{r} = x\hat{\imath} + y\hat{\jmath}$
- Fixed direction basis vectors $\hat{\imath}, \hat{\jmath}$
- $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$
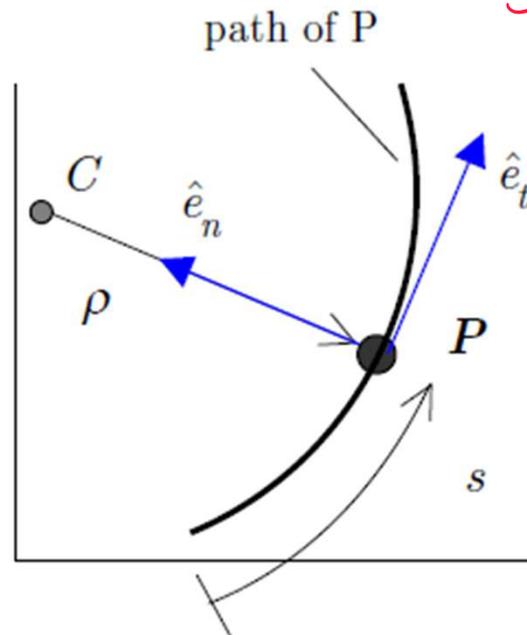- $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

good for horizontal & vertical motion

Path description:
- Distance along path: $s(t)$
- $\hat{e}_t, \hat{e}_n$ depend on path geometry *← move & rotate*
- $\vec{v} = v\hat{e}_t$
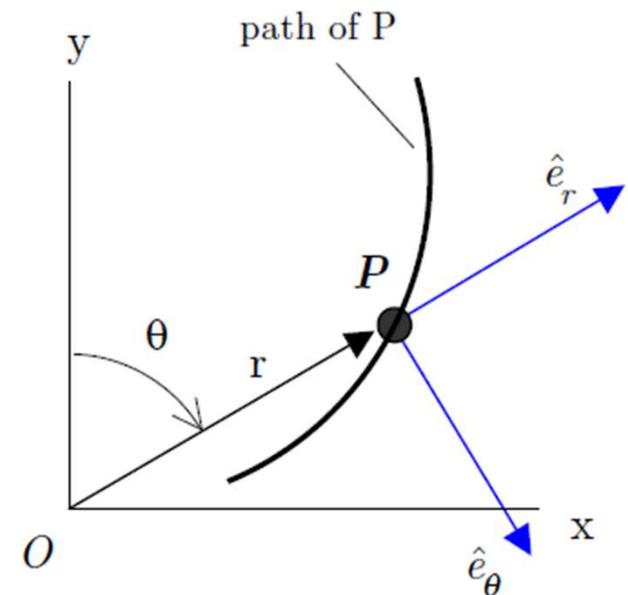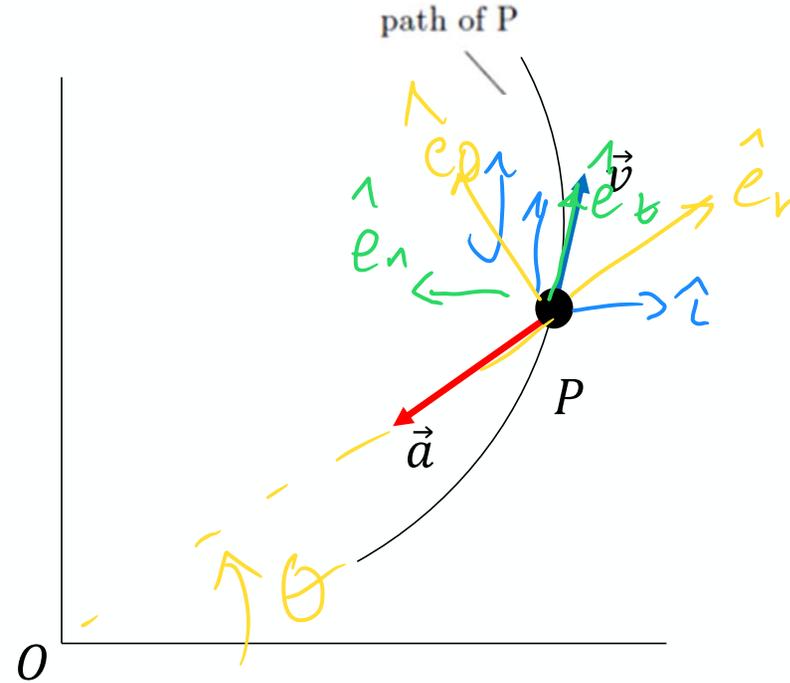- $\vec{a} = \dot{v}\hat{e}_t + \dfrac{v^2}{\rho}\hat{e}_n$

good for speed/turning

Polar description:
- Position: $\vec{r} = r\hat{e}_r$
- $\hat{e}_r, \hat{e}_\theta$ change with particle motion
- $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$
- $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

good for central observer

Different Descriptions – Same Motion

Velocity: $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = v\,\hat{e}_t = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

Acceleration: $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = \dot{v}\hat{e}_t + \dfrac{v^2}{\rho}\hat{e}_n = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

convert between descriptions

Joint Description: Combined Usage of Kinematic Descriptions

Example: Car moving around a turn

The motion of a car is described in Cartesian coordinates

Questions we care about:

- Is the car speeding up or slowing down?

  $\dot{V}, \dot{V}$

- How hard is the car turning?

  $\dfrac{V^2}{\rho}$

path descriptions

GPS coords:
$(39.2, -84.7)$

ρ

C

y

x

Example: Cheering for a runner

Tracking app gives path information: $\rightarrow S(t)$ } path
- Distance along course
- Current speed and split times

  $\checkmark$ $\checkmark$

Questions we care about:
- Where to stand to cheer – runner's absolute position at a time (x, y) $\leftarrow$ cartesian

$r, \theta \leftarrow$ polar

$S(t)$

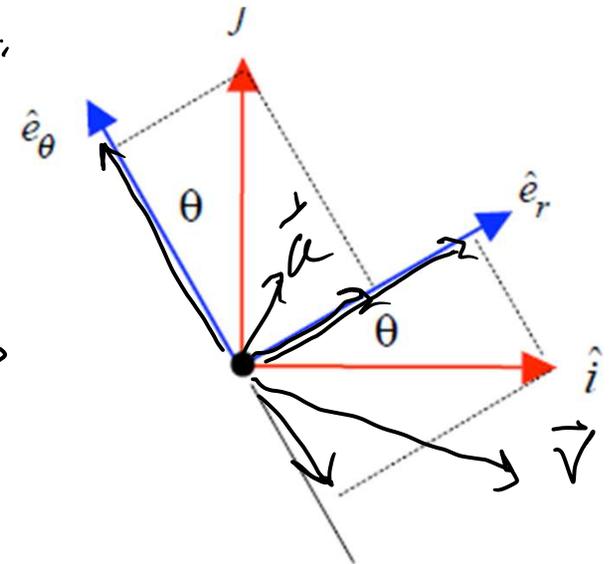θ

r

National Park

Thomas Rd

MOTIVATING EXAMPLE:

Suppose that the velocity and acceleration of a particle are known in terms of their polar coordinates as: $\vec{v} = (10\hat{e}_r - 20\hat{e}_\theta)$ m/s and $\vec{a} = (3\hat{e}_r + 2\hat{e}_\theta)$ m/s^2, where the orientation of the polar unit vectors are shown below relative to a set of Cartesian vectors. From this we want to find the Cartesian components of velocity and acceleration when $\theta = 36.87°$

3-4-5 tri.

Solution:

Write **target** basis vectors **in terms of given** basis vectors

$$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta = 0.8\,\hat{e}_r - 0.6\,\hat{e}_\theta$$

$$\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = 0.6\,\hat{e}_r + 0.8\,\hat{e}_\theta$$

Calculate velocity components by finding projection of $\vec{v}$ onto $\hat{i}, \hat{j}$

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} \qquad \vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$$

$$\dot{x} = \vec{v}\cdot\hat{i} = (10\hat{e}_r - 20\hat{e}_\theta)\cdot(0.8\hat{e}_r - 0.6\hat{e}_\theta)$$

$$\ddot{x} = \vec{a}\cdot\hat{i}$$
$$= 8 + 12 = 20\ \text{m/s}$$

$$\dot{y} = \vec{v}\cdot\hat{j} = (3e_r + 2e_\theta)\cdot(0.6\hat{e}_r + 0.8\hat{e}_\theta)$$

$$\ddot{y} = \vec{a}\cdot\hat{j}$$
$$= 1.8 + 1.6 = 3.4\ \text{m/s}$$

$$\vec{v} = 20\hat{i} + 3.4\hat{j}$$

Projection of $\vec{b}$ onto $\vec{a} = |\vec{b}||\hat{e}_a|\cos\theta = \vec{b}\bullet\hat{e}_a$

$\vec{b}$

θ

$\vec{a}$

projection of $\vec{b}$ onto $\vec{a}$

Component extraction via projection

Cartesian:
$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$
$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$

$$\dot{x} = \vec{v} \bullet \hat{\imath} \qquad \dot{y} = \vec{v} \bullet \hat{\jmath}$$
$$\ddot{x} = \vec{a} \bullet \hat{\imath} \qquad \ddot{y} = \vec{a} \bullet \hat{\jmath}$$

Polar:
$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\dot{r} = \vec{v} \bullet \hat{e}_r \qquad r\dot{\theta} = \vec{v} \bullet \hat{e}_\theta$$
$$\ddot{r} - r\dot{\theta}^2 = \vec{a} \bullet \hat{e}_r \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = \vec{a} \bullet \hat{e}_\theta$$

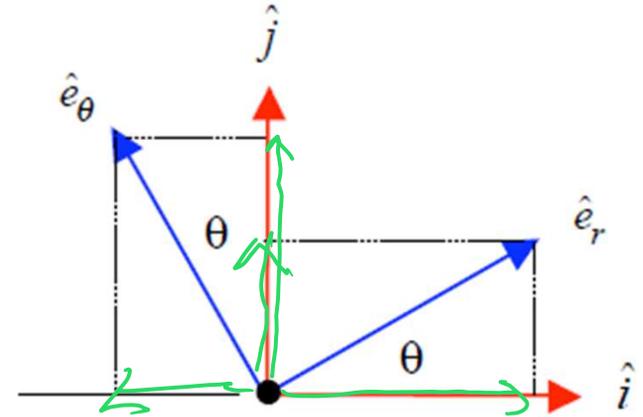1) target basis in terms of given
2) take projection

MOTIVATING EXAMPLE:

Suppose that the velocity and acceleration of a particle are known in terms of their polar coordinates as: $\vec{v} = (10\hat{e}_r - 20\hat{e}_\theta)$ m/s and $\vec{a} = (3\hat{e}_r + 2\hat{e}_\theta)$ m/s^2, where the orientation of the polar unit vectors are shown below relative to a set of Cartesian vectors. From this we want to find the Cartesian components of velocity and acceleration when $\theta = 36.87°$.

Solution:

Write **given** basis vectors **in terms of target** basis vectors

$$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\,\hat{j} = 0.8\hat{i} + 0.6\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\,\hat{j} = -0.6\hat{i} + 0.8\hat{j}$$

Directly substitute into expression for $\vec{v}$:

$$\vec{V} = (10\hat{e}_r - 20\hat{e}_\theta) = 10(0.8\hat{i} + 0.6\hat{j}) - 20(-0.6\hat{i} + 0.8\hat{j})$$

$$= 8\hat{i} + 6\hat{j} + 12\hat{i} - 16\hat{j} = 20\hat{i} - 10\hat{j} \text{ m/s}$$

$$\vec{a} (3\hat{e}_r + 2\hat{e}_\theta) = 3(0.8\hat{i} + 0.6\hat{j}) + 2(-0.6\hat{i} + 0.8\hat{j})$$

$$= 1.2\hat{i} + 3.4\hat{j} \text{ m/s}^2$$

Example 2

MOTIVATING EXAMPLE:

Suppose the velocity and acceleration of a particle are known in terms of their Cartesian components as: $\vec{v} = (30\hat{i} - 40\hat{j})$ m/s and $\vec{a} = (-10\hat{j})$ m/s^2. From this, we want to find the speed v, rate of change of speed $\dot{v}$ and the radius of curvature ρ of the path of the particle (path description variables).

$$v = |\vec{v}| = \sqrt{30^2 + (-40)^2} = \boxed{50 \text{ m/s}}$$

$$\rightarrow \vec{a} = \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n$$

$$\vec{v} = v\,\hat{e}_t \rightarrow \hat{e}_t = \frac{\vec{v}}{v} = \frac{30\hat{i} - 40\hat{j}}{50} = 0.6\hat{i} - 0.8\hat{j}$$

$$\dot{v} = \vec{a}\cdot\hat{e}_t = (-10\hat{j})\cdot(0.6\hat{i} - 0.8\hat{j}) = 8 \text{ m/s}^2$$

$$|a|^2 = \dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2 \Rightarrow \left(\frac{v^2}{\rho}\right)^2 = |a|^2 - \dot{v}^2$$

$$\rightarrow \rho = \frac{v^2}{\sqrt{|a|^2 - \dot{v}^2}} = \frac{50^2}{\sqrt{10^2 - 8^2}}$$

$$\boxed{= \frac{2500}{6} \text{ m}}$$

Example 1.C.2

Given: Pin P is constrained to move in the slotted guides that move at right angles to one another. At the instant shown, guide A moves to the right with a speed of v_A, a speed that is changing at a rate of $\dot{v}_A$. At the same time, B is moving downward with a speed of v_B with a rate of change of speed of $\dot{v}_B$.

↳ given cartesian

Find:

(a) The rate of change of speed of P at this instant; and

(b) The radius of curvature ρ of the path followed by P at this instant.

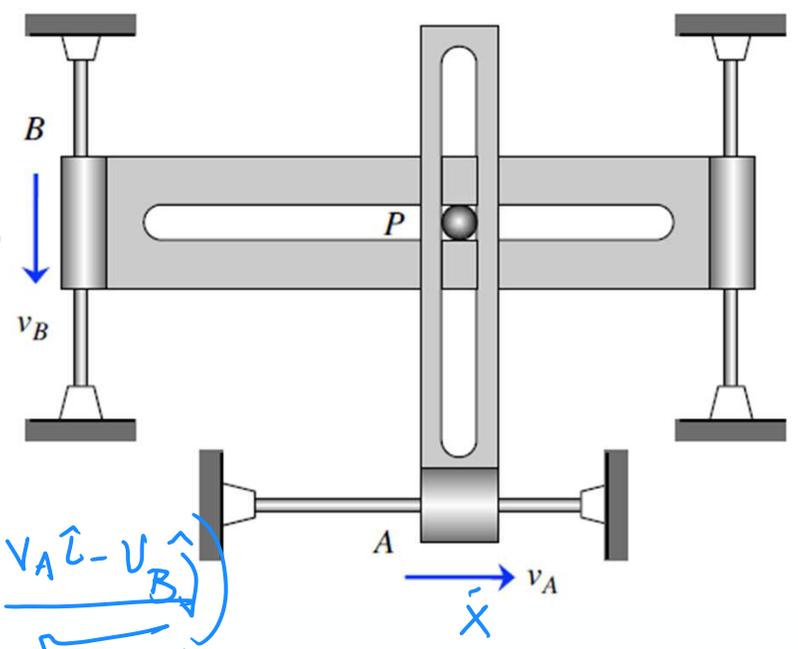$\dot{v}$ ← path coord.

Use the following parameters: $v_A = 0.2$ m/s, $v_B = 0.15$ m/s, $\dot{v}_A = 0.75$ m/s^2 and $\dot{v}_B = 0$.

$$\vec{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath} = V_A\,\hat{\imath} - V_B\,\hat{\jmath}$$

$$\vec{a} = \ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath} = \dot{V}_A\,\hat{\imath} - \dot{V}_B\,\hat{\jmath}$$

$$\vec{v} = v\,\hat{e}_t \quad\rightarrow\quad \boxed{\hat{e}_t = \frac{\vec{v}}{v} = \frac{V_A\,\hat{\imath} - V_B\,\hat{\jmath}}{\sqrt{V_A^2 + V_B^2}}}$$

a) $\vec{a} = \dot{v}\,\hat{e}_t + \dfrac{v^2}{\rho}\,\hat{e}_n$

$$\dot{v} = \vec{a}\cdot\hat{e}_t = \left(\dot{V}_A\,\hat{\imath} - \dot{V}_B\,\hat{\jmath}\right)\cdot\left(\frac{V_A\,\hat{\imath} - V_B\,\hat{\jmath}}{\sqrt{V_A^2 + V_B^2}}\right)$$

$$= \frac{1}{\sqrt{V_A^2 + V_B^2}}\left(\dot{V}_A V_A - \dot{V}_B V_B\right) \quad\leftarrow \text{sub \& solve}$$

Example 1.C.2.cont

b) $|a|^2 = \dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2 \rightarrow \rho = \dfrac{v^2}{\sqrt{|a|^2 - \dot{v}^2}}$

$|a|^2 = \dot{v}_A^2 + \dot{v}_B^2$

$\rightarrow$ $\boxed{\rho = \dfrac{v^2}{\sqrt{\dot{v}_A^2 + \dot{v}_B^2 - \dot{v}^2}} \Big\} \text{ from part A}}$

Example 1.C.4

$\ddot{v}_p = 0$

Given: At the bottom of a loop, an airplane P has a constant speed of v_P with the radius of curvature for the aircraft being ρ. The airplane is at a radial distance of r and at an angle of θ from a radar tracking station at O.

Find: Determine numerical values for $\ddot{r}$ and $\ddot{\theta}$ at this instant in time.

Use the following: $v_P = 75$ m/s, $\rho = 3000$ m, $r = 1000$ m and $\theta = 36.87°$.

target: $\hat{e}_r$, $\hat{e}_\theta$ given: $\hat{e}_t$, $\hat{e}_n$

from geometry:

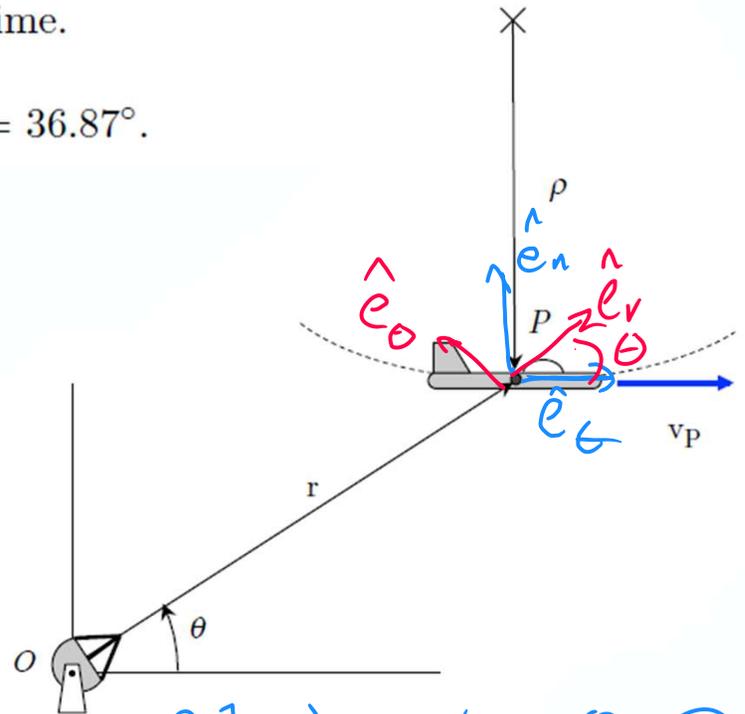$$\hat{e}_r = \cos\theta\, \hat{e}_t + \sin\theta\, \hat{e}_n$$

$$\hat{e}_\theta = -\sin\theta\, \hat{e}_t + \cos\theta\, \hat{e}_n$$

$$\vec{V} = V\hat{e}_t = \underline{\dot{r}\hat{e}_r} + r\dot{\theta}\hat{e}_\theta$$

$$\dot{r} = \vec{V}\cdot\hat{e}_r = (V\hat{e}_t)\cdot(\cos\theta\,\hat{e}_t + \sin\theta\,\hat{e}_n) = V\cos\theta$$

$$r\dot{\theta} = \vec{V}\cdot\hat{e}_\theta = (V\hat{e}_t)\cdot(-\sin\theta\hat{e}_t + \cos\theta\,\hat{e}_n) = -V\sin\theta$$

given θ, r sub & solve

$\dot{r}, \dot{\theta}$

Example 1.C.4.cont

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\underset{=0}{\quad} \qquad \underset{\text{given}}{\quad}$$

$$\vec{a} = \frac{v^2}{\rho}\hat{e}_n$$

$$(\ddot{r} - r\dot{\theta}^2) = \vec{a} \cdot \hat{e}_r = \left(\frac{v^2}{\rho}\hat{e}_n\right) \cdot \left(\cos\theta\,\hat{e}_t + \sin\theta\,\hat{e}_n\right)$$

$$(\ddot{r} - r\dot{\theta}^2) = \frac{v^2}{\rho}\sin\theta$$

$$(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \vec{a} \cdot \hat{e}_\theta = \left(\frac{v^2}{\rho}\hat{e}_n\right)\left(-\sin\theta\,\hat{e}_t + \cos\theta\,\hat{e}_n\right)$$

$$(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \frac{v^2}{\rho}\cos\theta$$

2 eqn, 2 unknowns → sub & solve

for $\ddot{r}, \ddot{\theta}$

ME 274: Basic Mechanics II

Lecture 2: Point Kinematics, Path Description

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Housekeeping

- Brightspace/Gradescope access?
- Lecture notes: https://www.purdue.edu/freeform/me274/quizzes-2/
- Hw 1 due **Tonight, 11:59 pm –**submit to Gradescope
- Hw 2 released today

Describing motion: particle kinematics – for all coordinate systems:

- Position, $\vec{r}(t)$
- Velocity, $\vec{v}(t)$ → $\vec{v} = \dfrac{d\vec{r}}{dt}$
- Acceleration, $\vec{a}(t)$ → $\vec{a} = \dfrac{d^2\vec{r}}{dt^2}$

Cartesian description: The path of particle P is expressed in terms of **x** and **y** components
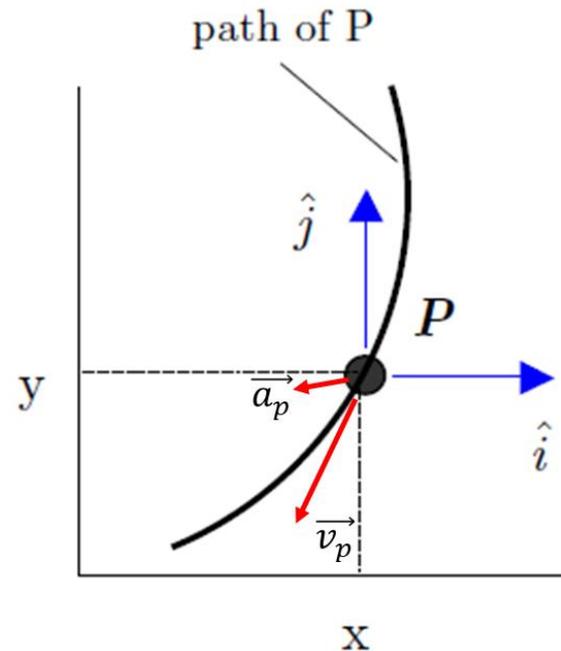
- $\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath}$
- $\vec{v} = \dfrac{d\vec{r}}{dt} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$
- $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

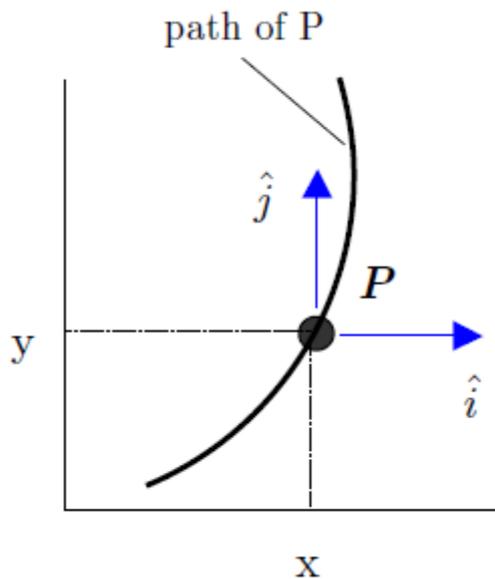Where $\hat{\imath}, \hat{\jmath}$ are **constant** basis vectors

Differentiation requires the **chain rule:**

if $x = x(t)$ and $y = f(x)$:

- $\dot{x} = \dfrac{dx}{dt}$,
- $\dot{y} = \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dot{x}\dfrac{dy}{dx}$
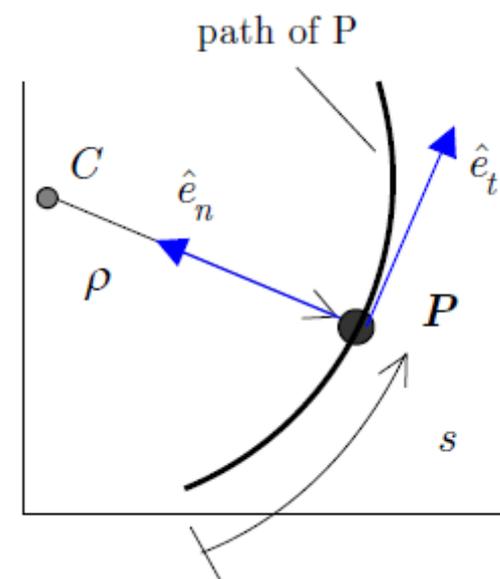
Choosing a Kinematic Description

Cartesian description:
- Position: $x(t), y(t)$
- Fixed direction basis vectors $\hat{\imath}, \hat{\jmath}$

Works best when:
- Motion aligns with horizontal/vertical directions
- Geometry is naturally written in x and y

Path description:
- Position defined by s(t)
- Tangent and normal basis vectors $\hat{e}_t, \hat{e}_n$ depend on path geometry

Works best when:
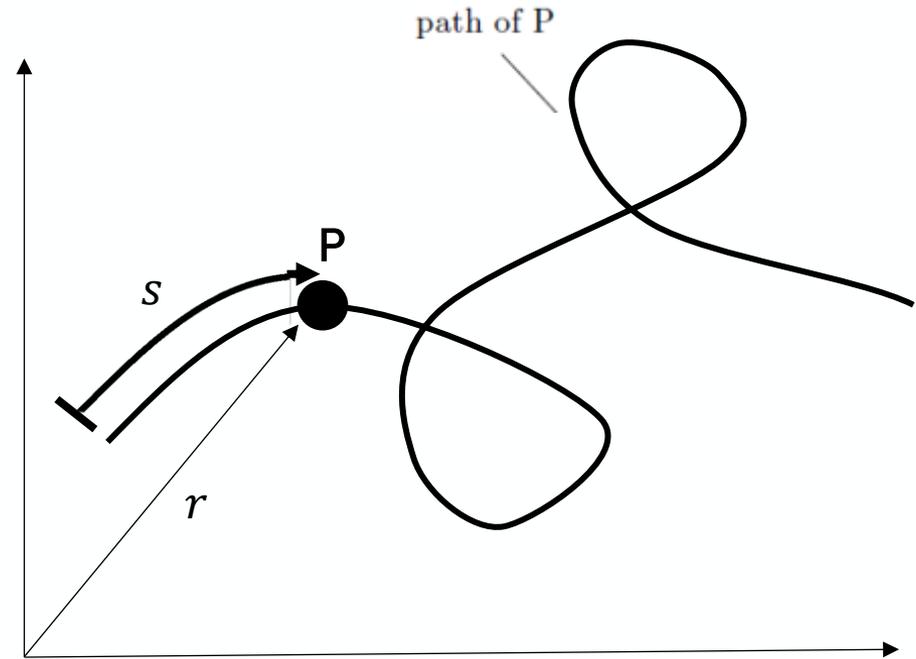- Motion constrained to a known path
- We care about speed and curvature

Motivating Example: Roller Coasters

If a particle, P, moves along a known path, the position, $\vec{r}$, is **known in terms of the distance**, s, the particle has moved **along the path**.

- $s = s(t)$ ← Scalar (arc length)
- $\vec{r} = \vec{r}(s)$ ← Vector

path of P

P

s

r

Remember for all descriptions

$$\vec{v} = \frac{d\vec{r}}{dt}$$

To solve for velocity in the path description, we use the **chain rule of differentiation:**

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$

So what do these mean physically?

Velocity vector: $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{ds}\dfrac{ds}{dt}$

$\dfrac{ds}{dt} \rightarrow$ the rate of change of distance with respect to time = **speed, v**

$$\frac{d\vec{r}}{ds} = \lim_{\Delta s \to 0}\left(\frac{\Delta\vec{r}}{\Delta s}\right)$$

- $\Delta\vec{r} \rightarrow$ change in particle position
- $|\Delta\vec{r}| \rightarrow$ chord length
- $\Delta s \rightarrow$ arc length

Magnitude of $\dfrac{d\vec{r}}{ds}$:
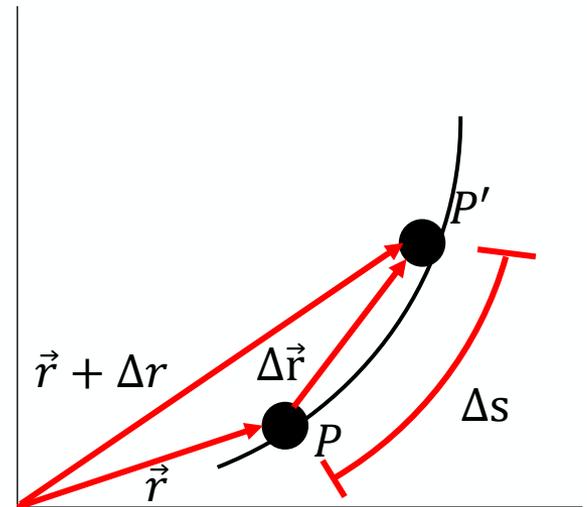
As $\Delta s \to 0$, $|\Delta\vec{r}| \to \Delta s$,

meaning $\dfrac{|\Delta\vec{r}|}{\Delta s} = 1$, making $\dfrac{d\vec{r}}{ds}$ a **unit vector**

Direction of $\dfrac{d\vec{r}}{ds}$:

As $\Delta s \to 0$, $\Delta\vec{r}$ becomes **tangent** to the path

Therefore: $\dfrac{d\vec{r}}{ds}$ **is a tangent unit vector, $\hat{e}_t$,** and $\vec{v} = \dfrac{d\vec{r}}{ds}\dfrac{ds}{dt} = v\hat{e}_t$

Acceleration and curvature

Differentiate wrt time:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\,\hat{e}_t)$$

Use the product rule:

$$\vec{a} = \dot{v}\hat{e}_t + v\frac{d\hat{e}_t}{dt}$$

Use the chain rule:

$$\frac{d\hat{e}_t}{dt} = \frac{d\hat{e}_t}{ds}\frac{ds}{dt} \quad \leftarrow \text{remember } \frac{ds}{dt} = v$$

Using chain rule and path geometry:

$$\frac{d\hat{e}_t}{ds} = \frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds}$$

Therefore:

$$\vec{a} = \dot{v}\hat{e}_t + v^2\frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds} \quad \leftarrow \text{we can simplify further}$$

Acceleration and curvature

$$\vec{a} = \dot{v}\hat{e}_t + v^2 \frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds}$$

$\hat{e}_t$ and $\hat{e}_n$ can be related to $\hat{\imath}$ and $\hat{\jmath}$ as:

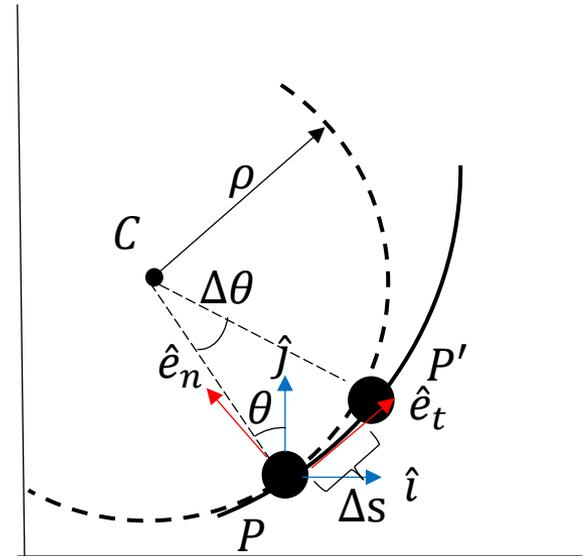$$\hat{e}_n = -\sin\theta\,\hat{\imath} + \cos\theta\hat{\jmath}$$
$$\hat{e}_t = cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath}$$

Therefore:

$$\frac{d\hat{e}_t}{d\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\hat{\jmath} = \hat{e}_n$$

From path geometry:

$$ds = \rho d\theta \quad \Rightarrow \quad \frac{ds}{d\theta} = \frac{1}{\rho}$$
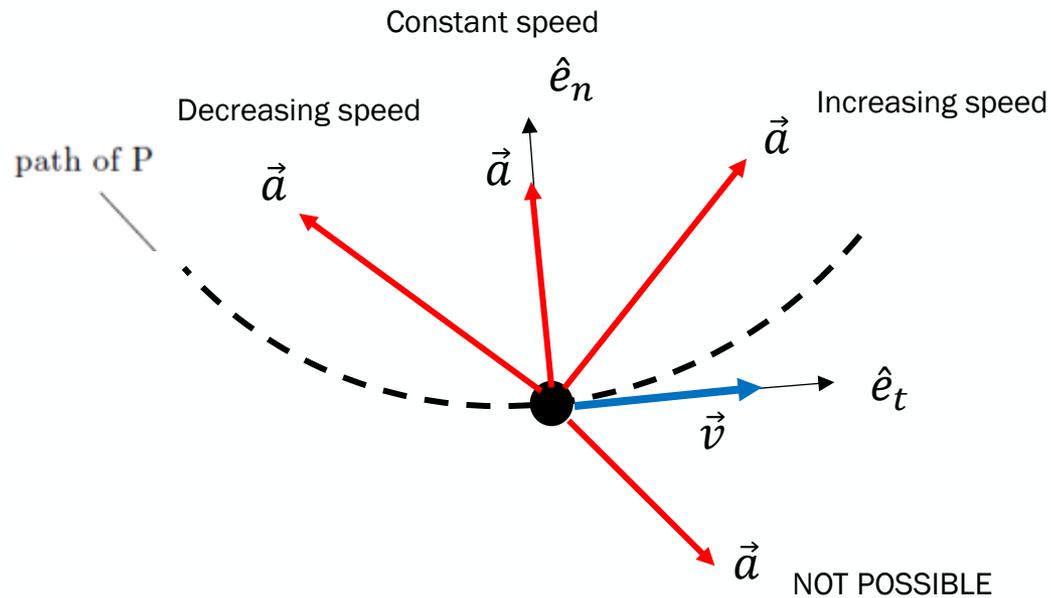

C = "center of curvature"

ρ = "radius of curvature"

Combining terms:

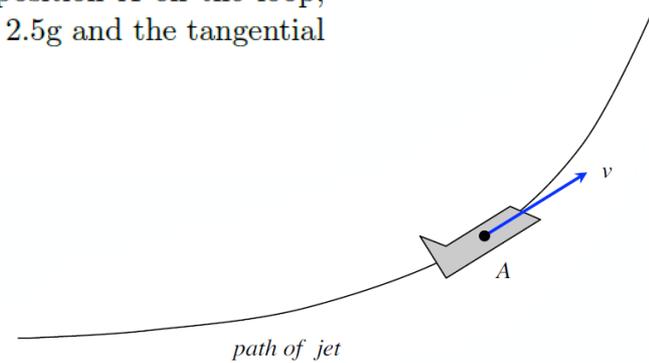$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

- Velocity: $\vec{v} = v\hat{e}_t$

- Acceleration: $\vec{a} = \dot{v}\hat{e}_t + \dfrac{v^2}{\rho}\hat{e}_n$
  - $\hat{e}_n$ is always pointed inward
  - $\dot{v} > 0 \rightarrow$ particle is speeding up
  - $\dot{v} < 0 \rightarrow$ particle is slowing down
  - $\dot{v} = 0 \rightarrow$ particle speed is constant

Given: A jet is flying on the path shown below with a speed of v. At position A on the loop, the speed of the jet is $v = 600$ km/hr, the magnitude of the acceleration is 2.5g and the tangential component of acceleration is $a_t = 5$ m/s^2.

Find: The radius of curvature of the path of the jet at A.

path of jet

A

v

Solution:

Remember $\vec{a} = \dot{v}\hat{e}_t + \dfrac{v^2}{\rho}\hat{e}_n$

We are given $|\vec{a}|$, $a_t = \dot{v}$, and v

Find an expression for $|\vec{a}|$:

$$|\vec{a}| = \sqrt{\dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2}$$

Rearrange and solve for ρ:

$$|\vec{a}|^2 = \dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2$$

$$\frac{v^2}{\rho} = \sqrt{|\vec{a}|^2 - \dot{v}^2}$$ ⟵ Has to be positive

$$\rho = \frac{v^2}{\sqrt{|\vec{a}|^2 - \dot{v}^2}}$$ ⟵ Sub in known values

Given: An automobile P is entering a freeway along a "clothoid-shaped" entrance ramp whose radius of curvature ρ is given by $\rho = (a + bs)^{-1}$, where a and b are constants, and s is the distance traveled along the entrance ramp. The speed of P is known as a function of position s on the entrance ramp to be: $v(s) = c + ds$, where c and d are constants.

Use the following parameters in your work: $a = 0.005/\text{ft}$, $b = 1 \times 10^{-5}/\text{ft}^2$, $c = 25$ ft/s and $d = 0.25/\text{s}$.

Find:

(a) Determine the velocity and acceleration vectors for P. Express these vectors in terms of their path coordinates, and in terms of, at most: s, a, b, c and d.

Solution:
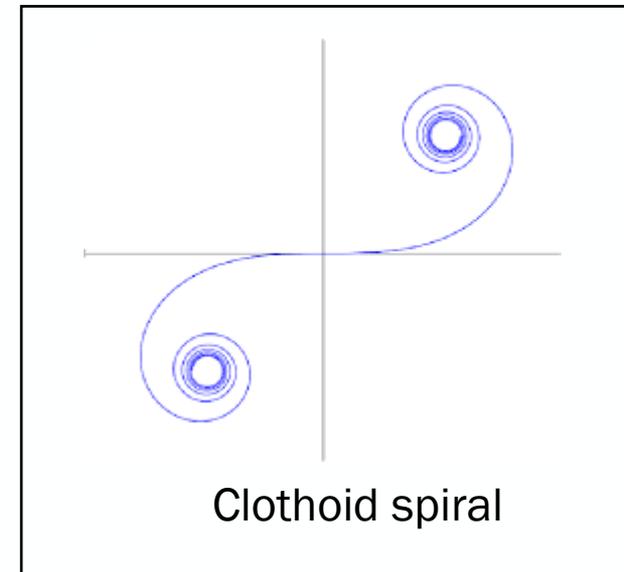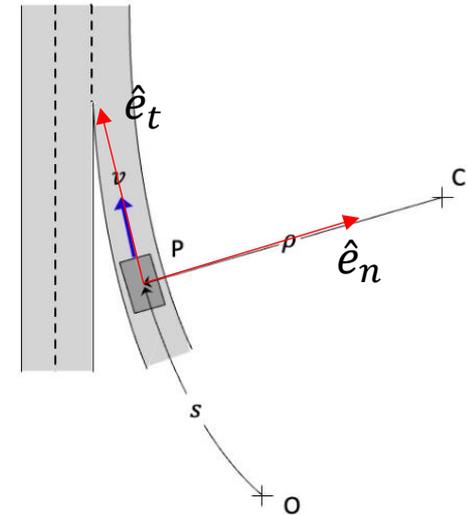
Velocity$\rightarrow \vec{v} = v\hat{e}_t = (c + ds)\hat{e}_t$

Acceleration $\rightarrow \vec{a} = \dot{v}\hat{e}_t + \dfrac{v^2}{\rho}\hat{e}_n$

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$$

$$\dot{v} = (c + ds)d$$

$$\frac{v^2}{\rho} = (c + ds)^2(a + bs)$$

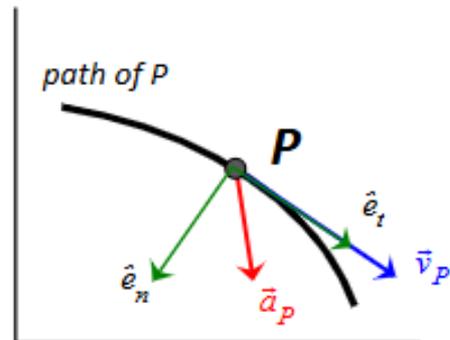$$\rightarrow \vec{a} = (c + ds)d\ \hat{e}_t + (c + ds)^2(a + bs)\ \hat{e}_n$$

Clothoid spiral

Summary: Particle Kinematics – Path Description

1. *PROBLEM*: Motion of a point described in path variables.

2. *FUNDAMENTAL EQUATIONS*:

$$\vec{v}_P = v_P \hat{e}_t = velocity\ of\ P$$

$$\vec{a}_P = \dot{v}_P \hat{e}_t + \frac{v_P^2}{\rho} \hat{e}_n = acceleration\ of\ P$$

path of P

where $\hat{e}_t$ and $\hat{e}_n$ are unit vectors tangent and (inwardly) normal to the path.

3. *OBSERVATIONS: In regard to the path description kinematics, we see*

- Velocity is ALWAYS tangent to the path.

- Acceleration, in general, has BOTH normal and tangential components.

- Note that acceleration depends on three factors: speed v_P, rate of change of speed $\dot{v}_P$ and radius of curvature of the path ρ.

- Rate of change of speed is the projection of acceleration onto the unit tangent vector: $\dot{v}_P = \vec{a}_P \cdot \hat{e}_t$

- Rate of change of speed is NOT equal to the magnitude of acceleration:

$$\left|\vec{a}_P\right| = \sqrt{\dot{v}_P^2 + \left(v_P^2 / \rho\right)^2} \neq \left|\dot{v}_P\right|$$

ME 274: Basic Mechanics II

Lecture 5: Relative and Constrained Motion

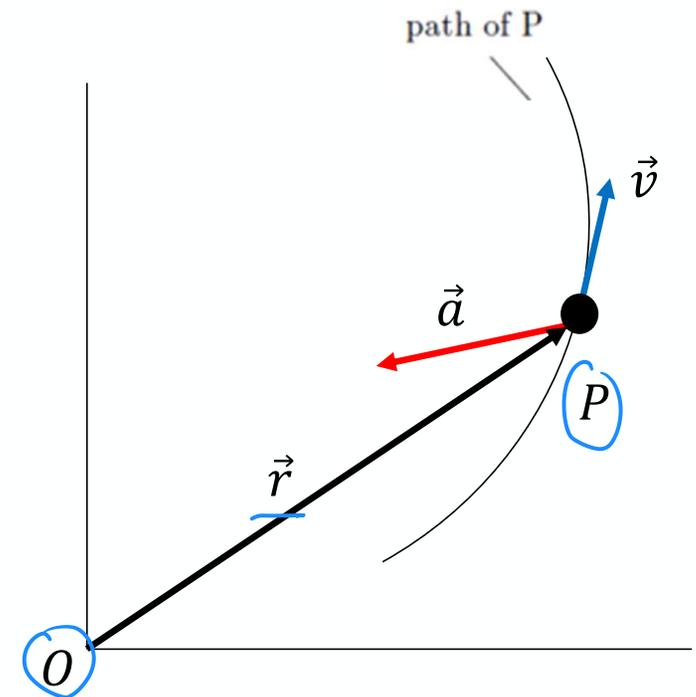**PURDUE UNIVERSITY** | School of Mechanical Engineering

Motion is measured relative to a point

So far, we have described:
- Motion of a single point
- Measured relative to a fixed reference point, O

Motion is not absolute → depends on the observer
- What do we do if we have multiple moving points?
- How do we describe motion of one body as seen from another?

path of P

$\vec{v}$

$\vec{a}$

P

$\vec{r}$

O

1D Example: trains crossing at a station

60 mph →   ← 60 mph

What is the speed of the red train observed from the platform? $-\ 60 mph$

What is the speed of the red train observed from the blue train? $-\ 120 mph$
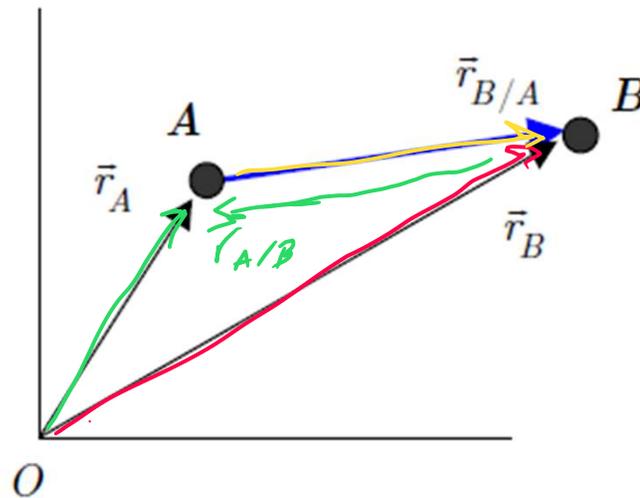
Relative position: where one point is as seen by another

The position of point B relative to point A is given by the vector $\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$

Note:
- $\vec{r}_{B/A}$ points from A TO B $B/A \Rightarrow B$ with respect to A
- $\vec{r}_{B/A} \neq \vec{r}_{A/B} \leftarrow$ same magnitude, opposite direction

Relative velocity and acceleration can be found by taking time derivatives

$$\vec{V}_{B/A} = \frac{d}{dt}\vec{r}_{B/A} = \frac{d\vec{r}_B}{dt} - \frac{d\vec{r}_A}{dt} = \vec{V}_B - \vec{V}_A$$

$$\vec{a}_{B/A} = \frac{d}{dt}\vec{V}_{B/A} = \frac{d\vec{V}_B}{dt} - \frac{d\vec{V}_A}{dt} = \vec{a}_B - \vec{a}_A$$

Example 1.D.1

Given: At the instant shown, car B is traveling with a speed of 50 km/hr and is slowing down at a rate of 10 km/hr². Car A is moving with a speed of 80 km/hr, a speed that is increasing at a rate of 10 km/hr². At this instant, A and B are traveling in the same direction.

$\dot{V}_B = -10$ $V_B = 50$

$\dot{V}_A = 10$ $V_A = 80$

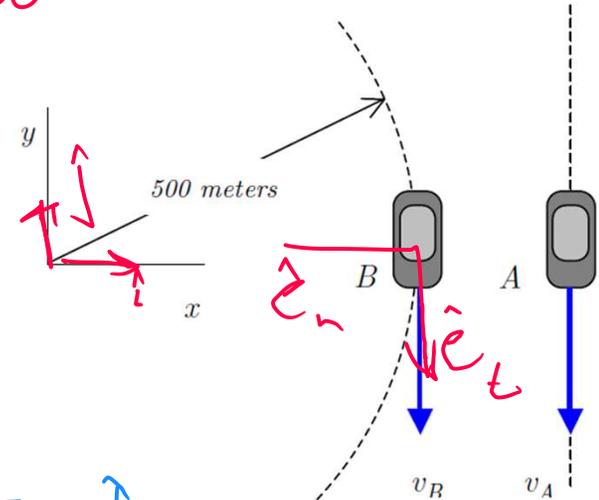**Find:** What acceleration does a passenger in car A observe for car B?

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A$$

$$\vec{a}_B = \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n$$

$$\vec{a}_B = -10\,\hat{e}_t + \frac{50^2}{(0.5)}\,\hat{e}_n$$

$$= -10\,\hat{e}_t - 5000\,\hat{e}_n$$

$$\vec{a}_B = 10\,\hat{j} - 5000\,\hat{i}$$

$$\vec{a}_A = -V_A\,\hat{j} = -10\,\hat{j}$$

$$\hat{e}_t = -\hat{j}$$
$$\hat{e}_n = -\hat{i}$$

$$\vec{a}_{B/A} = 10\,\hat{j} - 5000\,\hat{i} - (-10\,\hat{j})$$

$$\boxed{= -5000\,\hat{i} + 20\,\hat{j} \ \text{km/hr}^2}$$

500 meters

y

x

$\hat{e}_n$

$\hat{e}_t$

B   A

v_B v_A

Example 1.D.2

Given: Jet B is traveling due north with a speed of $v_B = 600$ km/hr. Passengers on jet B observe A to be flying sideways and moving due east.

Find: Determine:

(a) The speed of A; and

(b) The speed of A as observed by the passengers on jet B.

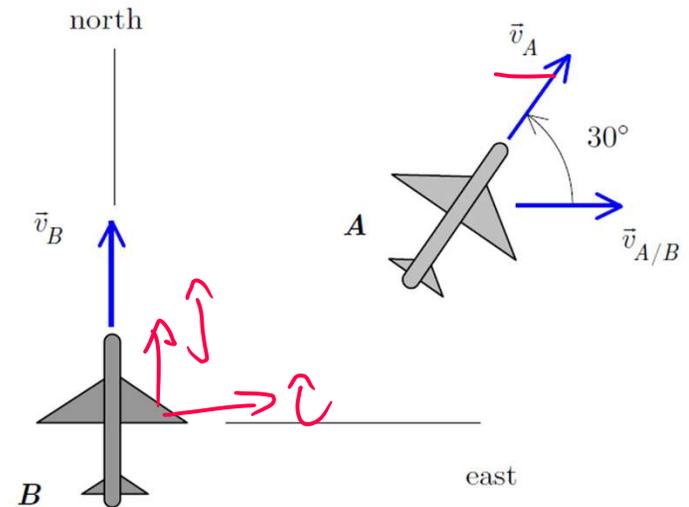$$\vec{V}_B = V_B \, \hat{j}$$

$$\vec{V}_{A/B} = V_{A/B} \, \hat{i}$$

$$\vec{V}_A = V_A \left(\cos\theta \, \hat{i} + \sin\theta \, \hat{j} \right)$$

$$\vec{V}_A = \vec{V}_B + \vec{V}_{A/B} = \underline{V_{A/B} \, \hat{i}} + \underline{V_B \, \hat{j}} = \underline{V_A \left(\cos\theta \, \hat{i} + \sin\theta \, \hat{j} \right)}$$
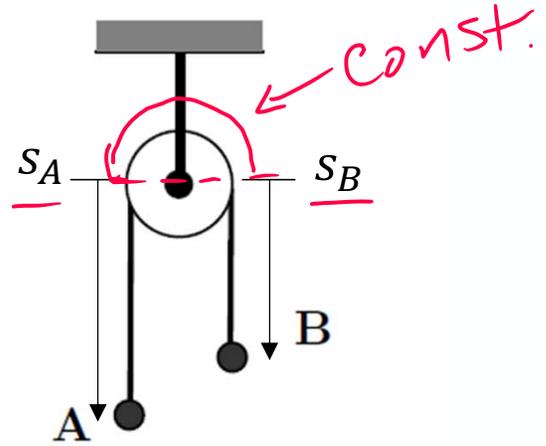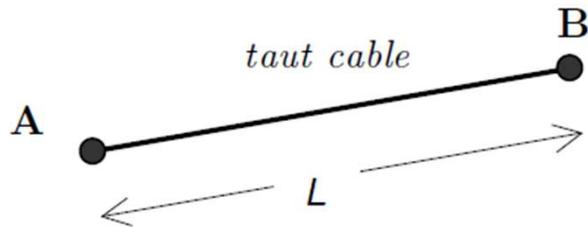
$$V_{A/B} = V_A \cos\theta$$

$$V_B = V_A \sin\theta \implies V_A = \frac{V_B}{\sin\theta} = \frac{60}{1/2} = 1200 \frac{km}{hr}$$

$$V_{A/B} = V_A \cos\theta = 1200 \cdot \frac{\sqrt{3}}{2} \frac{km}{hr}$$

Constrained Motion – Inextensible Cable

Constrained motion →the motion of one point depends on the motion of the other

$$\rightarrow \text{constraint eqns}$$

Important considerations:
- Length, L, does not change $\rightarrow \dfrac{dL}{dt} = 0$
- The cable is flexible & can go around pulleys
  - L remains constant, but the distance between A and B does not
  - The diameter of the pulley does not affect the constraint equation
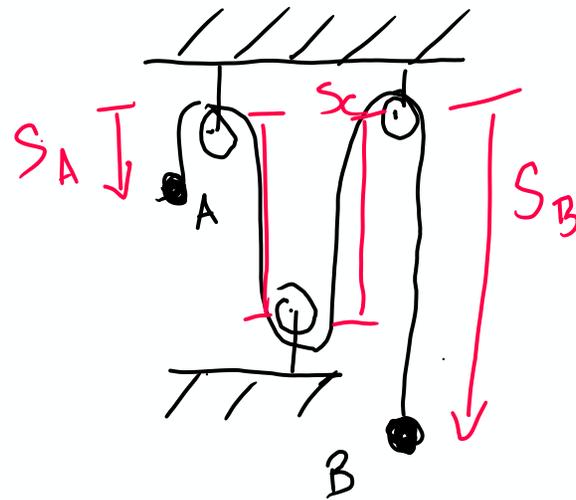
constraint eqn: $L = S_A + S_B + \text{const.}$

$$\frac{dL}{dt} = \frac{dS_A}{dt} + \frac{dS_B}{dt} + \cancel{\frac{d(\text{const})}{dt}}^{0}$$

$$0 = V_A + V_B$$

$$0 = a_A + a_B$$

Solving Inextensible Cable Problems

1. Carefully define a set of coordinates that describe the motion of the various particles in the system. → unique lengths of cable

2. For each cable, write an expression for its length, L, in terms of an appropriate set of coordinates defined above in step 1. $L = S_A + 2S_C + S_B$

3. Differentiate (with respect to time) the above expression for the cable length L and set $dL/dt = 0$ to determine the velocity constraint. $0 = V_A + 2V_C + V_B$

4. Differentiate again with respect to time to determine the acceleration constraint.

5. Repeat steps 2 through 4 for each cable in the system. $0 = a_A + 2a_C + a_B$

a note on connectors in pulley problems:

in these problems,
you may see
small "connector"
lengths
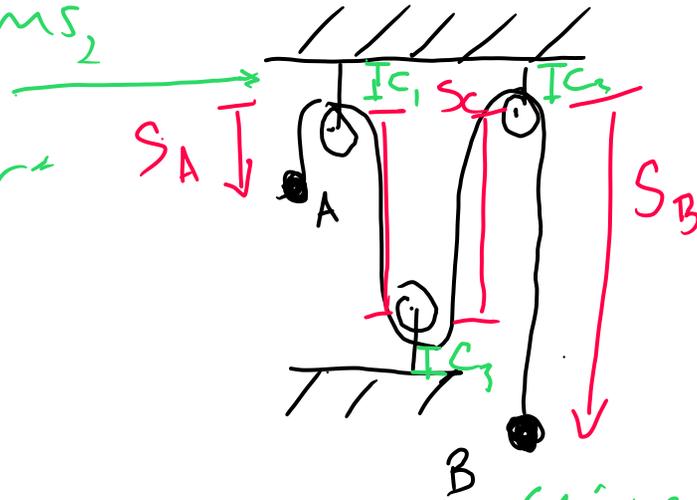
- note in many problems (like the one pictured)
you can define your length such that you do
not need to include the connectors in your
constraint eqns.

- unless otherwise specified, you can disregard these
connectors.

- if including them helps your book keeping, you can
label c_1, c_2, c_3 & group as a constant
  i.e. $L = S_A + 2S_c + S_B + \underbrace{(c_1 + c_2 + c_3)}_{= const}$

- when you take your derivatives for $\vec{v}$ or $\vec{a}$, this
  term $= 0$

Example 1.D.3

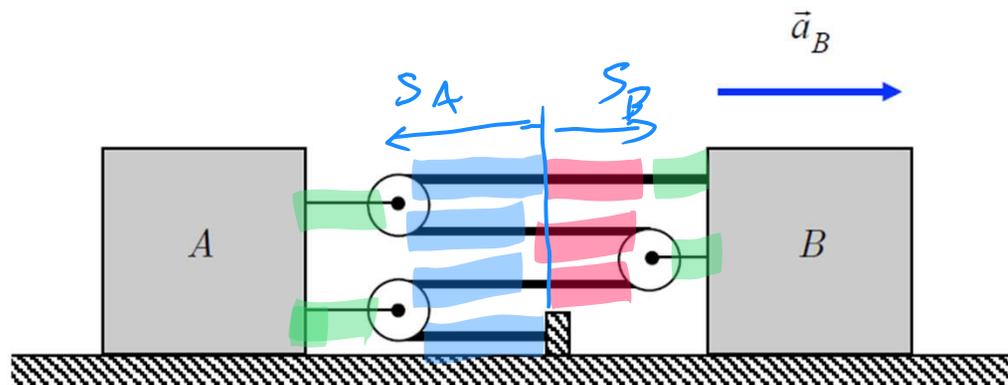**Given:** Block B has a constant acceleration of $a_B = 3$ m/s² to the right. At the instant shown, B has a velocity of 2 m/s to the right.

Find: Determine:
(a) The velocity of block A; and
(b) The acceleration of block A.

Solution:
1) Define a set of coordinates

2) Write an expression for cable length, L, in terms of coordinates

$$L = 4S_A + 3S_B$$

3) Differentiate the expression for L and set $\frac{dL}{dt} = 0$ to determine velocity constraint

$$0 = 4V_A + 3V_B \qquad V_A = -\frac{3V_B}{4} \Rightarrow -V_A = -\frac{3V_B}{4}(-2)$$

4) Differentiate again to determine acceleration constraint

$$0 = 4a_A + 3a_B \Rightarrow a_A = -\frac{3a_B}{4}$$

Example 1.D.4

Given: Blocks B and C are connected by a single inextensible cable, with this cable being wrapped around pulleys at D and E. In addition, the cable is wrapped around a pulley attached to block A as shown. Assume the radii of the pulleys to be small. Blocks B and C move downward with speeds of $v_B = 6$ ft/s and $v_C = 18$ ft/s, respectively.

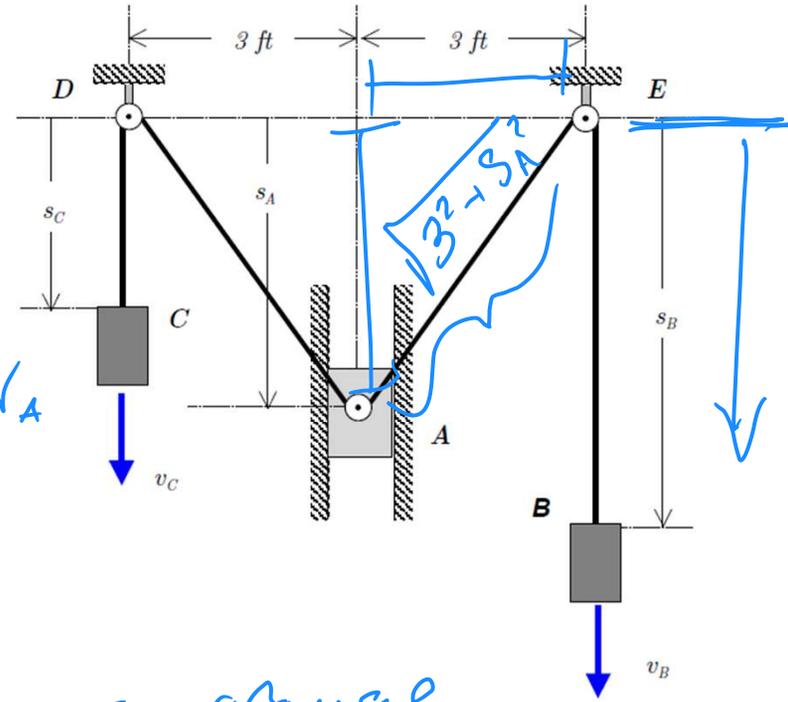**Find:** Determine the velocity of block A when $s_A = 4$ ft.

Constraint eqn:

$$L = S_C + 2\sqrt{3^2 + S_A^2} + S_B$$

$$\frac{dL}{dt} = 0 = V_C + 2\left(\frac{1}{2}\right)\left(9 + S_A^2\right)^{-1/2}\left(2S_A\right)V_A + V_B$$

$$0 = V_C + \frac{2S_A V_A}{\sqrt{9 + S_A^2}} + V_B \rightarrow \text{rearrange } + \text{ solve for } V_A$$

(Figure labels: D, E, 3 ft, 3 ft, s_C, s_A, s_B, C, A, B, v_C, v_B, $\sqrt{3^2 + S_A^2}$)

Example 1.D.4

Given: Block A moves with an acceleration of $\ddot{x}_A = a_A = 0.44 \text{ m/s}^2$.

Find: Determine the acceleration of block B.

$$L = 2x_B + (const - x_A)$$

$$2x_B + const - x_A + 2c$$

$$0 = 2v_B - v_A$$

$$0 = 2a_B - a_A$$

$\Big\}$ sub in & solve