ME 274: Basic Mechanics II

Week 2 *– Wednesday, January 21*

Particle kinematics: Joint Descriptions

Instructor: Manuel Salmerón

Attendance!

- Log in with your Purdue email (NOT Gmail)
- You have 20 seconds to answer each question
- The questions will only appear in the slides

Access:

Question 1

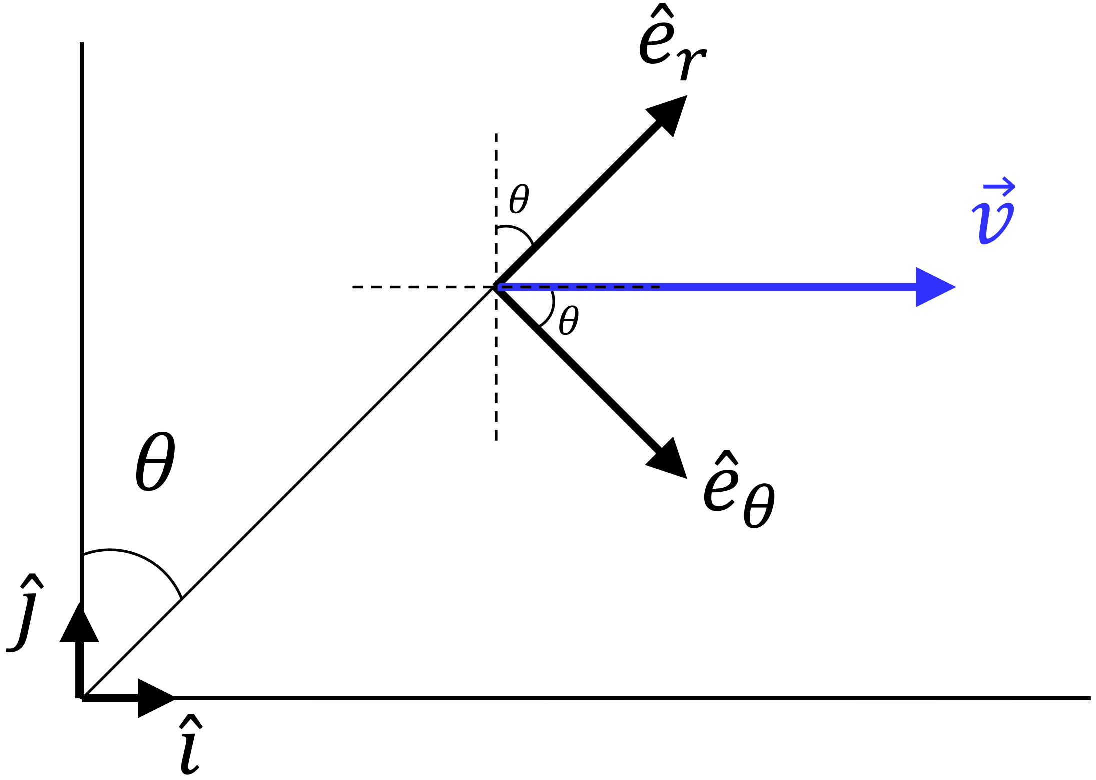
The scalar projection of $\vec{v}$ onto $\hat{e}_r$ is:

a) $\|\vec{v}\| \sin\theta$

b) $\vec{v} \cdot \hat{j}$

c) $\vec{v} \cdot \hat{\imath}$

d) $\|\vec{v}\| \cos\theta$



Question 2

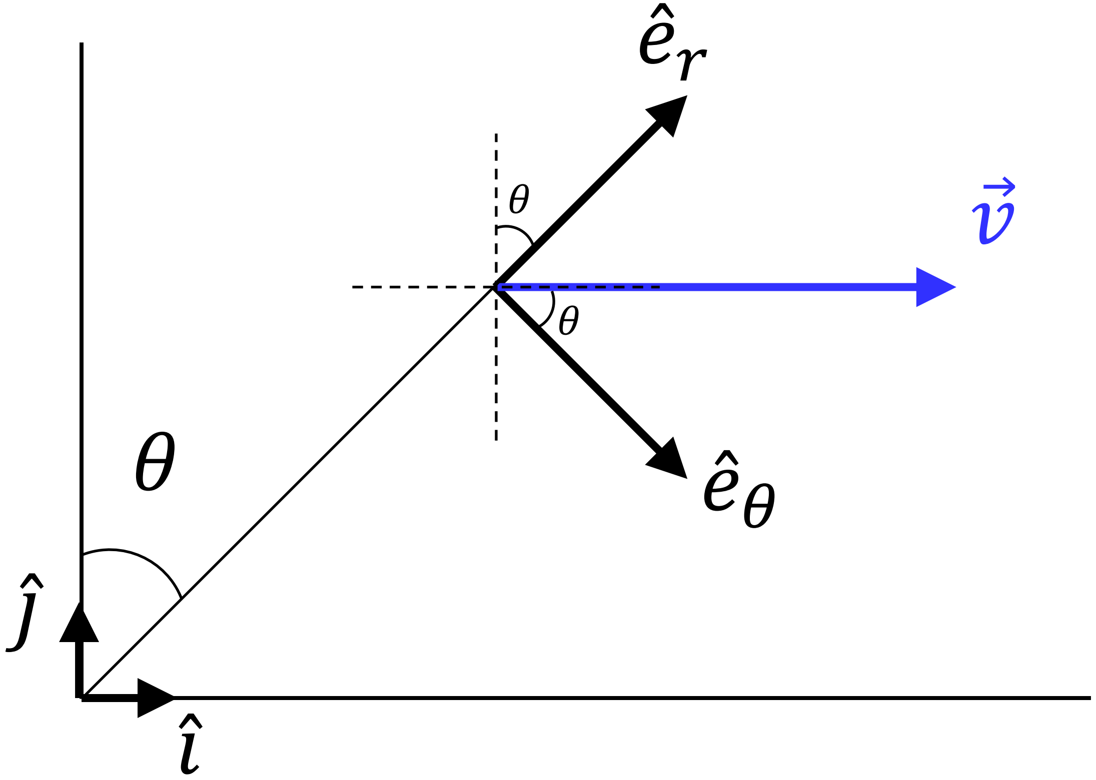
The scalar projection of $\vec{v}$ onto $\hat{e}_\theta$ is:

a) $\|\vec{v}\| \sin\theta$

b) $\vec{v} \cdot (\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$

c) $\vec{v} \cdot (\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})$

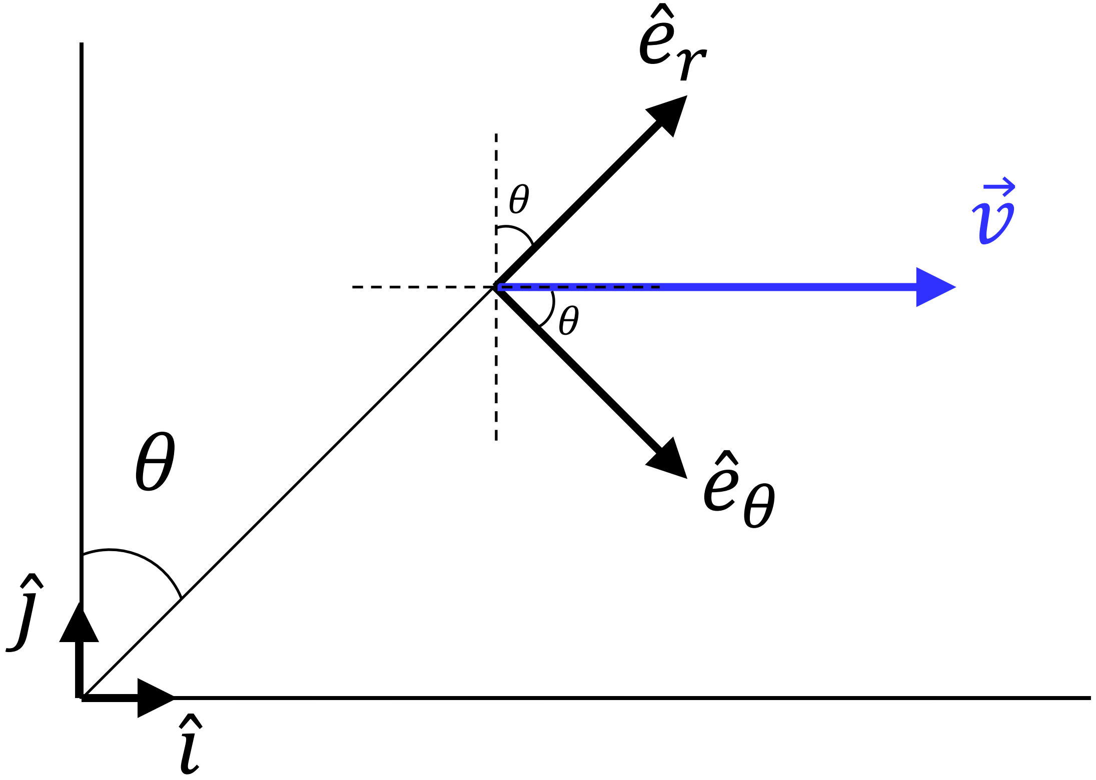
d) $\vec{v} \cdot (-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$



Correct answers

Q1: a) $\|\vec{v}\| \sin\theta$

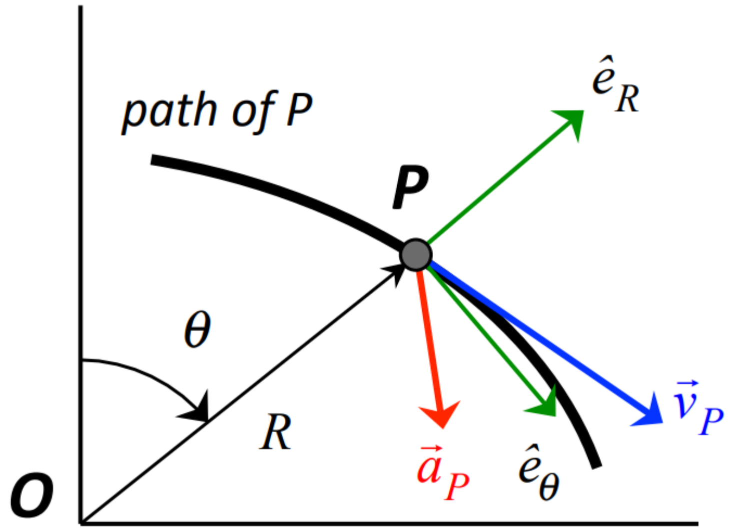
Q2: c) $\vec{v} \cdot (\cos\theta\, \hat{\imath} - \sin\theta\, \hat{\jmath})$



Today’s Agenda

1. Recap: polar coordinates
2. Joint Descriptions
3. Examples

1. Recap: Polar Coordinates



Kinematic Equations for Polar Coordinates:

$$\vec{v}(t) = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$$

$$\vec{a}(t) = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{e}_\theta$$

To keep in mind:

- You are free to choose the observation point O
- $\hat{e}_R$ always points OUTWARD from O to P
- $\hat{e}_\theta$ is perpendicular to $\hat{e}_R$ and in direction of increasing θ

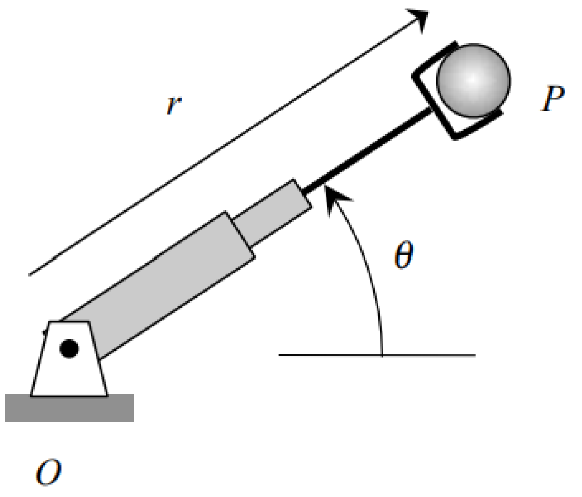
Additional Lecture Example 1.4

Given: A rotating and telescoping robotic arm is gripping a small sphere P in its end effector. The arm is rotating counterclockwise with a constant angular speed of $\dot{\theta}$. The arm is extending such that the radial distance from O to P is related to the rotation angle θ by the following equation:

$$r(\theta) = R_0 + R_1 \cos 2\theta$$

where r and θ are given in terms of meters and radians, respectively.

Find: Determine the velocity and acceleration of the sphere P. Write your answers as vectors in terms of the polar unit vectors $\hat{e}_r$ and $\hat{e}_\theta$.



Use the following parameters in your analysis: $R_0 = 2$ m, $R_1 = 0.5$ m, $\theta = \pi/2$ rad and $\dot{\theta} = 2$ rad/s.

Solution:

We need $\vec{v}$ and $\vec{a}$:

$$\vec{v}(t) = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a}(t) = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

What do we have? What are we missing?

$\dot{r}$ = need!

r = given!

$\dot{\theta}$ = given!

$\ddot{r}$ = need!

$\ddot{\theta} = 0$

Additional Lecture Example 1.4

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -2R_1\dot{\theta}\sin 2\theta$$ **(chain rule)**

$$\ddot{r} = \frac{d}{dt}\left(-2R_1\dot{\theta}\sin 2\theta\right) = -2R_1\frac{d}{dt}\left(\dot{\theta}\sin 2\theta\right)$$ **(constant out)**

$$\ddot{r} = -2R_1\left[\dot{\theta}\frac{d}{dt}(\sin 2\theta) + \sin 2\theta\frac{d}{dt}\left(\dot{\theta}\right)\right]$$ **(product rule)**

$$\ddot{r} = -2R_1\left[\dot{\theta}\frac{d}{d\theta}(\sin 2\theta)\frac{d\theta}{dt} + \ddot{\theta}\sin 2\theta\right]$$ **(chain rule)**

$$\ddot{r} = -2R_1\left[2\dot{\theta}^2\cos 2\theta + \ddot{\theta}\sin 2\theta\right]$$ **(solution)**

2. Joint Descriptions

velocity vector

$$\vec{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} \quad ; \quad Cartesian$$
$$= v\,\hat{e}_t \quad ; \quad path$$
$$= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \quad ; \quad polar$$

acceleration vector

$$\vec{a} = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} \quad ; \quad Cartesian$$
$$= \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n \quad ; \quad path$$
$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta \quad ; \quad polar$$

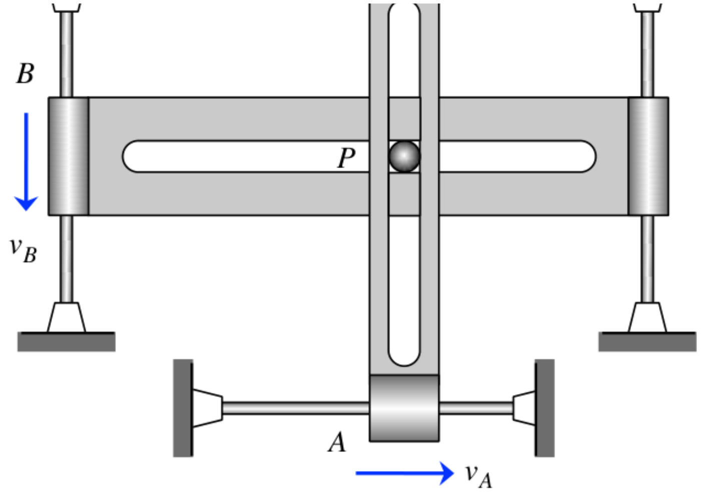
- Given that we are in description **A**, how can we go to description **B**?
- Steps:
  1. Write unit vectors of **A** in terms of the unit vectors of **B**
  2. Use vector projections or coefficient balancing to find unknowns

Lecture Book Example 1.C.2

Given: Pin P is constrained to move in the slotted guides that move at right angles to one another. At the instant shown, guide A moves to the right with a speed of v_A, a speed that is changing at a rate of $\dot{v}_A$. At the same time, B is moving downward with a speed of v_B with a rate of change of speed of $\dot{v}_B$.

Find:
(a) The rate of change of speed of P at this instant; and
(b) The radius of curvature ρ of the path followed by P at this instant.

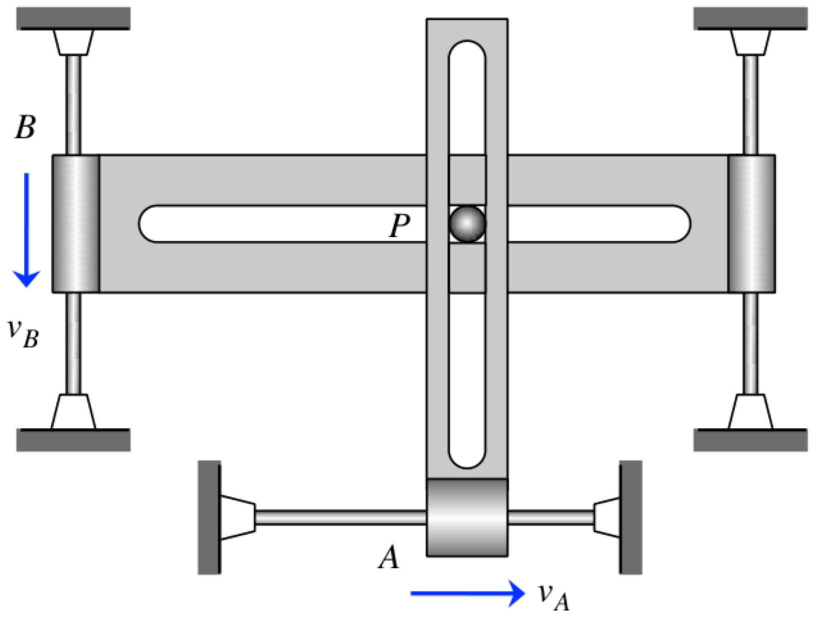
Use the following parameters: $v_A = 0.2$ m/s, $v_B = 0.15$ m/s, $\dot{v}_A = 0.75$ m/s^2 and $\dot{v}_B = 0$.



Lecture Book Example 1.C.2

Given: v_A, v_B, $\dot{v}_A$, $\dot{v}_B$

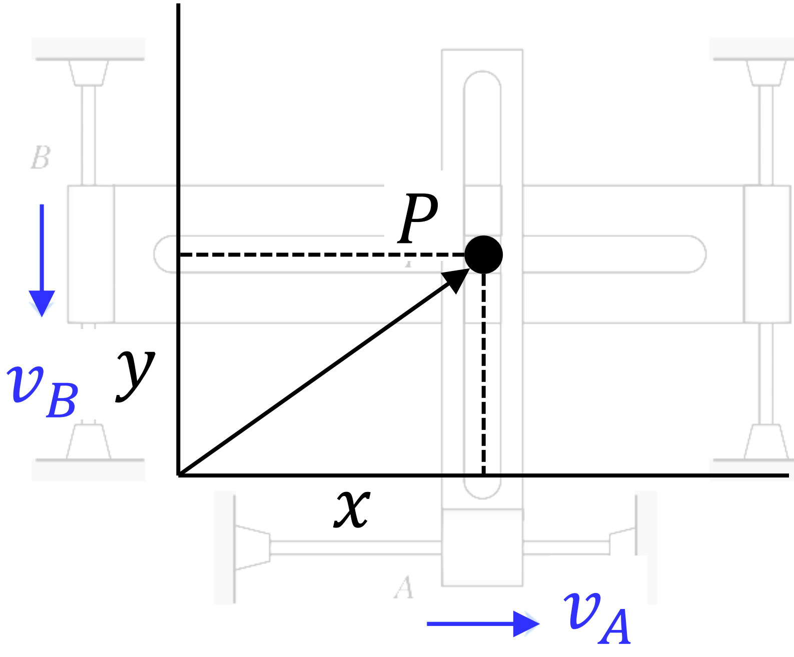
Find: (a) $\dot{v}$, and (b) ρ



Lecture Book Example 1.C.2

Given: $v_A,\ v_B,\ \dot{v}_A,\ \dot{v}_B$

Find: (a) $\dot{v}$, and (b) ρ



Solution:

We will likely need to go to the path description ($\hat{e}_t$, $\hat{e}_n$)

Fundamental equations:

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = v\hat{e}_t = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(\ddot{r} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

From the velocity equations:

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = v_A\hat{\imath} - v_B\hat{\jmath} = v\hat{e}_t$$

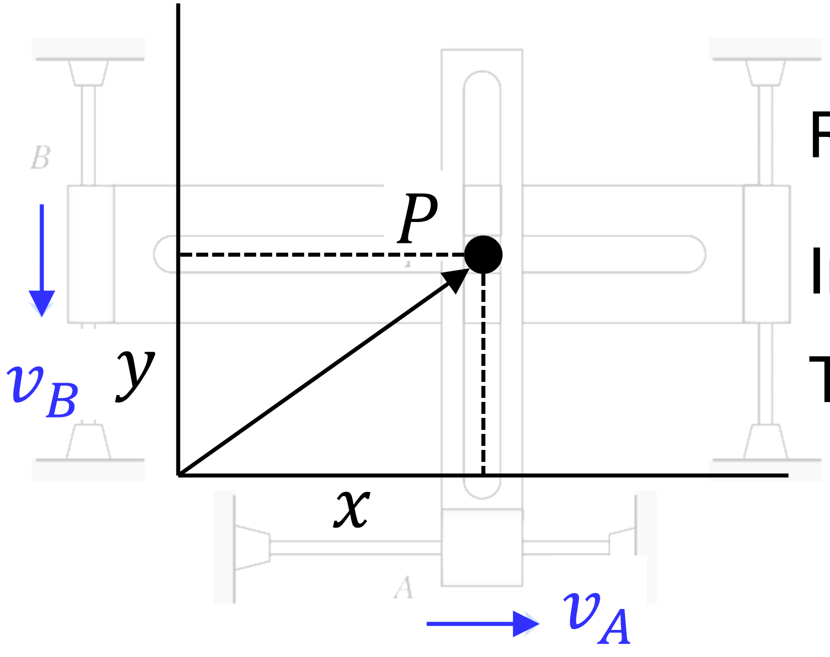
The vector $\vec{v}$ is the same in both descriptions:

$$\|\vec{v}\| = \sqrt{v_A^2 + v_B^2} = v$$

Lecture Book Example 1.C.2

Given: v_A, v_B, $\dot{v}_A$, $\dot{v}_B$

Find: (a) $\dot{v}$, and (b) ρ



Solution:

Knowing $\vec{v} = v_A\hat{\imath} - v_B\hat{\jmath}$ and $v = \sqrt{v_A^2 + v_B^2}$:

$$\hat{e}_t = \frac{\vec{v}}{v}$$

Remember that $\dot{v}$ appears in $\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$

In Cartesian coordinates: $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = \dot{v}_A\hat{\imath} - \dot{v}_B\hat{\jmath}$

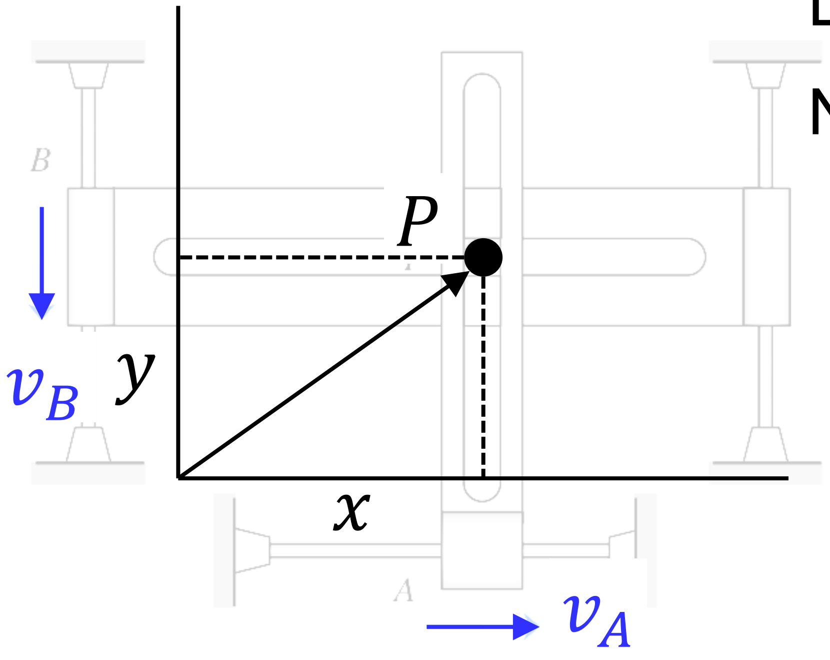
Thus:

$$\vec{a} = \dot{v}_A\hat{\imath} - \dot{v}_B\hat{\jmath} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

Lecture Book Example 1.C.2

Given: $v_A,\ v_B,\ \dot{v}_A,\ \dot{v}_B$

Find: (a) $\dot{v}$, and (b) ρ



Solution:

$$\vec{a} = \dot{v}_A\hat{\imath} - \dot{v}_B\hat{\jmath} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

Do we need $\hat{e}_n$ to get $\dot{v}$?

No: $\dot{v}$ is the scalar projection of $\hat{a}$ onto $\hat{e}_t$

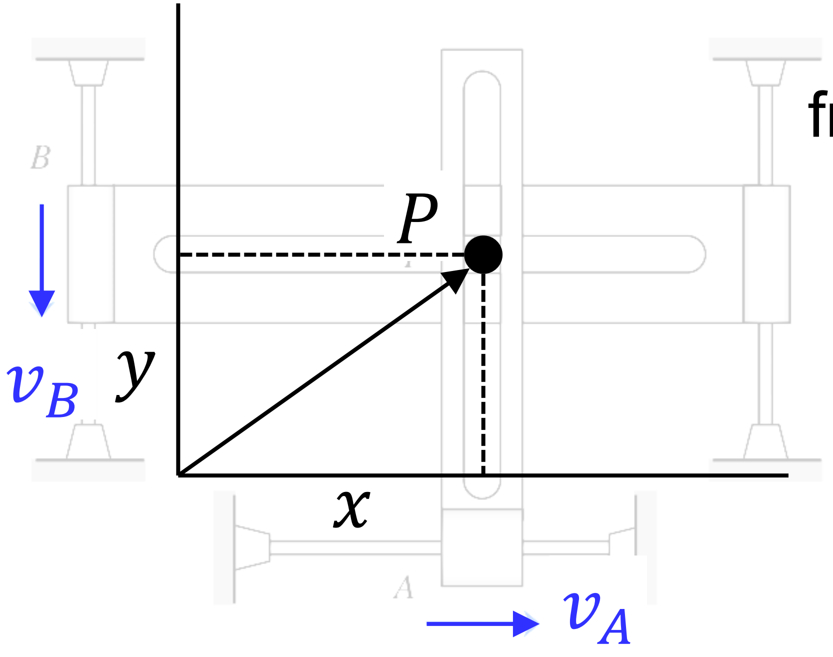
$$\dot{v} = \vec{a}\cdot\hat{e}_t = (\dot{v}_A\hat{\imath} - \dot{v}_B\hat{\jmath})\cdot\left(\frac{v_A\hat{\imath} - v_B\hat{\jmath}}{v}\right)$$

$$\dot{v} = \frac{1}{v}(v_A\dot{v}_A + v_B\dot{v}_B)$$

Lecture Book Example 1.C.2

Given: $v_A,\ v_B,\ \dot{v}_A,\ \dot{v}_B$

Find: (a) $\dot{v}$, and (b) ρ



Solution:

The vector $\vec{a}$ is the same in both descriptions:

$$\|\vec{a}\| = \sqrt{\dot{v}_A^2 + \dot{v}_B} = \sqrt{\dot{v}^2 + \frac{v^4}{\rho^2}},$$

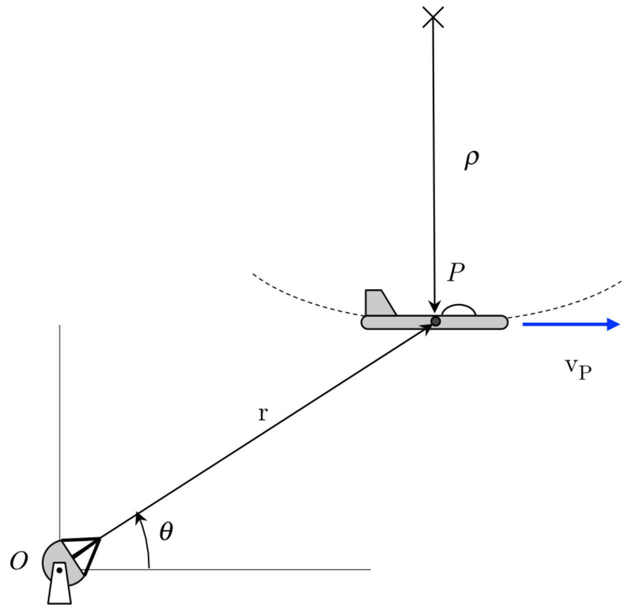
from where we can clear ρ.

Lecture Book Example 1.C.4

Given: At the bottom of a loop, an airplane P has a constant speed of v_P with the radius of curvature for the aircraft being ρ. The airplane is at a radial distance of r and at an angle of θ from a radar tracking station at O.

Find: Determine numerical values for $\ddot{r}$ and $\ddot{\theta}$ at this instant in time.

Use the following: $v_P = 75$ m/s, $\rho = 3000$ m, $r = 1000$ m and $\theta = 36.87°$

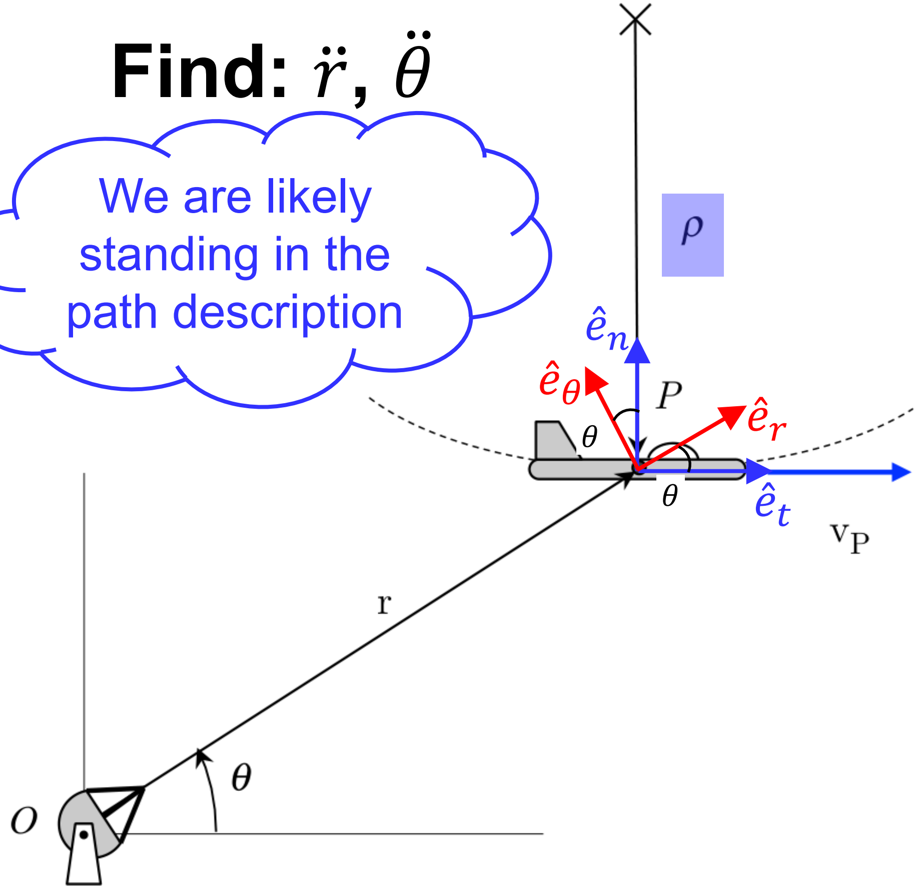


Lecture Book Example 1.C.4

Given: $v_P \equiv$ **const,** ρ, r, θ

Find: $\ddot{r}$, $\ddot{\theta}$

We are likely standing in the path description



We will **surely** need to go to the polar description ($\hat{e}_r$, $\hat{e}_\theta$)

Solution:

Fundamental equations:

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = v\hat{e}_t = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

From the figure:

$$\hat{e}_n = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta$$

$$\hat{e}_t = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$$

NOTE: See lecture book for a solution in terms of $\hat{e}_r$ and $\hat{e}_\theta$

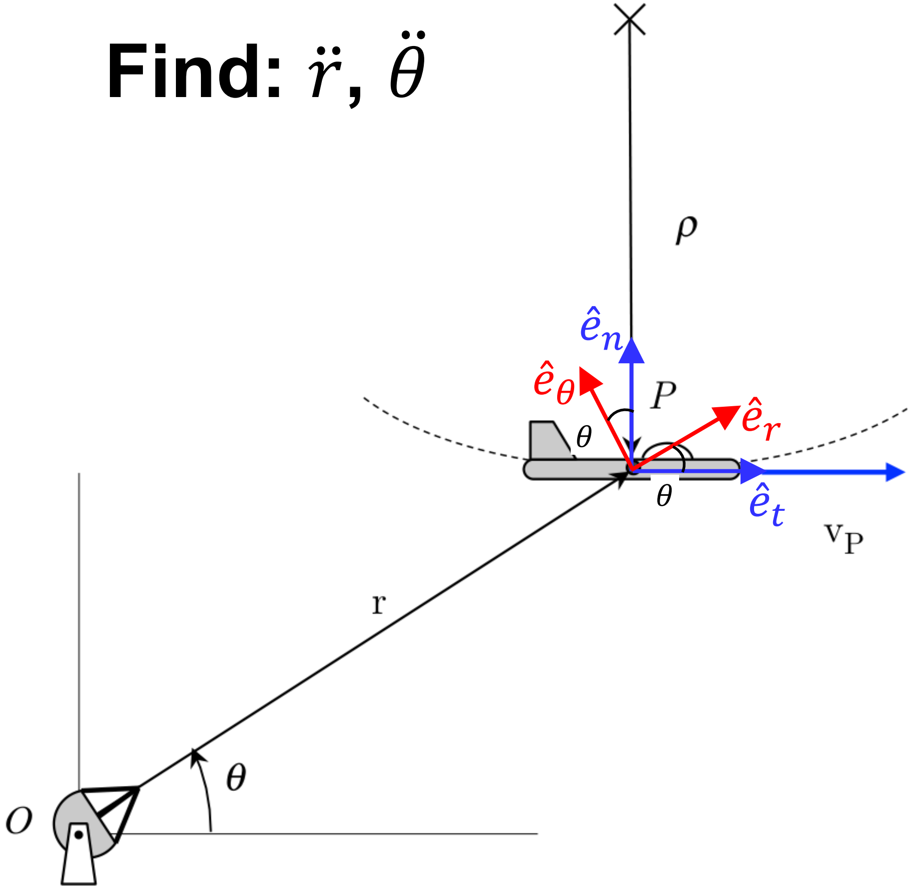
Substitute into $\vec{v}$-path and equate to $\vec{v}$-polar:

$$\vec{v} = v_P\hat{e}_t = v_P(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta) = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

Lecture Book Example 1.C.4

Given: $v_P \equiv$ **const,** ρ**,** r**,** θ

Find: $\ddot{r}$**,** $\ddot{\theta}$



Solution:

$$\vec{v} = v_P \hat{e}_t = v_P(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta) = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

Balancing coefficients:

$$\dot{r} = v_P\cos\theta \qquad \dot{\theta} = -\frac{v}{r}\sin\theta$$

Now, the acceleration:

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta = \frac{v^2}{\rho}(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta)$$

Balancing coefficients:

$$\ddot{r} - r\dot{\theta}^2 = \frac{v^2}{\rho}\sin\theta \Rightarrow \ddot{r} = \frac{v^2}{\rho}\sin\theta + r\dot{\theta}^2 = 3.63\ \text{m}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{v^2}{\rho}\cos\theta \Rightarrow \ddot{\theta} = \frac{1}{r}\left(\frac{v^2}{\rho}\cos\theta - 2\dot{r}\dot{\theta}\right) = 0.01\ \text{rad}$$

ME 274: Basic Mechanics II

Week 2 *– Friday, January 23*

Particle kinematics: Constrained and relative motion

Instructor: Manuel Salmerón

About late homework submissions

From the syllabus:

- No homework submitted after the due date/time will be accepted
- If technical issues prevent you from submitting by the due time, email this submission to the lead TA (or the instructor) **prior** to the due time.

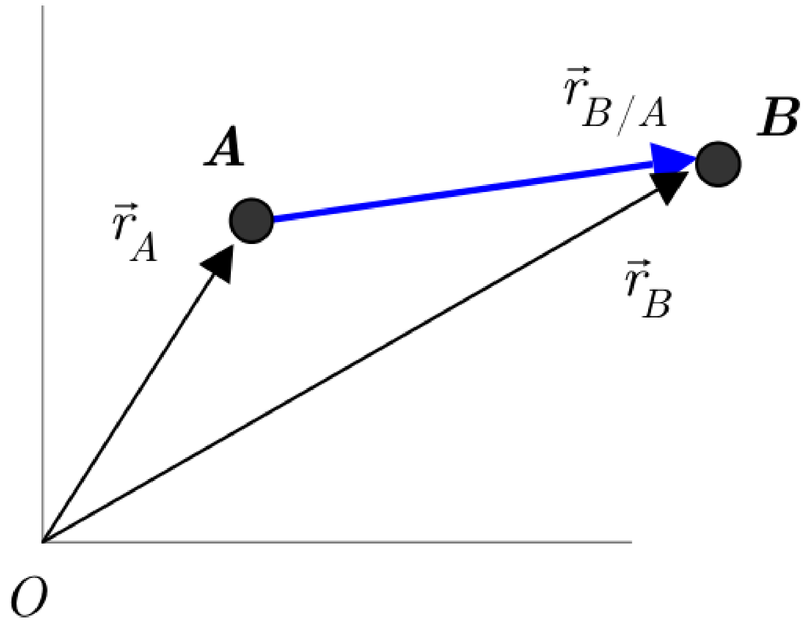
My email: salmeron@purdue.edu

Lead TA: Yeongun Ki, yki@purdue.edu

Today’s Agenda

1. Relative Motion
2. Example
3. Constrained Motion
4. Examples

1. Relative Motion



$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A :$$ position of particle B relative to position of particle A

$$\frac{d}{dt}$$

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A :$$ velocity of particle B with respect to particle A

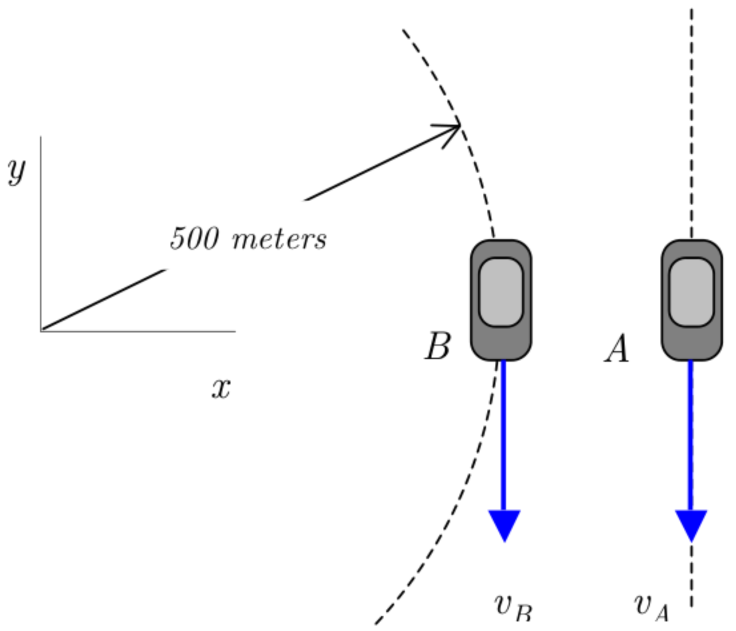
$$\frac{d}{dt}$$

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A :$$ acceleration of particle B with respect to particle A

2. Example 1.D.1

Given: At the instant shown, car B is traveling with a speed of 50 km/hr and is slowing down at a rate of 10 km/hr^2. Car A is moving with a speed of 80 km/hr, a speed that is increasing at a rate of 10 km/hr^2. At this instant, A and B are traveling in the same direction.

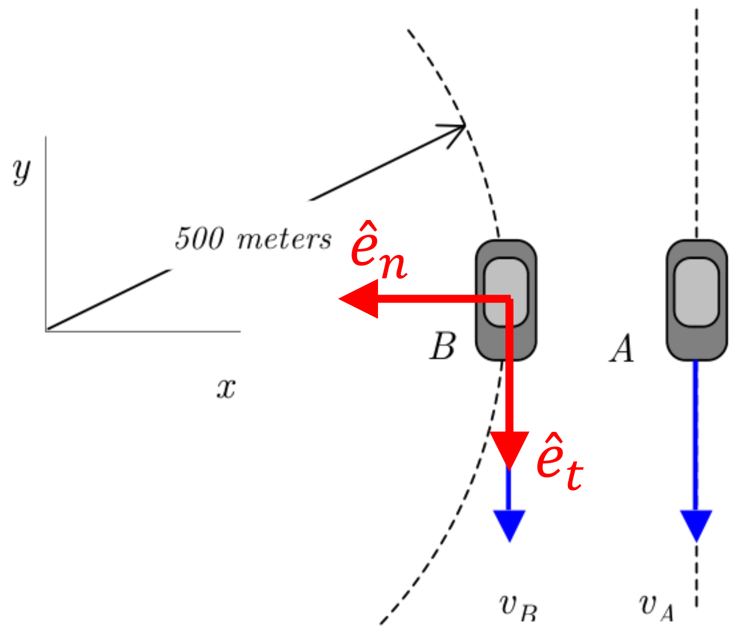
Find: What acceleration does a passenger in car A observe for car B?



2. Example 1.D.1

Given: $v_B = 50$ km/h, $\dot{v}_B = 10$ km/h² $(-)$, $v_A = 80$ km/h, $\dot{v}_A = 10$ km/h²

Find: acceleration of B with respect to A, $\vec{a}_{B/A}$



Solution:

We suspect $\vec{a}_B$ has a normal component:

$$\vec{v}_B = v_B \hat{e}_t$$

$$\vec{a}_B = -v_B \hat{e}_t + \frac{v_B^2}{\rho} \hat{e}_n$$

Car A is in rectilinear motion:

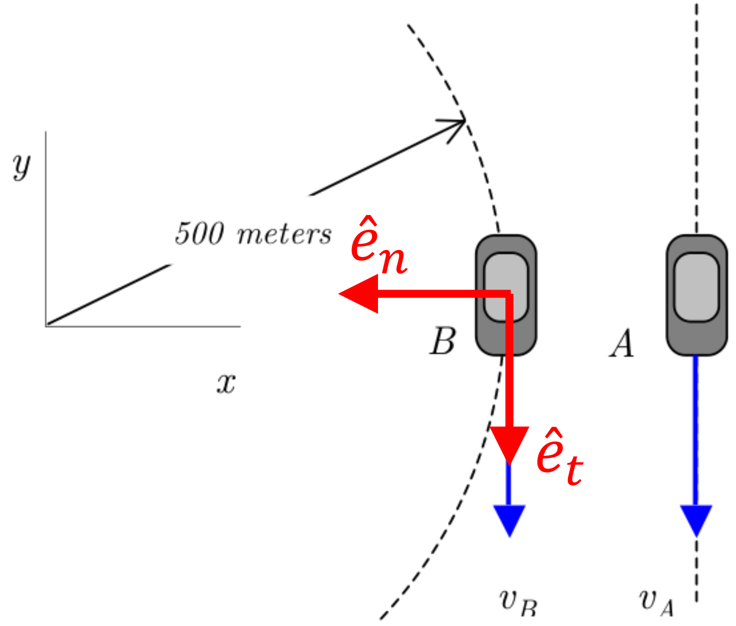
$$\vec{v}_A = v_A \hat{e}_t$$

$$\vec{a}_A = \dot{v}_A \hat{e}_t$$

2. Example 1.D.1

Given: $v_B = 50$ km/h, $\dot{v}_B = 10$ km/h² (–), $v_A = 80$ km/h, $\dot{v}_A = 10$ km/h²

Find: acceleration of B with respect to A, $\vec{a}_{B/A}$



Solution:

Acceleration of B with respect to A:

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A \qquad \textbf{(definition)}$$

$$\vec{a}_{B/A} = \left(-v_B\hat{e}_t + \frac{v_B^2}{\rho}\hat{e}_n\right) - (\dot{v}_A\hat{e}_t) \qquad \textbf{(plug in } \vec{a}_B \textbf{ and } \vec{a}_A\textbf{)}$$

$$\vec{a}_{B/A} == (-\dot{v}_B - \dot{v}_A)\hat{e}_t + \frac{v_B^2}{\rho}\hat{e}_n \qquad \textbf{(group components)}$$

Substituting the given values:

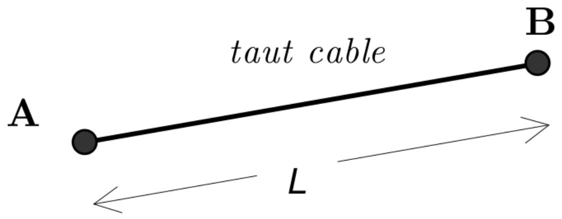
$$\vec{a}_{B/A} = -20\hat{e}_t + 5\hat{e}_n \text{ m/s}^2$$

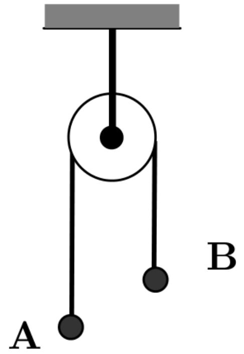
NOTE: website's solution uses $\hat{\imath}$ and $\hat{\jmath}$. Answers are equivalent.

3. Constrained Motion

The motion of one point depends on the motion of another point

For example, taut, inextensible cables:





Solution steps:

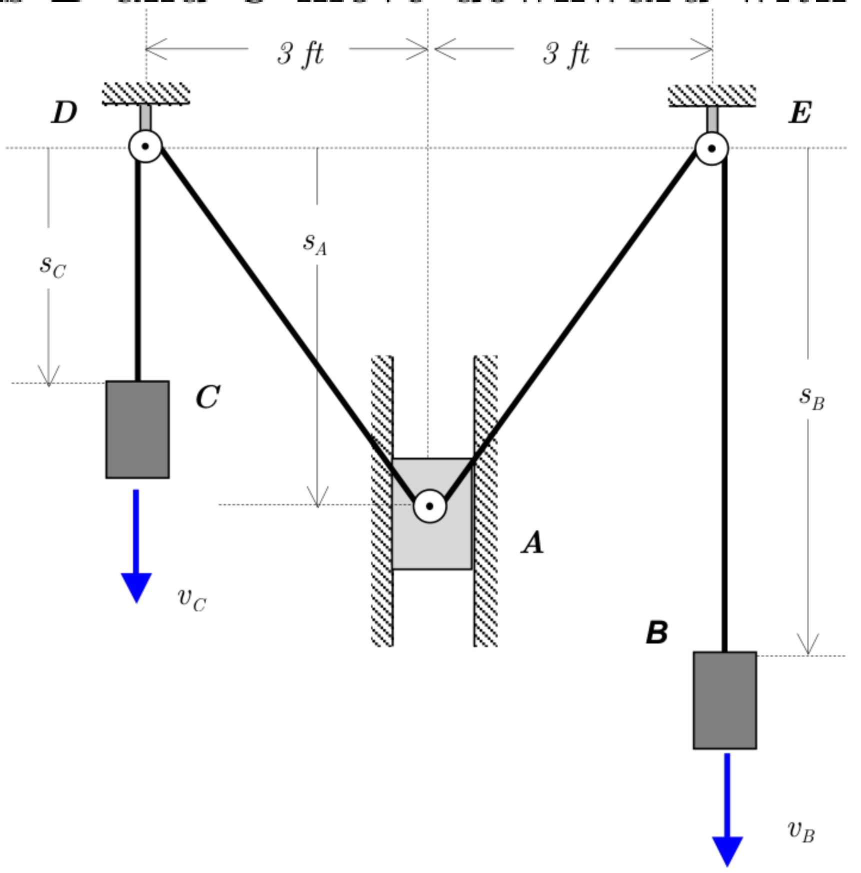
1. Define coordinates
2. Write the equation(s) of L
3. Velocity constraint: $\frac{dL}{dt} = 0$
4. Acceleration constraint: $\frac{d^2L}{dt^2} = 0$

Repeat for each cable/rope in the system

4. Example 1.D.6

Given: Blocks B and C are connected by a single inextensible cable, with this cable being wrapped around pulleys at D and E. In addition, the cable is wrapped around a pulley attached to block A as shown. Assume the radii of the pulleys to be small. Blocks B and C move downward with speeds of $v_B = 6$ ft/s and $v_C = 18$ ft/s, respectively.

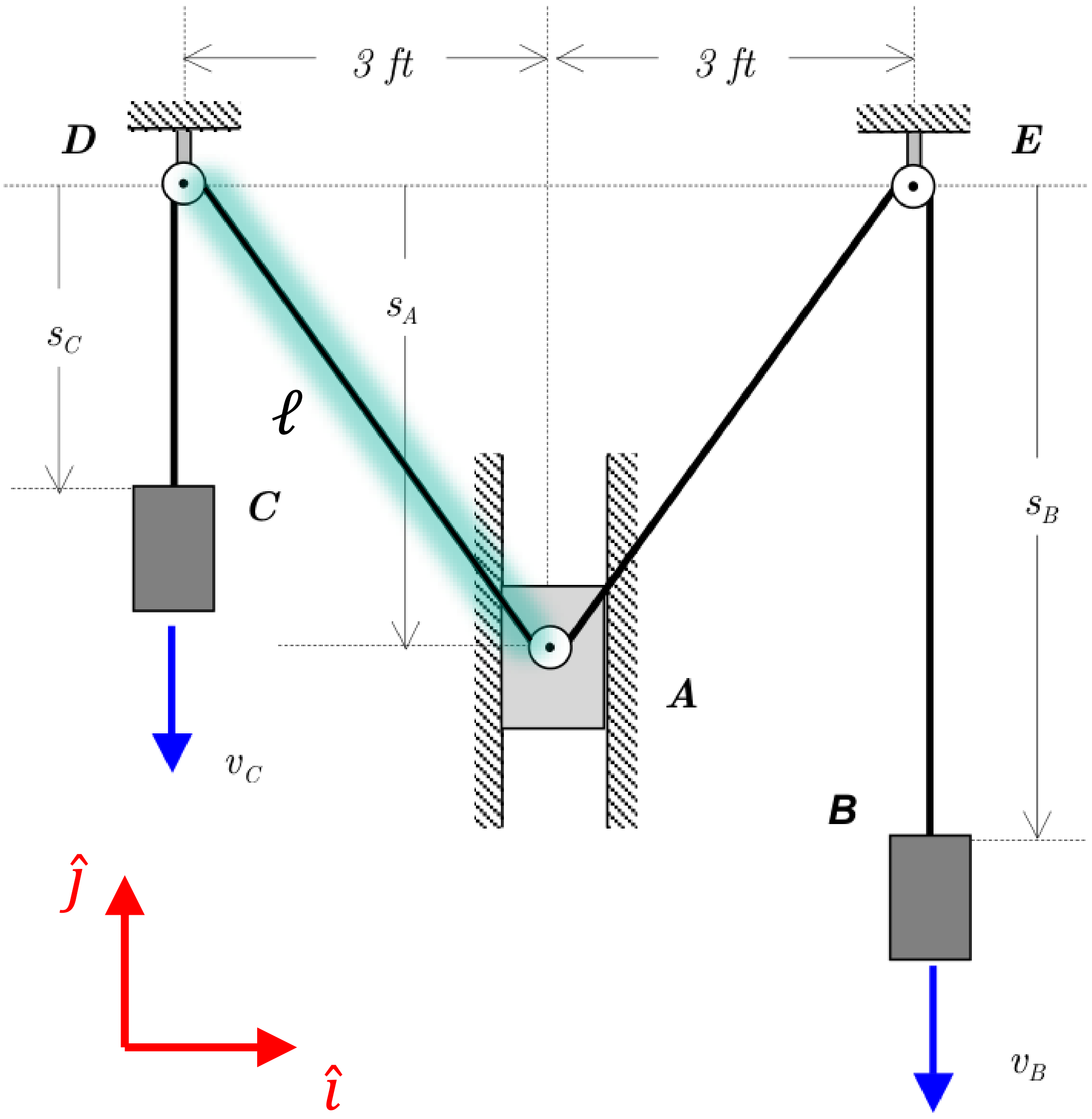
Find: Determine the velocity of block A when $s_A = 4$ ft.



4. Example 1.D.6

Given: $v_B = 6$ ft/s, $v_C = 18$ ft/s

Find: $\vec{v}_A$ when $s_A = 4$ ft



Solution:

Rope equation:

$$L = 2\ell + s_B + s_C$$

where $\ell = \sqrt{s_A^2 + 3^2}$ is known.

Velocity constraint:

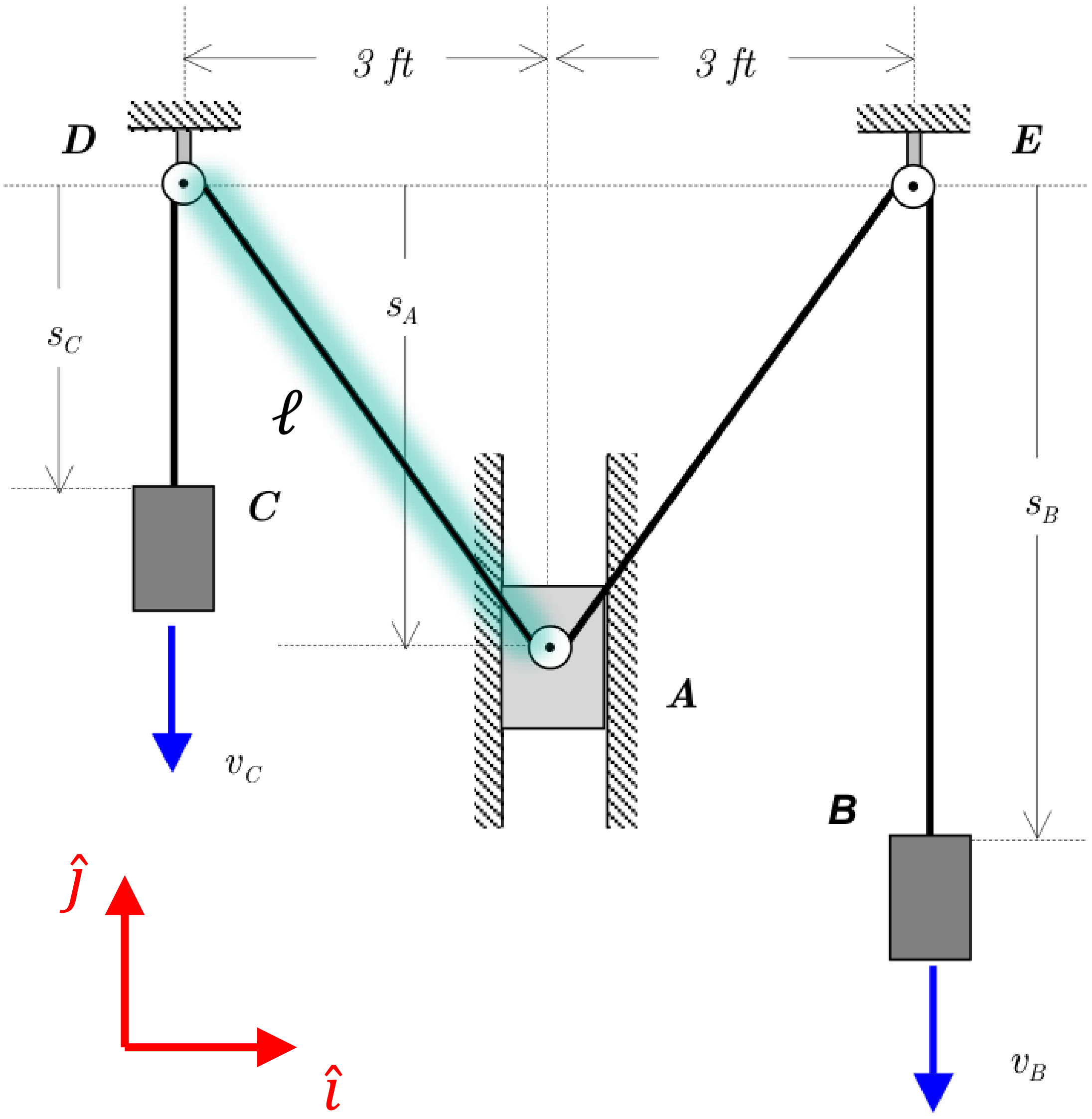
$$\frac{d}{dt}(2\ell + s_B + s_C) = 0$$

$$\frac{d\ell}{dt} = \frac{d\ell}{ds_A}\frac{ds_A}{dt} = \frac{2s_A}{2\sqrt{s_A^2 + 9}}v_A$$

4. Example 1.D.6

Given: $v_B = 6$ ft/s, $v_C = 18$ ft/s

Find: $\vec{v}_A$ when $s_A = 4$ ft



Solution:

Substitute into the velocity constraint:

$$v_B + v_C + 2\frac{s_A v_A}{\sqrt{s_A^2 + 9}} = 0$$

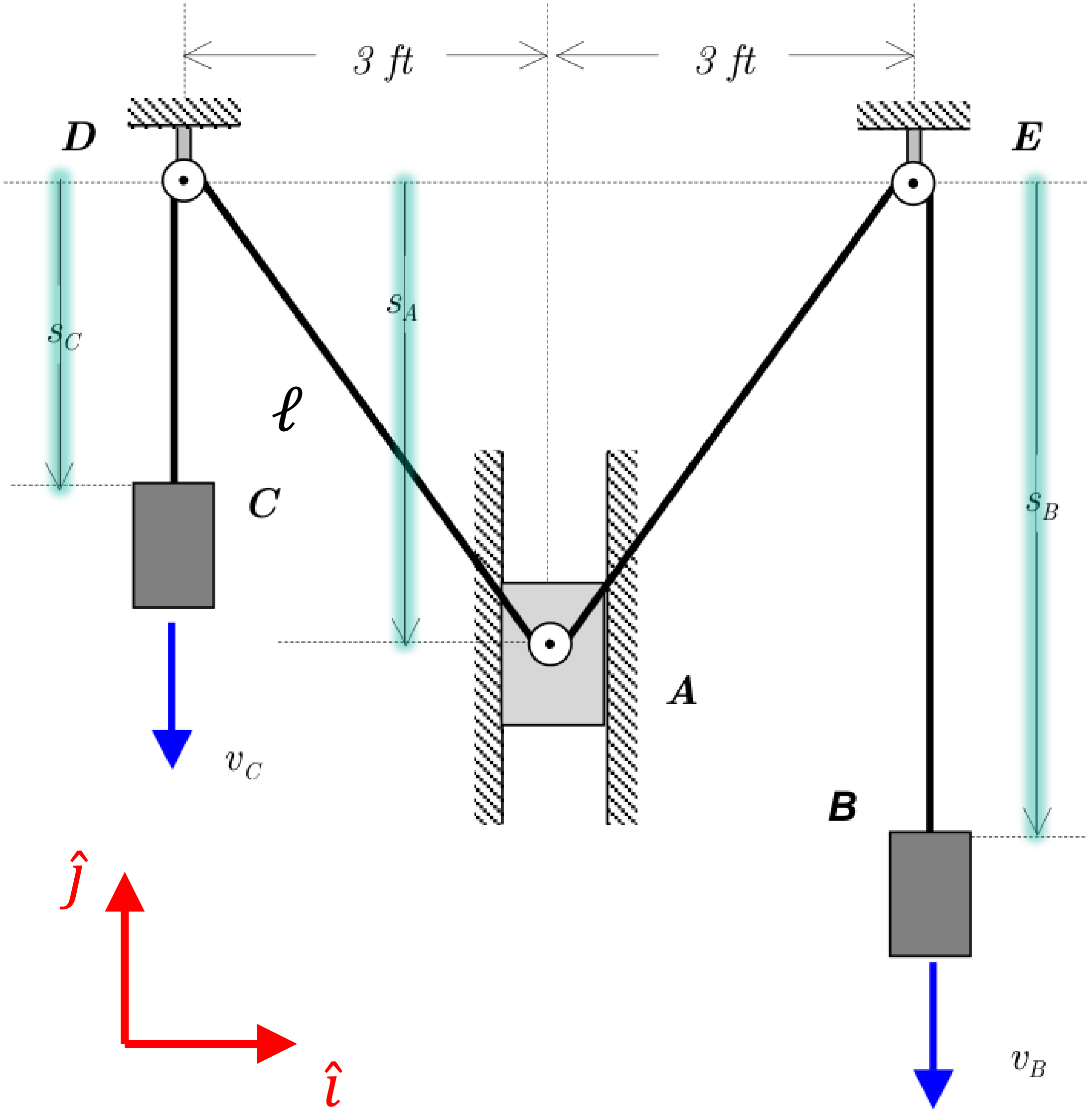
Clear for v_A:

$$v_A = \frac{-v_B - v_C}{2s_A}\sqrt{s_A^2 + 9}$$

4. Example 1.D.6

Given: $v_B = 6$ ft/s, $v_C = 18$ ft/s

Find: $\vec{v}_A$ when $s_A = 4$ ft



Solution:

$$v_A = \frac{-v_B - v_C}{2s_A}\sqrt{s_A^2 + 9}$$

How to decide the direction?

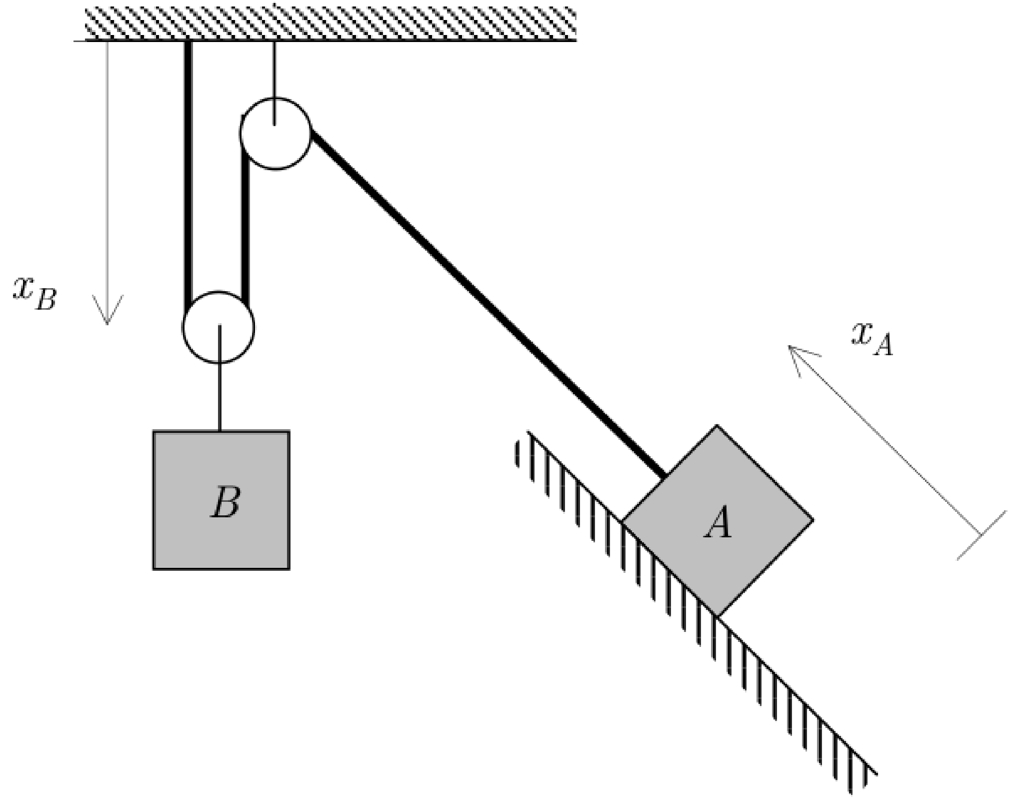
When B and C go down **along their respective reference lines** ($v_B > 0$, $v_C > 0$), A goes up ($v_A < 0$). Thus:

$$\vec{v}_A = \left(\frac{v_C + v_B}{2s_A}\sqrt{s_A^2 + 9}\right)\hat{\jmath}$$

4. Example 1.D.4

Given: Block A moves with an acceleration of $\ddot{x}_A = a_A = 0.44$ m/s^2

Find: Determine the acceleration of block B.



Solution:

$$\vec{a} = -\frac{a_A}{2}\hat{j} = -0.22\hat{j}$$