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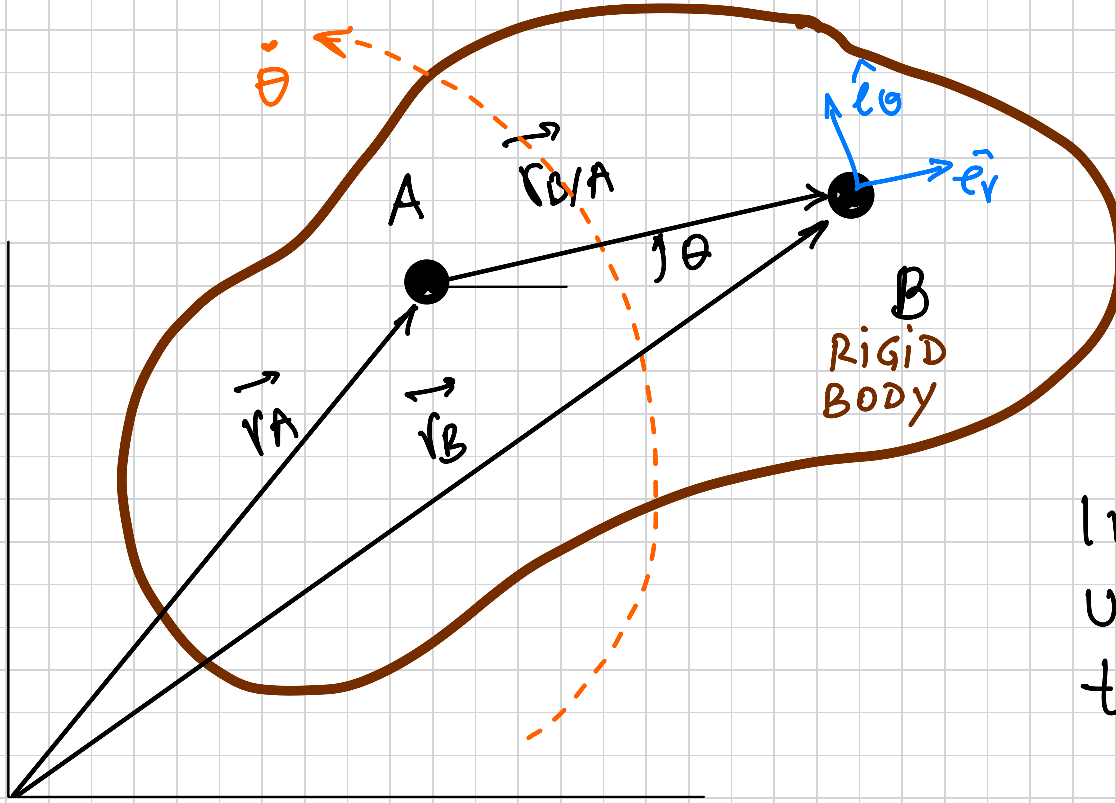
KINEMATICS OF RIGID BODY (2D)
(a.k.a. potatoes kinematics)

We know that the position of point B seen by A is:

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

$\vec{r}_B$



$|\vec{r}_{B/A}| = r$ = constant

θ variable (can change)

Intuitive to use polar description to express $\vec{r}_{B/A}$

$$\vec{r}_{B/A} = \boxed{r}\,\boxed{\hat{e}_r} = \vec{r}_B - \vec{r}_A$$

Aligned w/ $\vec{r}_{B/A}$

$|\vec{r}_{B/A}|$ distance from A to B

Let's revisit $\vec{v}_{B/A}$ and $\vec{a}_{B/A}$ from previously defined results:

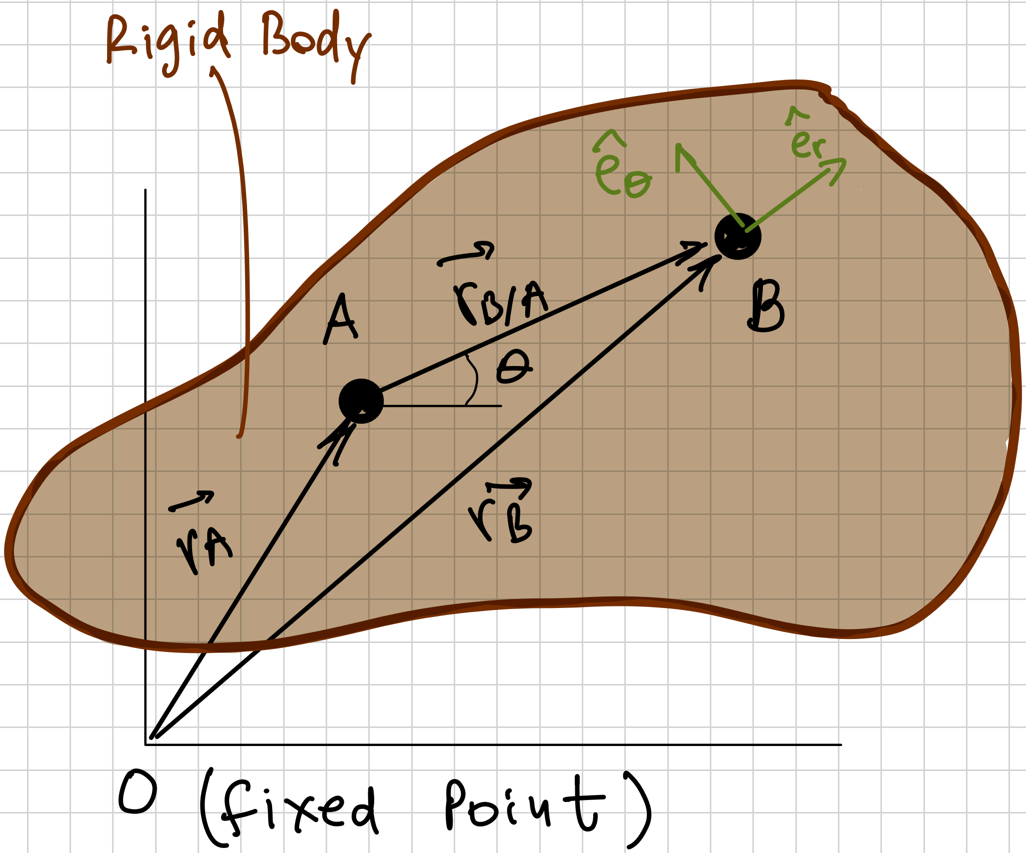
$$\vec{v}_{B/A} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta = \vec{v}_B - \vec{v}_A$$

$$\vec{a}_{B/A} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta = \vec{a}_B - \vec{a}_A$$

We will use these relationships specifically for rigid bodies, that is, objects for which the distance between two points on the object remains fixed

Kinematic Equations of Planar Rigid Body Motion

$$\left.\begin{array}{lll} |\vec{r}_{B/A}| & \rightarrow & \text{constant} \\ \theta & \rightarrow & \text{can change} \end{array}\right\} \Rightarrow \vec{r}_{B/A} \text{ not constant}$$



Since A and B lie on the same rigid body, $|\vec r_{B/A}|$ is constant, and $\dot r = 0$ for all time (: $\dot\theta \neq 0$); so:

$$\vec v_{B/A} = \vec v_B - \vec v_A$$

$$= \cancel{\dot r}^{\,0}\,\hat e_r + r\dot\theta\,\hat e_\theta$$

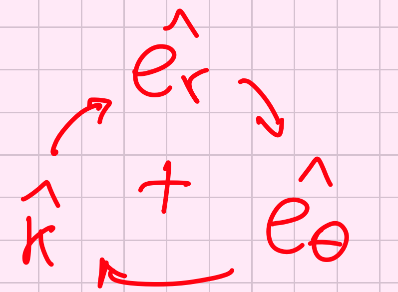
$$= r\dot\theta\,\hat e_\theta$$

$$= r\dot\theta\,(\hat k \times \hat e_r)$$

$$= \underbrace{\boxed{\dot\theta\,\hat k}}_{w\hat k} \times \underbrace{\boxed{r\,\hat e_r}}_{\vec r_{B/A}} \quad , \quad \dot\theta = \omega$$

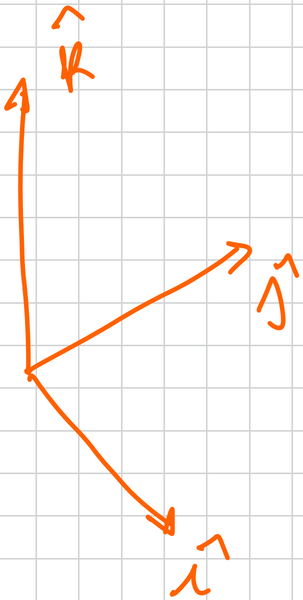
$$, \ \vec w = w\hat k : \text{ angular velocity}$$

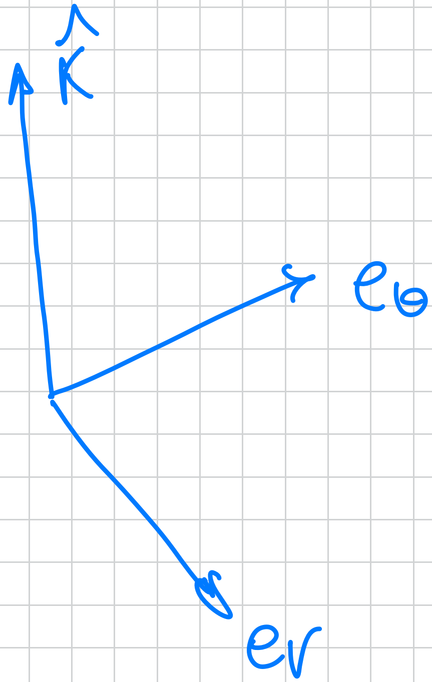
$$\Rightarrow \vec v_{B/A} = \vec\omega \times \vec r_{B/A}$$



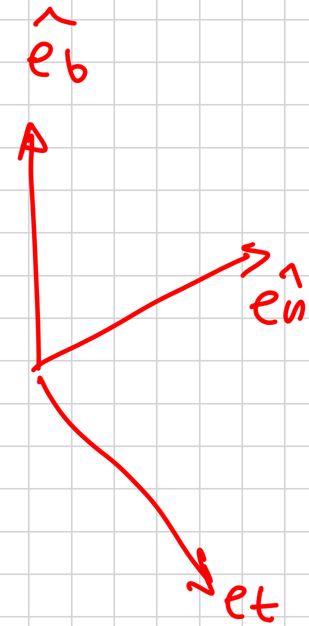
$\hat e_\theta = \hat k \times \hat e_r$

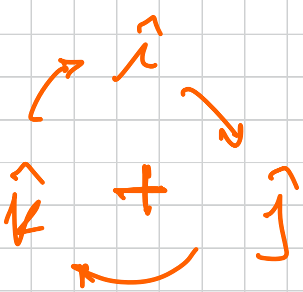
This applies to any coordinate system (even 3D)





Polar





$\hat{i} \times \hat{j} = \hat{k}$

$\hat{j} \times \hat{k} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}$

Let's repeat the procedure for acceleration:

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A$$

$$= (\cancelto{0}{\ddot{r}} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\cancelto{0}{\dot{r}}\dot{\theta})\,\hat{e}_\theta$$

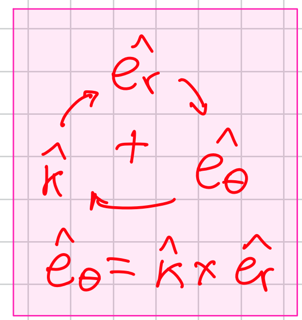
$$= -r\dot{\theta}^2\,\hat{e}_r + r\ddot{\theta}\,\hat{e}_\theta$$

Here, let's rewrite $\hat{e}_r$ and $\hat{e}_\theta$ in terms of cross products of $\hat{k}$ and $\hat{e}_r$, so they can be related with angular rates of change

$$\hat{e}_\theta = \hat{k} \times \hat{e}_r$$

$$\hat{e}_r = -\hat{k} \times \hat{e}_\theta$$

$$= -\hat{k} \times (\hat{k} \times \hat{e}_r)$$



$$\Rightarrow \vec{a}_{B/A} = -r\dot{\theta}^2\left(-\hat{k} \times \hat{e}_\theta\right) + r\ddot{\theta}\left(\hat{k} \times \hat{e}_r\right)$$

$$= r\dot{\theta}^2\left[\hat{k} \times \left(\hat{k} \times \hat{e}_r\right)\right] + r\ddot{\theta}\left(\hat{k} \times \hat{e}_r\right)$$

Rearranging conveniently:

$$= \underbrace{\boxed{(\dot{\theta}\,k)}}_{\omega\hat{k}} \times \left[\underbrace{\boxed{(\dot{\theta}\,k)}}_{\omega\hat{k}} \times \underbrace{\boxed{(r\hat{e}_r)}}_{\vec{r}_{B/A}}\right] + \underbrace{\boxed{(\ddot{\theta}\,\hat{k})}}_{\alpha\hat{k}} \times \underbrace{\boxed{r\hat{e}_r}}_{\vec{r}_{B/A}}$$

Finally:

$$\vec{a}_{B/A} = \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right) + \vec{\alpha} \times \vec{r}_{B/A} = \vec{a}_B - \vec{a}_A$$

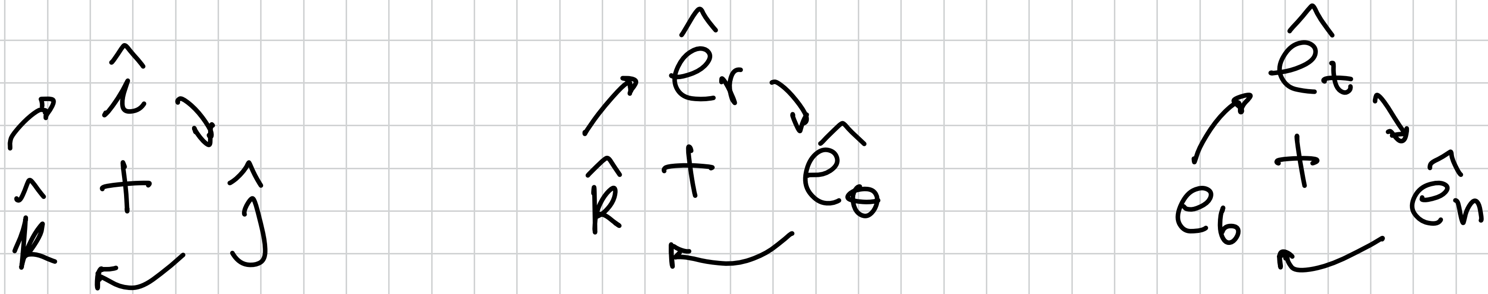
From this, we can rewrite the velocity and acceleration vectors of point B seen from an observer in point A (on the R.B.)

$$\vec{v}_B = \underbrace{\boxed{\vec{v}_A}}_{\text{translation}} + \underbrace{\boxed{\vec{\omega} \times \vec{r}_{B/A}}}_{\text{rotation, relative motion}}$$

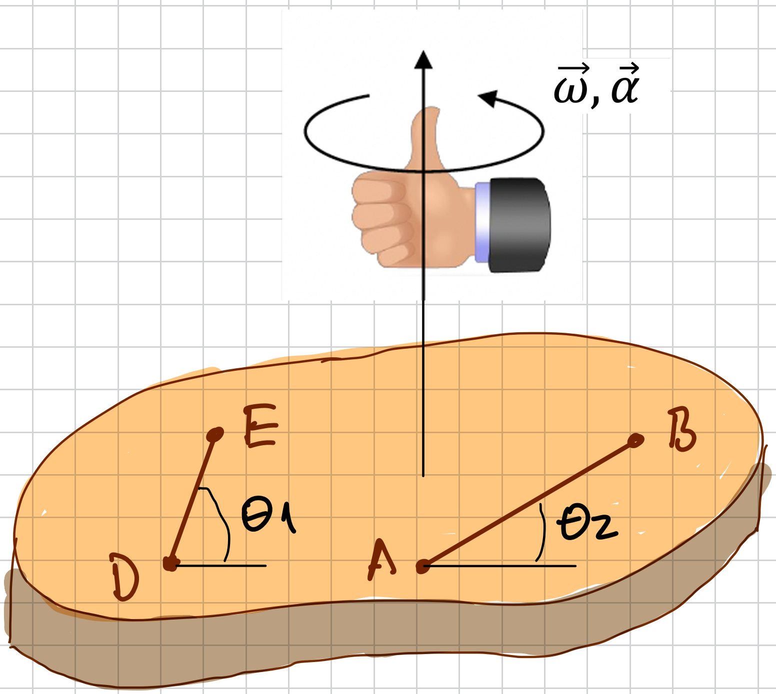
$$\vec{a}_B = \underbrace{\boxed{\vec{a}_A}}_{\text{translation}} + \underbrace{\boxed{\vec{\omega} \times [\vec{\omega} \times \vec{r}_{B/A}] + \vec{\alpha} \times \vec{r}_{B/A}}}_{\text{rotation, relative motion}}$$

Recall that $\vec{r}_{B/A}$ extends from A to B. Even though we expressed it in terms of polar coordinates, we could've used any system. In general problem-solving, we will use Cartesian description to relate A and B ($\hat{\imath}$ and $\hat{\jmath}$)

Use the unit vector circles



Angular acceleration & angular velocity



⚠ Use right-hand rule to determine direction of $\vec{\omega}$ & $\vec{\alpha}$

$$\theta_1 = \theta_2 + c$$

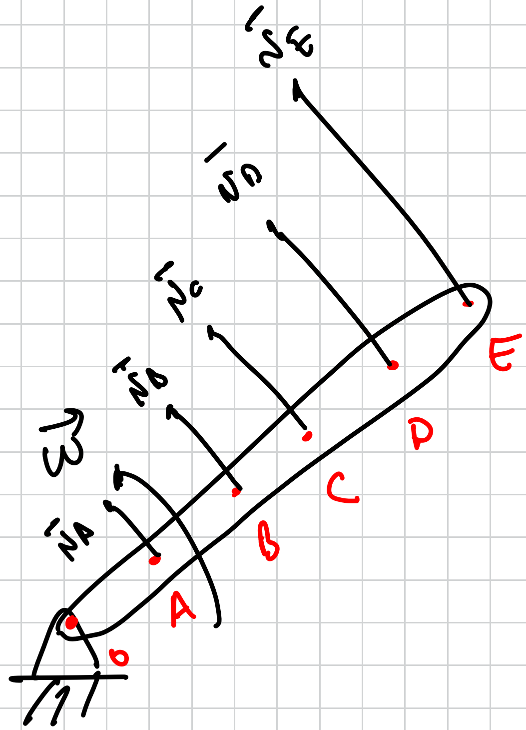
$$\omega = \dot{\theta}_1 = \dot{\theta}_2 \quad , \quad \alpha = \ddot{\theta}_1 \, , \, \ddot{\theta}_2$$

Derivation Remarks

→ $\vec{\omega}$ and $\vec{\alpha}$ describe the angular motion of the rigid body and are the same for any pair of points on the rigid body (e.g.: DE, AB, DB, ...)

→ A rigid body has one AND ONLY ONE angular velocity and ONLY ONE angular acceleration

→ $\hat{k}$ is the direction of the axis about which the rigid body rotates The signs $\pm$ provide a sense of the direction of rotation ($\pm \omega \hat{k}$) and whether it's speeding up or slowing down ($\pm \alpha \hat{k}$)



All points have same $\vec{\omega}$

Simplifications for Planar Motion

If A and B are in a plane perpendicular to $\hat{k}$, (i.e., xy plane):

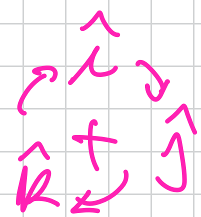
$$\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) = (\omega\hat{k}) \times [(\omega\hat{k}) \times (x\hat{\imath} + y\hat{\jmath})]$$

$$= \omega\hat{k} \times (\omega x\hat{\jmath} - \omega y\hat{\imath})$$

$$= -\omega^2 x\hat{\imath} - \omega^2 y\hat{\jmath}$$

$$= -\omega^2 (x\hat{\imath} + y\hat{\jmath})$$

$$= -\omega^2 \vec{r}_{B/A}$$



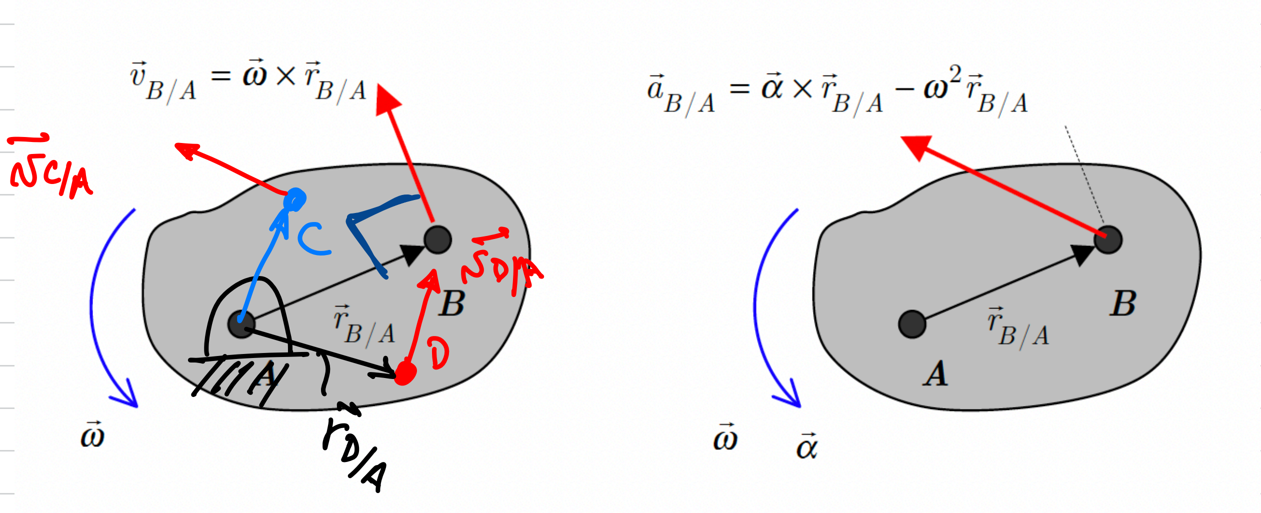
So, $$\boxed{\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A}}$$

For planar Motion

If one point is fixed (e.g.: A), then

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

Since $\vec{\omega} \times \vec{r}_{B/A}$ is perpendicular to $\vec{r}_{B/A}$, $\vec{v}_B \perp \vec{r}_{B/A}$ when the body rotates @



$\vec{v}_{B/A}$ always perpendicular to $\vec{r}_{B/A}$

$\vec{a}_{B/A}$ not always perpendicular to $\vec{r}_{B/A}$

Example 2.A.1

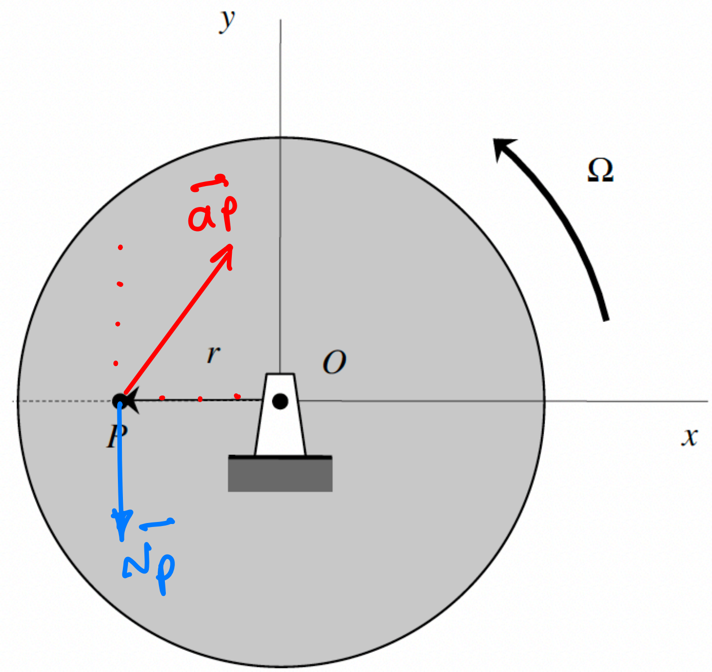
Given: The disk shown is rotating at a non-constant rate of Ω about a fixed axis passing through its center O. At a particular instant, the acceleration vector of point P on the disk is $\vec{a}_P$.

Find: Determine:

(a) The angular velocity of the disk at this instant; and $\rightarrow \Omega\hat{k}$

(b) The angular acceleration of the disk at this instant.

Use the following parameters in your analysis: $\vec{a}_P = 3\hat{\imath} + 4\hat{\jmath}$ m/s^2 and $r = 0.4$ m. Also, be sure to write your answers as vectors.



Solution

$$\vec{v}_P = \cancel{\vec{v}_O}^{\,0} + \vec{\omega} \times \vec{r}_{P/O}$$

$$\vec{v}_P = \Omega\hat{k} \times (-r\hat{\imath})$$

$$= \Omega r(-\hat{\jmath})$$

$$\vec{v}_P = -\Omega r\hat{\jmath}$$

I don't know about

I know about

$$\vec{a}_P = \cancel{\vec{a}_O}^{\,0} + \vec{\alpha} \times \vec{r}_{P/O} + \vec{\omega} \times [\vec{\omega} \times \vec{r}_{P/O}]$$

$$= \alpha\hat{k} \times (-r\hat{\imath}) + \omega\hat{k} \times [\omega\hat{k} \times (-r\hat{\imath})]$$

$$= -\alpha r\,\hat{j} + \omega\hat{k} \times (-\omega r\,\hat{j})$$

$$3\hat{i} + 4\hat{j} = -\alpha r\,\hat{j} + \omega^2 r\,\hat{i}$$

$\hat{j}$: $\quad 4 = -\alpha r \Rightarrow \alpha = -\frac{4}{r} = -\frac{4}{0.4} = -10\ \frac{rad}{s^2}$

$\hat{i}$: $\quad 3 = \omega^2 r \Rightarrow \omega = \sqrt{\frac{3}{r}} = \sqrt{\frac{3}{0.4}} \quad 2.73\ \frac{rad}{s}$