JOINT DESCRIPTION OF MOTION

1/21/26

In this section we combine different descriptions of motion for convenient use:

So far, we derived:

Cartesian		Path		Polar
$\vec{r} = x\hat{\imath} + y\hat{\jmath}$	$\longrightarrow$	$r[s(t)]$	$\longrightarrow$	$r\hat{e}_r$
$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$	$\longrightarrow$	$v\hat{e}_t$	$\longrightarrow$	$\dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$
$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$	$\longrightarrow$	$\dot{r}\hat{e}_t + r\dot{\theta}\hat{e}_n$	$\longrightarrow$	$(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

Descriptions other than Cartesian are tools to simplify the analysis. Regardless of system, velocity is velocity and acceleration is acceleration in any description.

This means that all three descriptions of velocity refer to the same velocity <u>vector</u> similarly for acceleration.

Some problems will require the combined use of two ore more kinematic descriptions. As such, we will need to convert from/to one another.

Procedure for converting:

1) Write unit vectors of Description 1 in terms of Description 2

2) Use vector projections or coefficient balance to find unknown kinematic descriptions.

⚠ To project, vectors need to be in the same system!

Think about the motion of a car described in $x(t)\hat{i} + y(t)\hat{j}$. If I need to know the rate of change of the speed or the sideway acceleration, I might need to use Path coordinates (makes much more sense!)

Some case examples:

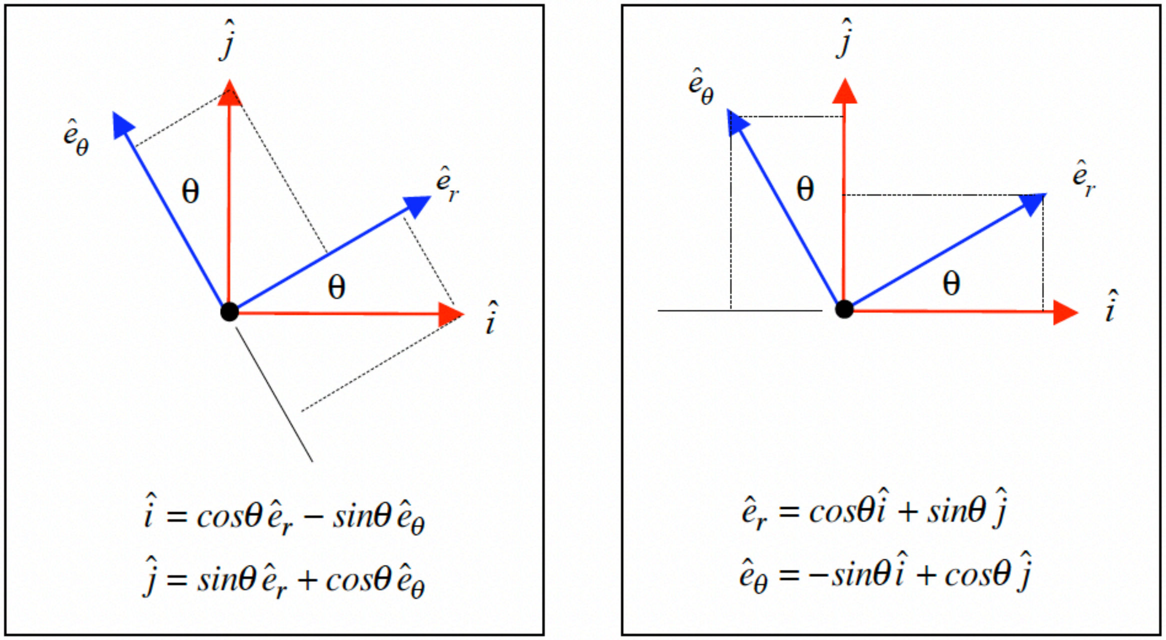
- Car on curved exit ramp (clothoid example)
- Roller coasters. Shapes usually given in Cartesian cooldinates, but path is used to calculate accelerations
- Satellite orbit relative to ground (given in polar but translated to Cartesian)

Example:

Suppose that the velocity and acceleration of a particle are known in terms of their polar coordinates as: $\vec{v} = (10\hat{e}_r - 20\hat{e}_\theta)$ m/s and $\vec{a} = (3\hat{e}_r + 2\hat{e}_\theta)$ m/s^2, where the orientation of the polar unit vectors are shown below relative to a set of Cartesian vectors. From this we want to find the Cartesian components of velocity and acceleration when $\theta = 36.87°$.

$\sin\theta = 3/5 = 0.6$

$\cos\theta = 4/5 = 0.8$



Solution:

1) Method of projection:

$$\hat{i} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta = 4/5\,\hat{e}_r - 3/5\,\hat{e}_\theta$$

$$\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta = 3/5\,\hat{e}_r + 4/5\,\hat{e}_\theta$$

Let's project unit vectors $(\hat{\imath},\hat{\jmath})\big|_{\hat{e}_r,\hat{e}_\theta}$ into $\vec{v}\big|_{\hat{e}_r,\hat{e}_\theta}$

$$\dot{x} = \hat{\imath}\big|_{\hat{e}_r,\hat{e}_\theta} \cdot \vec{v}\big|_{\hat{e}_r,\hat{e}_\theta}$$

$$\dot{x} = (0.8\,\hat{e}_r + 0.6\,\hat{e}_\theta)\cdot(10\,\hat{e}_r - 20\,\hat{e}_\theta) = 8 - 12 = -4 \ \tfrac{m}{s}$$

$$\dot{y} = \hat{\jmath}\big|_{\hat{e}_r,\hat{e}_\theta} \cdot \vec{v}\big|_{\hat{e}_r,\hat{e}_\theta}$$

$$\dot{y} = (0.6\,\hat{e}_r + 0.8\,\hat{e}_\theta)\cdot(10\,\hat{e}_r - 20\,\hat{e}_\theta) = 6 - 16 = -10 \ \tfrac{m}{s}$$

$$\therefore \quad \vec{v} = -4\,\hat{\imath} - 10\,\hat{\jmath} \ \ m/s \quad = 10\,\hat{e}_r - 20\,\hat{e}_\theta \ \tfrac{m}{s}$$

Similarly for acceleration:

$$\ddot{x} = \hat{\imath} \cdot \vec{a} = (0.8\hat{e}_r + 0.6\hat{e}_\theta)\cdot(3\hat{e}_r + 2\hat{e}_\theta)$$

$$\ddot{x} = 12/5 - 6/5 = 6/5 \ m/s^2$$

$$\ddot{y} = \hat{\jmath} \cdot \vec{a} = (0.8\hat{e}_r + 0.6\hat{e}_\theta)\cdot(3\hat{e}_r + 2\hat{e}_\theta)$$

$$\ddot{y} = 9/5 + 8/5 = 17/5 \ m/s^2$$

$$\therefore \quad \vec{a} = 6/5\,\hat{\imath} + 17/5\,\hat{\jmath} \ m/s^2 = 3\hat{e}_r - 2\hat{e}_\theta \quad m/s^2$$

2) Method of substitution : Let's use the figure on the right to write $\hat{e}_r$ and $\hat{e}_\theta$ in terms of $\hat{\imath}$ and $\hat{\jmath}$

$$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} = 0.8\,\hat{\imath} + 0.6\,\hat{\jmath}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} = -0.6\,\hat{\imath} + 0.8\,\hat{\jmath}$$

Then,

$$\vec{v} = 10\,\hat{e}_r - 20\,\hat{e}_\theta$$

$$= 10\left(0.8\,\hat{\imath} + 0.6\,\hat{\jmath}\right) - 20\left(-0.6\,\hat{\imath} + 0.8\,\hat{\jmath}\right)$$

$$= \hat{\imath} + 6\,\hat{\jmath} + 12\,\hat{\imath} - 16\,\hat{\jmath} = 20\,\hat{\imath} - 10\,\hat{\jmath} \ \text{m/s}$$

$$\vec{a} = 3\,\hat{e}_r + 2\,\hat{e}_\theta$$

$$= 3\left(0.8\,\hat{\imath} + 0.6\,\hat{\jmath}\right) + 2\left(-0.6\,\hat{\imath} + 0.8\,\hat{\jmath}\right)$$

$$\vec{a} = 1.2\,\hat{\imath} + 3.4\,\hat{\jmath}$$

Example:

Suppose the velocity and acceleration of a particle are known in terms of their Cartesian components as: $\vec{v} = (30\hat{\imath} - 40\hat{\jmath})$ m/s and $\vec{a} = (-10\hat{\jmath})$ m/s^2. From this, we want to find the speed v, rate of change of speed $\dot{v}$ and the radius of curvature ρ of the path of the particle (path description variables).

Solution:

Velocity is velocity in any description.

$\Rightarrow$ speed $v = \sqrt{30^2 + 40^2} = 50$ m/s

Now, since $\vec{v} = v\,\hat{e}_t$, I can obtain the unit vector $\hat{e}_t$ simply by

$$\hat{e}_t = \frac{\vec{v}}{|\vec{v}|} = \frac{30\hat{\imath} - 40\hat{\jmath}}{50} = \boxed{0.6\,\hat{\imath} - 0.8\,\hat{\jmath}}$$

unit vector $\hat{e}_t$ expressed in cartesian

Next, let's obtain $\dot{v}$ by projecting $\hat{e}_t$ onto $\vec{a}$:

$$\dot{v} = \hat{e}_t\Big|_{\hat{\imath},\hat{\jmath}} \cdot \vec{a}\Big|_{\hat{\imath},\hat{\jmath}} = (0.6\hat{\imath} - 0.8\hat{\jmath})\cdot(-10\hat{\jmath}) = 8 \text{ m/s}^2$$

Now recall that $\vec{a} = \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n$

$$\Rightarrow \quad |\vec{a}|^2 = \dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2 \quad \Rightarrow \quad |\vec{a}|^2 - \dot{v}^2 = \left(\frac{v^2}{\rho}\right)^2$$

Solving for ρ:

$$\rho = \frac{v^2}{\sqrt{|\vec{a}| - \dot{v}^2}} = \frac{50^2}{\sqrt{100-64}} = \frac{2500}{6} \ m$$

Key strategy:

This is not a memory workout. Rather try to understand the kinematic relationships behind one description or the other and work from there.

In general, the directional vector of a vector (unit vector) is $\frac{\vec{v}}{|\vec{v}|}$, For dynamics is useful $\Rightarrow$ $\vec{v} = v\,\hat{e}_t \Rightarrow \hat{e}_t = \frac{\vec{v}}{|\vec{v}|}$

Example 1.C.2

Given: Pin P is constrained to move in the slotted guides that move at right angles to one another. At the instant shown, guide A moves to the right with a speed of v_A, a speed that is changing at a rate of $\dot{v}_A$. At the same time, B is moving downward with a speed of v_B with a rate of change of speed of $\dot{v}_B$.

indications of Cartesian

Find:

(a) The rate of change of speed of P at this instant; and

(b) The radius of curvature ρ of the path followed by P at this instant.

indications of path desc.

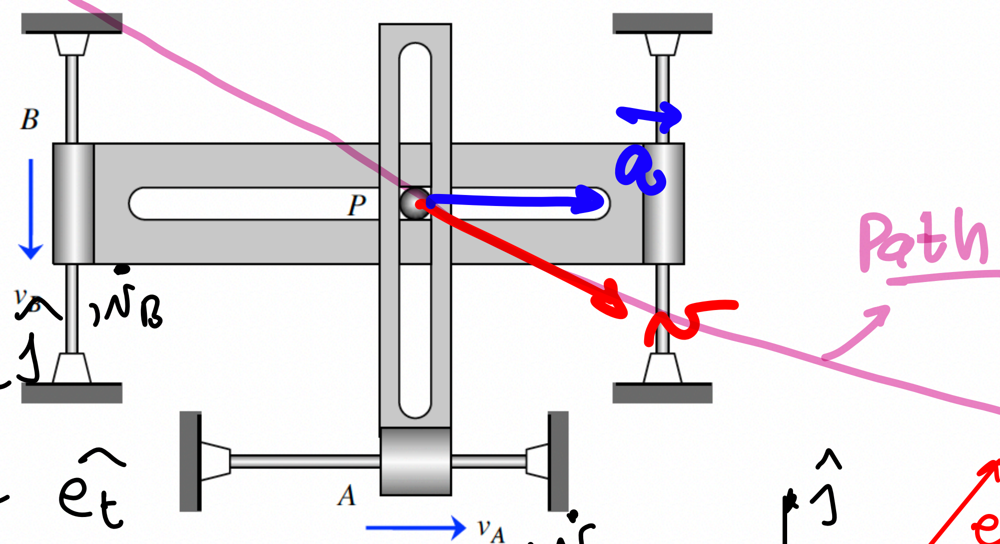
Use the following parameters: $v_A = 0.2$ m/s, $v_B = 0.15$ m/s, $\dot{v}_A = 0.75$ m/s^2 and $\dot{v}_B = 0$.

Solution:

$$\vec{v} = v_A \hat{i} - v_B \hat{j}$$

Direction of any vector is

$$\frac{\vec{v}}{|\vec{v}|} = \frac{v_A}{\sqrt{v_A^2 + v_B^2}} \hat{i} - \frac{v_B}{\sqrt{v_A^2 + v_B^2}} \hat{j}$$

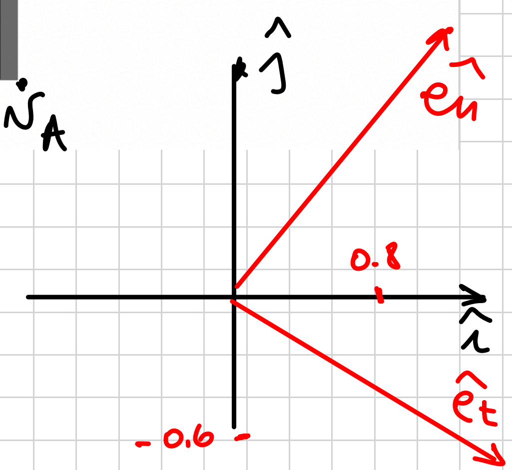


This also happens to be $\hat{e}_t$

$$\vec{v} = v \hat{e}_t \Rightarrow \hat{e}_t = \frac{\vec{v}}{|\vec{v}|}$$

$$\text{speed} = |\overrightarrow{\text{velocity}}|$$

$$\Rightarrow \quad \hat{e}_t = 0.8 \hat{i} - 0.6 \hat{j}$$



$$\hat{e}_n = 0.6\,\hat{i} + 0.8\,\hat{j} \quad \text{(from geometry)}$$

Once $\hat{e}_n$ and $\hat{e}_t$ are known, finding $\dot{v}$, and ρ is just a matter of projection.

$$\vec{a} = \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n = \dot{v}_A\,\hat{i} + \cancel{\dot{v}_B}\,\hat{j} \quad 0$$

$$\dot{v} = \hat{e}_t\Big|_{\hat{i},\hat{j}} \cdot \vec{a}\Big|_{\hat{i},\hat{j}} = \underbrace{(0.8\,\hat{i} - 0.6\,\hat{j})}_{\hat{e}_t} \cdot \underbrace{(0.75\,\hat{i})}_{\vec{a}} = (0.8)(0.75)$$

$$\dot{v} = 0.6 \ \mathrm{m/s^2}$$

$$\frac{v^2}{\rho} = \hat{e}_n\Big|_{\hat{i}\hat{j}} \cdot \vec{a}\Big|_{\hat{i}\hat{j}} = \underbrace{(0.6\,\hat{i} + 0.8\,\hat{j})}_{\hat{e}_n} \cdot \underbrace{(0.75\,\hat{i})}_{\vec{a}} = (0.6)(0.75)$$

$$\frac{v^2}{\rho} = 0.45 \ \mathrm{m/s^2}$$

Solving for ρ:

$$\rho = \frac{(0.2^2 + 0.15^2)}{0.45} = 0.138 \ \mathrm{m}$$

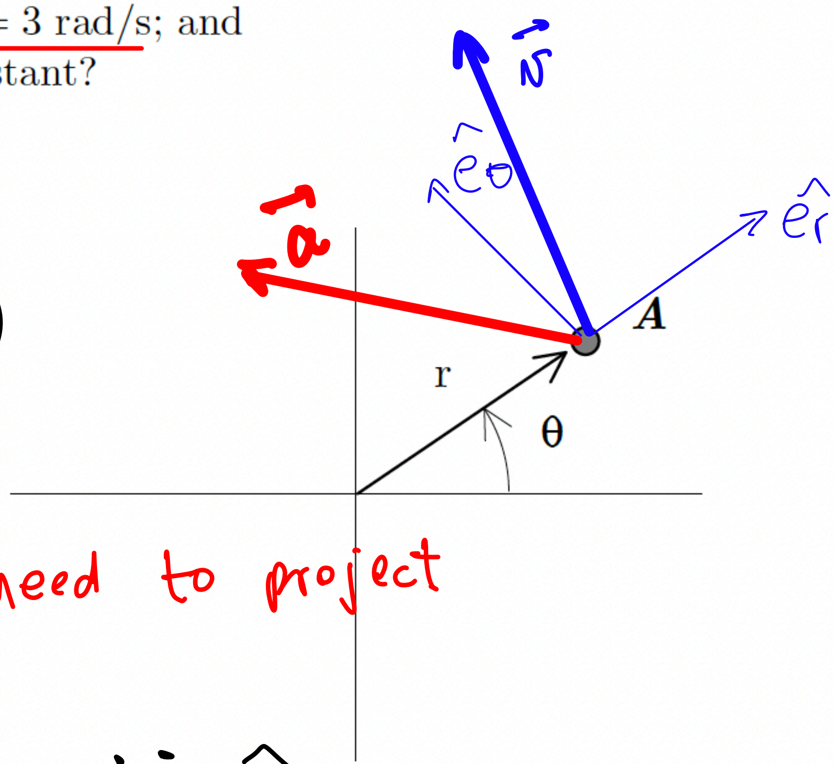
Example 1.C.3

Given: The path of point P is given in polar coordinates as: $r = 4 + 2\theta$, where r is given in meters and θ is given in radians. The angle θ is increasing at a constant rate of $\dot{\theta}$.

Find: Determine:

(a) The acceleration vector of P when $\theta = \pi$, if $\dot{\theta} = 3$ rad/s; and

(b) Is the speed of P increasing, decreasing or constant?



Given: $r = 4 + 2\theta$ (m)

θ (rad)

$\dot{\theta}$ (rad/s)

Find $\vec{a}$, $\dot{v}$

Problem asks for $\dot{v}$ $\Rightarrow$ need to project from polar to path!

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\dot{r} = 2\cancel{\dot{\theta}}^{3} = 6 \text{ m/s}$$

$$\ddot{r} = 2\cancel{\ddot{\theta}}^{0} = 0$$

$$\vec{a} = -r\dot{\theta}^2\,\hat{e}_r + 2\dot{r}\dot{\theta}\,\hat{e}_\theta$$

$$\vec{a} = -(4+2\theta)(3)^2\,\hat{e}_r + 2(6)(3)\,\hat{e}_\theta$$

$$\vec{a} = -(36+6\pi)\,\hat{e}_r + 36\,\hat{e}_\theta$$

$$\vec{a} = -54.84\,\hat{e}_r + 36\,\hat{e}_\theta$$

Now, for velocity:

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta = 6\hat{e}_r + (4+2\theta)3\,\hat{e}_\theta$$

$$\vec{v} = 6\hat{e}_r + (12+6\pi)\hat{e}_\theta = 6\hat{e}_r + 30.84\,\hat{e}_\theta$$

I need $\hat{e}_t$ in terms of $\hat{e}_r$ and $\hat{e}_\theta$.

$$\Rightarrow \hat{e}_t = \frac{\vec{v}\,|_{\hat{e}_r,\hat{e}_\theta}}{|\vec{v}\,|_{\hat{e}_r,\hat{e}_\theta}|} = \frac{6}{\sqrt{6^2+(12+6\pi)^2}}\,\hat{e}_r + \frac{12+6\pi}{\sqrt{6^2+(12+6\pi)^2}}\,\hat{e}_\theta$$

$$\hat{e}_t = 0.19\,\hat{e}_r + 0.98\,\hat{e}_\theta$$

Only, now I can do projection of $\vec{a}$ onto $\hat{e}_t$

$$\dot{v} = \vec{a} \cdot \hat{e}_t = \left[-(36+6\pi)\hat{e}_r + 36\hat{e}_\theta\right] \cdot (0.19\hat{e}_r + 0.98\hat{e}_\theta)$$

$\dot{v} = 24.85 \ m/s^2 > 0 \Rightarrow$ speed of P INCREASING!